Complexity of the Logic for Multiagent Systems with Restricted Modal Context

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Abstract. In this paper we continue our research in computational complexity of fragments of a particular belief-desire-intention (BDI) logic for cooperative problem solving (CPS) called TeamLog. The logic in question combines propositional multi-modal logic to express static aspects of CPS with propositional dynamic logic used for dynamic aspects of CPS. We concentrate here on the static fragment of the logic.
The complexity of the satisfiability problem for the static fragment of TeamLog is known to be EXPTIME-complete, even if modal depth is bounded by a constant ≥ 2. We show a restriction on modal context of formulas that makes the problem PSPACE-complete without setting boundary on modal depth. Bounding modal depth, additionally, leads to NPTIME-completeness.

1 Introduction

In this paper, we investigate the complexity of the satisfiability problem of a teamwork logic called TeamLog [1–3]. Although the results and methods of this paper may be applied to a complexity analysis of many multi-modal logics combining different, but interrelated agent attitudes, the jumping point of the paper is a theory of teamwork presented in [1], and briefly introduced below.

In teamwork, when a team of agents aims to work together in a planned and coherent way, the group as a whole needs to present a common collective attitude over and above individual attitudes of team members. Collective motivational attitudes towards a common goal are essential for achieving a sensible organization of cooperation [1, 3]. In the approach undertaken in TeamLog, the fundamental role of collective intention is to consolidate a group as a cooperating team, while collective commitment leads to team action, i.e., coordinated realization of individual actions by agents that have committed to do them according to a team plan (see [3]). The formal theory constructed in TeamLog takes a viewpoint of the system developer who wants to reason about, specify and verify a multiagent system.

The logical framework of TeamLog is squarely multi-modal, in the sense that different operators are combined and may interfere [1, 3]. We have shown in [4] that the individual part of the theory of teamwork, called TeamLog$^{\text{ind}}$

is PSPACE-complete and the full system, modelling a subtle interplay between individual and group attitudes, is EXPTIME-complete, and remains so even if propositional dynamic logic is added. It turns out that for many interesting formulas appearing in human reasoning, satisfiability tends to be easier to compute than suggested by the worst-case labels like "PSPACE-complete" and "EXPTIME-complete" [5]. Therefore it is worthwhile to investigate by which reasonable means the complexity of the teamwork theory may be reduced. In [6] and [7] we investigated the complexity class of the satisfiability problem if modal depth or number of propositional symbols is bounded by a constant. In the case of TeamLog$^{\mathrm{ind}}$ fragment modal depth reduction leads to NPTIME-completeness, while in the case of the full TeamLog framework the problem remains EXPTIME-complete, if the boundary is ≥ 2. In this paper we propose a restriction for modal context of formulas for which the satisfiability problem is PSPACE-complete.

2 Logical Framework

As mentioned before, the TeamLog framework uses multi-modal logics to formalize static aspects of multi agent systems, that is agents' informational and motivational attitudes. In original papers where the framework is introduced these modalities are given names that make it easy to recognize which attitudes are represented by them. In this paper however we have chosen to use a more compact notation, which is more convenient for presentation of proofs and algorithms. In the following presentation we will state to which attitudes the modal operators in this new notation refer. Below we give a summary of correspondence between the notation used here and the standard TeamLog notation.

$$\begin{array}{rl} \mathrm{BEL}(i,\varphi) \equiv [i]_{\mathrm{B}}\varphi & \mathrm{E\text{-}BEL}_G(\varphi) \equiv [G]_{\mathrm{B}}\varphi \\ \mathrm{GOAL}(i,\varphi) \equiv [i]_{\mathrm{G}}\varphi & \mathrm{C\text{-}BEL}_G(\varphi) \equiv [G]_{\mathrm{B}}^{+}\varphi \\ \mathrm{INT}(i,\varphi) \equiv [i]_{\mathrm{I}}\varphi & \mathrm{E\text{-}INT}_G(\varphi) \equiv [G]_{\mathrm{I}}\varphi \\ & \mathrm{M\text{-}INT}_G(\varphi) \equiv [G]_{\mathrm{I}}^{+}\varphi \end{array}$$

2.1 The Language and Notation

Although this work is concerned with logics for multiagent systems, it deals with several systems of multi-modal logics and it will be convenient to abstract out from aforementioned attitudes and to concentrate on groups of modal operators interrelated with different axioms.

Formulas are defined with respect to a fixed finite set of agents. The inductive definition of the language is given below.

Definition 1 (Language). The definition is based on the following two sets:

- an enumerable set $\mathcal{P}$ of propositional symbols,
- a finite, nonempty set $\mathcal{A}$ of *agents*, denoted by numerals $1, \ldots, n$.

The language $\mathcal{L}$ is the smallest set satisfying the following:

- $\mathcal{P} \subseteq \mathcal{L}$,
- if $\psi \in \mathcal{L}$ then $\neg\psi \in \mathcal{L}$,
- if $\{\psi_1, \psi_2\} \subseteq \mathcal{L}$ then $\{\psi_1 \wedge \psi_2\} \subseteq \mathcal{L}$,
- if $\psi \in \mathcal{L}$ and $i \in \mathcal{A}$ then $\{[i]_{\mathrm{B}}\psi, [i]_{\mathrm{G}}\psi, [i]_{\mathrm{I}}\psi\} \in \mathcal{L}$,
- if $\psi \in \mathcal{L}$ and $G \subseteq \mathcal{A}$ then $\{[G]_{\mathrm{B}}\psi, [G]_{\mathrm{I}}\psi\} \in \mathcal{L}$.
- if $\psi \in \mathcal{L}$ and $G \subseteq \mathcal{A}$ then $\{[G]^+_{\mathrm{B}}\psi, [G]^+_{\mathrm{I}}\psi\} \in \mathcal{L}$.

The standard propositional constants and connectives $\top$, $\bot$, $\wedge$, $\rightarrow$ and $\leftrightarrow$ are defined in the usual way.

Modalities $[i]_{\mathrm{B}}$, $[i]_{\mathrm{G}}$ and $[i]_{\mathrm{I}}$ correspond to agent's i beliefs, goals and intentions, respectively. Moreover $[G]_{\mathrm{B}}$ and $[G]_{\mathrm{I}}$ correspond to general beliefs and general intentions in the group of agents G, respectively. Modalities $[G]^+_{\mathrm{B}}$ and $[G]^+_{\mathrm{I}}$ correspond to common beliefs and mutual intentions within the group of agents G, respectively. These modalities are closely related to iteration $[\cdot]^*$ in PDL (we used + superscript, since it refers to "one or more" repetition rather than "zero or more", as in case of $*$). We call these modalities *iterated modalities*.

We will use the sign $\Box$ to denote any operator of the form $[\cdot]_{\mathrm{O}}$ or $[\cdot]^+_{\mathrm{O}}$. Given a finite set of formulas Φ, $\bigwedge \Phi$ will be used to denote the conjunction of all formulas in Φ (if $\Phi = \varnothing$ then $\bigwedge \Phi \equiv \top$). The set of all subformulas of a given formula φ will be denoted by $\mathrm{Sub}(\varphi)$, and $\neg\mathrm{Sub} = \mathrm{Sub}(\varphi) \cup \{\sim\psi : \psi \in \mathrm{Sub}(\varphi)\}$, where $\sim\psi$ denotes ξ if ψ is negated formula of the form $\neg\xi$ and $\neg\psi$ if ψ is not negated.

Throughout the paper we use the notion of the modal depth defined below.

Definition 2 (Modal depth). The *modal depth* of a formula, denoted by $\mathrm{dep}(\cdot)$ is defined inductively as follows:

- $\mathrm{dep}(p) = 0$, where $p \in \mathcal{P}$,
- $\mathrm{dep}(\neg\psi) = \mathrm{dep}(\psi)$,
- $\mathrm{dep}(\psi_1 \wedge \psi_2) = \max\{\mathrm{dep}(\psi_1), \mathrm{dep}(\psi_2)\}$,
- $\mathrm{dep}(\Box\psi) = \mathrm{dep}(\psi) + 1$.

Let $\Phi \subseteq \mathcal{L}$, then $\mathrm{dep}(\Phi) = \max\{\mathrm{dep}(\psi) : \psi \in \Phi\}$, if $\Phi \neq \varnothing$, and $\mathrm{dep}(\varnothing) = 0$.

The following restriction on modal context of formulas will be crucial.

R A formula φ satisfies the restriction iff for any $[G]^+_{\mathrm{O}}\psi \in \mathrm{Sub}(\varphi)$ there is no formula $[i]_{\mathrm{O}}\xi \in \mathrm{Sub}(\psi)$ with $i \in G$ and no formula $[G']^+_{\mathrm{O}}\xi \in \mathrm{Sub}(\psi)$ with $G' \cap G \neq \varnothing$.

When applied to beliefs and common beliefs of agents (i.e. modal operator $[G]^+_{\mathrm{B}}$), this restriction could be seen as forbidding common introspection within a group of agents. When applied to intentions and mutual intentions (i.e. modal operator $[G]^+_{\mathrm{I}}$), this restriction forbids group of agents to have mutual intentions towards intentions of agents within this group.

2.2 Kripke Model

The semantics of logics under consideration is traditionally defined with use of the notion of a Kripke model.

Definition 3 (Kripke model).
A Kripke model is a tuple
$\mathcal{M} = (W, \{B_i : i \in \mathcal{A}\}, \{G_i : i \in \mathcal{A}\}, \{I_i : i \in \mathcal{A}\}, Val)$*, such that*

1. W *is a set of possible worlds, or states;*
2. *For all* $i \in \mathcal{A}$*, it holds that* $B_i, G_i, I_i \subseteq W \times W$*. They stand for the accessibility relations for each agent with respect to* beliefs, goals*, and* intentions, *respectively. For example,* $(s, t) \in B_i$ *means that* t *is an epistemic alternative for agent* i *in state* s*.*
3. $Val : \mathcal{P} \times W \to \{0, 1\}$ *is a valuation function that assigns the truth values to atomic propositions in states.*

Definition 4 (Kripke interpretation). *Kripke interpretation* is a pair $(\mathcal{M}, w)$, where $\mathcal{M}$ is a Kripke model and $w \in W$ (W is the set of worlds of the model $\mathcal{M}$).

Let $\{R_i : i \in \mathcal{A} \text{ and } R_i \subseteq W \times W\}$ be a collection of binary relations on W, $G \subseteq \mathcal{A}$ and $R \subseteq W \times W$. We will use R_G to denote $\bigcup_{i \in G} R_i$ and R^+ to denote the transitive closure of R. Moreover, let G^+ denote the set of all non empty sequences over the set G and let $i_1, \ldots, i_m \in G^+$. We will use $R_{i_1,\ldots,i_m}$ to denote the composition of relations $R_{i_1}, \ldots, R_{i_m}$, i.e. $(u, v) \in R_{i_1,\ldots,i_m}$ iff there is a sequence of states $u_0, \ldots u_m$ such that $u_0 = u$, $u_m = v$ and $(u_{j-1}, u_j) \in R_{i_j}$ for all $1 \leq j \leq m$. We will use $(R_i)^m$ as a short cut for $R_{i_1,\ldots,i_m}$ with $i_1 = \ldots = i_m$. Obviously $(R_G)^+ = \bigcup_{\sigma \in G^+} R_\sigma$.

2.3 Semantics

Semantics of formulas of the language $\mathcal{L}$ is defined as follows:

Definition 5 (Semantics of formulas). Let $(\mathcal{M}, w)$ be a Kripke interpretation, $i \in \mathcal{A}$ and $\mathcal{A} \supseteq G \neq \varnothing$. Semantics of formulas is defined inductively as follows:

- $(\mathcal{M}, w) \models p$, iff $w \in h(p)$,
- $(\mathcal{M}, w) \models \neg\psi$, iff it doesn't hold that $(\mathcal{M}, w) \models \psi$,
- $(\mathcal{M}, w) \models \psi_1 \wedge \psi_2$, iff $(\mathcal{M}, w) \models \psi_1$ and $(\mathcal{M}, w) \models \psi_2$,
- $(\mathcal{M}, w) \models [i]_{\mathrm{B}}\psi$, iff $(\mathcal{M}, v) \models \psi$ for all v such that $(w, v) \in B_i$, (analogically for $[i]_{\mathrm{G}}\psi$ and $[i]_{\mathrm{I}}\psi$),
- $(\mathcal{M}, w) \models [G]_{\mathrm{B}}\psi$, iff $(\mathcal{M}, v) \models \psi$ for all v such that $(w, v) \in B_G$, (analogically for $[G]_{\mathrm{I}}\psi$),
- $(\mathcal{M}, w) \models [G]^+_{\mathrm{B}}\psi$, iff $(\mathcal{M}, v) \models \psi$ for all v such that $(B_G)^+ (w, v)$ (analogically for $[G]_{\mathrm{I}}\psi$).

Let $(\mathcal{M}, w)$ be a Kripke interpretation. A formula φ is *satisfied* in $(\mathcal{M}, w)$ if $(\mathcal{M}, w) \models \varphi$. A formula is *satisfiable* if there is an interpretation $(\mathcal{M}, w)$ such that $(\mathcal{M}, w) \models \varphi$.

2.4 Axiom System

The modal logic considered here is a *normal multimodal logic*, i.e. all modalities satisfy axiom **K** and generalisation rule **GEN**. Moreover different families of modalities, distinguished in the notation adopted here by subscripts B, G or I satisfy different additional axioms. For B family it is a system $\mathrm{KD45_n}$, for G family it is a system $\mathrm{K_n}$ and for I it is a system $\mathrm{KD_n}$. There are also additional axioms interrelating different families of operators, as presented below. The whole system is presented and discussed in more details in [3].

Definition 6 (Axiom system). In what follows $j \in \mathcal{A}$ and $G \subseteq \mathcal{A}$, where $\mathcal{A}$ is a set of agents.

P1	All instantiations of propositional tautologies;
K	$[j]_{\mathrm{O}}(\varphi \to \psi) \to ([j]_{\mathrm{O}}\varphi \to [j]_{\mathrm{O}}\psi)$, where O is either B, G or I;
D	$[j]_{\mathrm{O}}\psi \to \langle j\rangle_{\mathrm{O}}\psi$, where O is either B or I;
4(B,O)	$[j]_{\mathrm{O}}\psi \to [j]_{\mathrm{B}}[j]_{\mathrm{O}}\psi$, where O is either B, G or I;
5(B,O)	$\langle j\rangle_{\mathrm{O}}\psi \to [j]_{\mathrm{B}}\langle j\rangle_{\mathrm{O}}\psi$, where O is either B, G or I;
GI	$[j]_{\mathrm{I}}\psi \to [j]_{\mathrm{G}}\psi$;
S1	$[G]_{\mathrm{O}}\psi \leftrightarrow \bigwedge_{i\in G}[i]_{\mathrm{O}}\psi$, where O is either B or I;
S2	$[G]^{+}_{\mathrm{O}}\psi \leftrightarrow [G]_{\mathrm{O}}(\psi \wedge [G]^{+}_{\mathrm{O}}\psi)$, where O is either B or I;
MP	From φ and $\varphi \to \psi$, infer ψ;
GEN	From ψ infer $[j]_{\mathrm{O}}\psi$, where O is either B, G or I;
IND	From $\varphi \to [G]_{\mathrm{O}}(\psi \wedge \varphi)$ infer $\varphi \to [G]^{+}_{\mathrm{O}}\psi$, where O is either B or I.

In modal logics there exists a correspondence between axiom systems and properties of accessibility relations, in a sense that it can be shown that soundness and completeness hold for a given axiom system and a class of models where accessibility relation have certain properties. In case of axioms considered here there is the following correspondence between them and properties of accessibility relations (as before $j \in \mathcal{A}$).

PD	$\forall x \exists y O_j(x, y)$	(**D**);
P4(B,O)	$\forall x, y, z(B_j(x, y) \wedge O_j(y, z) \to O_j(x, z))$	(**4(B,O)**);
P5(B,O)	$\forall x, y, z(B_j(x, y) \wedge O_j(x, z) \to O_j(y, z))$	(**5(B,O)**);
PGI	$G_j \subseteq I_j$	(**GI**).

Correspondence between **D** and **PD** as well as **4(B,B)** and **P4(B,B)**, **5(B,B)** and **P5(B,B)** are well known facts (see for example [8]). Proofs of correspondence for **P4(B,O)** and **4(B,O)**, **P5(B,O)** and **5(B,O)**, **PC(B,O)** and **PGI** are also basic and can be found for example in [3].

The following fact, being a consequence of the property **P4(B,O)**, will be used later in the paper.

Fact 1. Let $O_j \subseteq W \times W$ satisfy property ***P4(B,O)***. Then for any $k > 0$ it holds that

$$\forall x, y, z\Big((B_j)^k (x, y) \wedge O_j(y, z) \to O_j(x, z)\Big).$$

Proof. Induction on k. If $k = 1$ then the fact holds, as it is the property **P4(B,O)**. If $k > 1$ then there is v such that $B_j(x, v)$ and $(B_j)^{k-1}(v, y)$. By the induction hypothesis it holds that $O_j(v, z)$, and thus, by property **P4(B,O)**, $O_j(x, z)$. $\square$

3 Complexity of the Satisfiability Problem

We will show that the satisfiability problem for TeamLog formulas with modal context restricted by **R** is PSPACE-complete. Moreover we will show that bounding modal depth in addition to **R** leads to NPTIME-completeness.

PSPACE-hardness of the problem follows immediately from the fact that the satisfiability problem for TeamLog$^{\mathrm{ind}}$is PSPACE-hard, so what remains to be shown is that the problem is in PSPACE. To do this we will present an algorithm for deciding the satisfiability of a restricted TeamLog formula that can be run in deterministic polynomial space. The algorithm checks the satisfiability of a formula by trying to construct a structure called a pre-tableau, which is a basis for an interpretation for φ, if the algorithm decides that φ is satisfiable.

The construction of the algorithm and related results are based on the method presented in [5]. Similar algorithm was used in [4], [6] and [7] where the complexity of TeamLog$^{\mathrm{ind}}$ and TeamLog was studied.

The method is centred around the well known notions of a *propositional tableau*, a *fully expanded propositional tableau* and a *tableau* designed for a particular system of multi-modal logic. Let us give adaptations of the most important definitions from [5] as a reminder:

Definition 7 (Propositional tableau). A *propositional tableau* is a set $\mathcal{T}$ of formulas such that:

1. if $\neg\neg\psi \in \mathcal{T}$ then $\psi \in \mathcal{T}$;
2. if $\varphi \wedge \psi \in \mathcal{T}$ then $\{\varphi, \psi\} \subseteq \mathcal{T}$;
3. if $\neg(\varphi \wedge \psi) \in \mathcal{T}$ then either $\neg\varphi \in \mathcal{T}$ or $\neg\psi \in \mathcal{T}$;
4. there is no formula ψ such that ψ and $\neg\psi$ are in $\mathcal{T}$.

Definition 8 (Fully expanded propositional tableau). A *fully expanded propositional tableau* is a propositional tableau $\mathcal{T}$ such that:

5. for all $\varphi \in \mathcal{T}$ and all $\psi \in \mathrm{Sub}(\varphi)$ either $\psi \in \mathcal{T}$ or $\sim\psi \in \mathcal{T}$;

Notice that according to Definition 8 fully expanded propositional tableau is a set of formulas that along with each formula ψ contained in it, contains also all its subformulas, each of them either in positive or negated form. [1] [2]

To deal with iterated modalities we define two additional conditions:

It1 If $[G]^+_{\mathrm{B}}\varphi \in \mathcal{T}$ then $\bigcup_{i\in G}\{[i]_{\mathrm{B}}\varphi, [i]_{\mathrm{B}}[G]^+_{\mathrm{B}}\varphi\} \subseteq \mathcal{T}$; similarly for $[G]^+_{\mathrm{I}}\varphi$ w.r.t. I_i.

It2 If $\neg[G]^+_{\mathrm{B}}\varphi \in \mathcal{T}$ then there exists a sequence $i_1, \ldots, i_m \in G^+$ such that $\neg[i_1]_{\mathrm{B}} \ldots [i_m]_{\mathrm{B}}\varphi \in \mathcal{T}$; similarly for $[G]^+_{\mathrm{I}}\varphi$ w.r.t. I_i.

A propositional tableau satisfying these conditions will be called an *expanded tableau*. Fully expanded propositional tableau satisfying these conditions will be called a *fully expanded tableau*.

[1] Negated in a sense of $\sim$, i.e. by negated form of $\neg\xi$ we mean ξ.

[2] Notice that not every propositional tableau is fully expanded. For example a set of formulas $\{\neg(p \wedge q), \neg p\}$ is a propositional tableau which is not fully expanded ($\{\neg(p \wedge q), \neg p, q\}$ is an example of fully expanded propositional tableau having this set as a subset).

Definition 9 (TeamLog **tableau).** A TeamLog *tableau* $\mathcal{T}$ is a tuple $\mathcal{T} = (W, \{B_i : i \in \mathcal{A}\}, \{G_i : i \in \mathcal{A}\}, \{I_i : i \in \mathcal{A}\}, L)$, where W is a set of states, B_i, G_i, I_i are binary relations on W, and L is a *labelling function* associating with each state $w \in W$ a set $L(w)$ of formulas, such that $L(w)$ is an expanded tableau. Here follow the two conditions that every modal tableau for our language must satisfy (see [5]):

6. If $[i]_{\mathrm{B}}\varphi \in L(w)$ and $(w, v) \in B_i$, then $\varphi \in L(v)$; similarly for $[i]_{\mathrm{G}}\varphi$ w.r.t. G_i and $[i]_{\mathrm{I}}\varphi$ w.r.t. I_i.
7. If $\neg[i]_{\mathrm{B}}\varphi \in L(w)$, then there exists a v with $(w, v) \in B_i$ and $\sim\varphi \in L(v)$; similarly for $[i]_{\mathrm{G}}\varphi$ w.r.t. G_i and $[i]_{\mathrm{I}}\varphi$ w.r.t. I_i.

Furthermore, a TeamLog tableau must satisfy the following additional conditions (for all $i \in \mathcal{A}$) related to axioms of $\text{TeamLog}^{\text{ind}}$:

TD	If $[i]_{\mathrm{B}}\varphi \in L(w)$, then either $\varphi \in L(w)$ or there exists $v \in W$ such, that $(w, v) \in B_i$; similarly for $[i]_{\mathrm{I}}\varphi$ w.r.t. I_i.
T45(B,O)	If $(w, v) \in B_i$ then $[i]_{\mathrm{O}}\varphi \in L(w)$ iff $[i]_{\mathrm{O}}\varphi \in L(v)$, where O is either B, G, or I.
TGI	If $(w, v) \in G_i$ and $[i]_{\mathrm{I}}\varphi \in L(w)$ then $\varphi \in L(v)$.

Condition **TD** corresponds to axiom **D**, **T45(B,O)** corresponds to axioms **4(B,O)** and **5(B,O)**, and **TGI** corresponds to axiom **GI**.

Given a formula φ we say that $\mathcal{T} = (W, \{B_i : i \in \mathcal{A}\}, \{G_i : i \in \mathcal{A}\}, \{I_i : i \in \mathcal{A}\}, L)$ is a TeamLog tableau for φ if $\mathcal{T}$ is a TeamLog tableau and there is a state $w \in W$ such that $\varphi \in L(w)$.

The following notation will be useful. Let Φ be a set of formulas, then

$$\begin{aligned} \neg\Phi &= \Phi \cup \{\sim\psi : \psi \in \Phi\}, \\ \Phi/\Box &= \{\psi : \Box\psi \in \Phi\}, \\ \Phi/\neg\Box &= \{\psi : \neg\Box\psi \in \Phi\}, \\ \Phi \cap \Box &= \{\Box\psi : \Box\psi \in \Phi\}. \end{aligned}$$

Let φ be a formula, then $\mathrm{Sub}(\varphi)$ denotes the set of all subformulas of φ.

Lemma 1. *A formula φ is* TeamLog *satisfiable iff there is a* TeamLog *tableau for φ.*

Proof. For the left to right implication assume that φ is satisfiable, and that $(\mathcal{M}, w) \models \varphi$ where $\mathcal{M} = (W, \{B_i : i \in \mathcal{A}\}, \{G_i : i \in \mathcal{A}\}, \{I_i : i \in \mathcal{A}\}, \mathit{Val})$ and $w \in W$. Consider $\mathcal{T} = (W, \{B_i : i \in \mathcal{A}\}, \{G_i : i \in \mathcal{A}\}, \{I_i : i \in \mathcal{A}\}, L)$, where $\psi \in L(s)$ iff $(\mathcal{M}, s) \models \psi$. Since $\varphi \in L(w)$, so it is enough to show that $\mathcal{T}$ is a TeamLog tableau. Using induction on the structure of formulas we will show that for any formula ψ and any $v \in W$ the properties of a TeamLog tableau are satisfied. Most cases are typical and are shown as usual, so we omit them here. We will concentrate on the cases related to iterated modalities. Assume $\psi = [G]^+_{\mathrm{I}}\xi$ and that $[G]^+_{\mathrm{I}}\xi \in L(v)$ for some $v \in W$. Then $(\mathcal{M}, v) \models [G]^+_{\mathrm{I}}\xi$ and so $(\mathcal{M}, v) \models [i]_{\mathrm{I}}\xi$ and $(\mathcal{M}, v) \models [i]_{\mathrm{I}}[G]^+_{\mathrm{I}}\xi$ for all $i \in G$. Thus $\{[i]_{\mathrm{I}}\xi, [i]_{\mathrm{I}}[G]^+_{\mathrm{I}}\xi\} \subseteq L(v)$. Now let $\psi = \neg[G]^+_{\mathrm{I}}\xi$ and let $\neg[G]^+_{\mathrm{I}}\xi \in L(v)$ for some $v \in W$. Then

$(\mathcal{M}, v) \models \neg[G]^+_{\mathrm{I}}\xi$, so there exist a nonempty sequence $i_1, \ldots, i_m$ of elements of G such that $(\mathcal{M}, v) \models \neg[i_1]_{\mathrm{I}} \ldots [i_m]_{\mathrm{I}}\xi$. Thus $\neg[i_1]_{\mathrm{I}} \ldots [i_m]_{\mathrm{I}}\xi \in L(v)$. Similar arguments can be used for $\psi = [G]^+_{\mathrm{B}}\xi$.

For the right to left direction let $\mathcal{T} = (W, \{B_i : i \in \mathcal{A}\}, \{G_i : i \in \mathcal{A}\}, \{I_i : i \in \mathcal{A}\}, L)$ be a tableau for φ. Let $\mathcal{M} = (W, \{\widetilde{B}_i : i \in \mathcal{A}\}, \{\widetilde{G}_i : i \in \mathcal{A}\}, \{\widetilde{I}_i : i \in \mathcal{A}\}, Val)$, where $Val(p, u) = 1$ if $p \in L(u)$ ($Val(p, u) = 0$ otherwise), and each $\widetilde{B}_i$, $\widetilde{G}_i$, $\widetilde{I}_i$ is a minimal relation having relations $\widehat{B}_i$, $\widehat{G}_i$ and $\widehat{I}_i$ (respectively) as a subset and satisfying properties **P4(B,O)** and **P5(B,O)**. Relations $\widehat{G}_i = G_i$, $\widehat{I}_i = I_i \cup \{(v, v) \in W \times W : \neg\exists u \in W.(v, u) \in I_i\}$ and $\widehat{B}_i = B_i \cup \{(v, v) \in W \times W : \neg\exists u \in W.(v, u) \in B_i\}$). We will show that for any $\psi \in \neg\mathrm{Sub}(\varphi)$ and $v \in W$, $\psi \in L(v)$ implies $(\mathcal{M}, v) \models \psi$. The proof is by induction on the structure of formulas. Most cases are shown like in the case of TeamLog$^{\mathrm{ind}}$tableau, without iterated modalities (see for example [6]). The only new cases are iterated modalities. Let $\psi = [G]^+_{\mathrm{I}}\xi$. By condition **It1** $\{[i]_{\mathrm{I}}\xi, [i]_{\mathrm{I}}[G]^+_{\mathrm{I}}\xi\} \subseteq L(v)$ for any $i \in G$. Moreover it can be easily shown, using induction on the length of sequences over G, that for any u such that $(v, u) \in \left(\widetilde{I}_G\right)^+$ it holds that $\{\xi, [i]_{\mathrm{I}}\xi, [i]_{\mathrm{I}}[G]^+_{\mathrm{I}}\xi\} \subseteq L(u)$. Thus, by the induction hypothesis, $(\mathcal{M}, u) \models \xi$ and so $(\mathcal{M}, v) \models [G]^+_{\mathrm{I}}\xi$. The case of $\psi = [G]^+_{\mathrm{B}}\xi$ can be shown analogically.

Now let $\psi = \neg[G]^+_{\mathrm{I}}\xi$. By condition **It2** there is a sequence $i_1, \ldots, i_m \in G^+$ such that $\neg[i_1]_{\mathrm{I}} \ldots [i_m]_{\mathrm{I}}\xi \in L(v)$. It can be easily seen, by property 7 of the TeamLog tableau, that there must exist a sequence of states $s_0, \ldots s_m$ such that $s_0 = v$, $(s_{j-1}, s_j) \in I_{i_j}$ (for all $1 \leq j \leq m$) and $\sim\xi \in L(s_m)$. Thus, by the induction hypothesis and the fact that $I_i \subseteq \widetilde{I}_i$, it holds that $(\mathcal{M}, s_m) \models\sim\xi$, and thus $(\mathcal{M}, s_m) \models \neg\xi$, so $(\mathcal{M}, v) \models \neg[G]^+_{\mathrm{I}}\xi$. The case of $\psi = [G]^+_{\mathrm{B}}\xi$ can be shown analogically. □

3.1 The Algorithm

To check the satisfiability, an algorithm is used that, given a formula φ, tries to construct a *pre-tableau* – a tree-like structure that forms the basis for a TeamLog tableau for φ and, further, for an interpretation for φ.[3]

Nodes of the pre-tableau are labelled with subsets of $\neg\mathrm{Sub}(\varphi)$. The nodes with labels being a fully expanded tableaux that are not blatantly inconsistent are called *states* and all the other nodes, that are not fully expanded tableaux, are called *internal nodes*. Given a node v, the *height* of v, denoted by $\mathrm{height}(v)$, is the number of nodes on the path from v to the root of pre-tableau, excluding v (so that $\mathrm{height}(root) = 0$). Additionally the notion of *state height* is defined for states. Given a state s, the *state height* of s, denoted by $\mathrm{s\text{-}height}(s)$ is the number of states no the path from s to the root of pre-tableau, excluding s.

Modifications of the algorithm from [5] are connected with the new axioms of the TeamLog logic, corresponding properties of accessibility relations and the fact that the algorithm deals with $[\cdot]^+_{\mathrm{O}}$ modalities.

[3] We omit a formal definition of a pre-tableau here, for details see [5].

Algorithm 1: DecideSatisfiability

Input: a formula φ
Output: the decision whether φ is satisfiable or not

```
/* A pre-tableau construction                                      */
```

Construct a tree consisting of single node `root`, with $L(\texttt{root}) = \{\varphi\}$;
repeat
  Let S be a set of all leaves of the tree with labelling sets that are not blatantly inconsistent;
  if *There is* $s \in S$ *such that* $L(s)$ *is not a propositional tableau and* $\psi \in L(s)$ *is a reason* **then**
    `FormPropositionalTableau(`s,ψ`)`;
  end
  else if *There is* $s \in S$ *such that* $L(s)$ *is not a fully expanded tableau and* $\psi \in L(s)$ *is a reason* **then**
    `FormFullyExpandedTableau(`s,ψ`)`;
  end
  else if *There is* $s \in S$ *such that* $L(s)$ *is a fully expanded tableau* **then**
    `CreateSuccessors(`s`)`;
  end
until *no change occurred* ;

`MarkNodes`;

if `root` *is marked* `satisfiable` **then**
  return `satisfiable`;
else
  return `unsatisfiable`;

Throughout the algorithm the notion of blatantly inconsistent set of formulas is used. A set of formulas Φ is *blatantly inconsistent* if for some formula ψ, both ψ and $\neg\psi$ are in Φ. Moreover, successors of states created by the algorithm are called $\mathrm{b_i}$-, $\mathrm{g_i}$-, and $\mathrm{i_i}$-successors respectively, depending on the relation between the state and the successor.

The following sets will be used to define labels of the newly created successors of a state (O is either {B, G or I}):

$$L^{[i]_{\mathrm{B}}}(s) = (L(s)/[i]_{\mathrm{B}}) \cup (L(s) \cap [i]_{\mathrm{O}}) \cup (L(s) \cap \neg[i]_{\mathrm{O}}) \cup ((L(s)/[i]_{\mathrm{I}}) \cap [G \cup \{i\}]^{+}_{\mathrm{I}})$$

$$L^{[i]_{\mathrm{I}}}(s) = L(s)/[i]_{\mathrm{I}},$$

$$L^{[i]_{\mathrm{G}}}(s) = (L(s)/[i]_{\mathrm{G}}) \cup L^{[i]_{\mathrm{I}}}(s),$$

$$L^{\neg[i]_{\mathrm{O}}}(s,\psi) = \{\sim\psi\} \; \cup \; L^{[i]_{\mathrm{O}}}(s).$$

It can be easily checked that a formula of the form $[\{i\}]^{+}_{\mathrm{B}}\psi$ is equivalent to $[i]_{\mathrm{B}}\psi$. Obviously any formula can be converted in linear time to a form that does not contain $[\{i\}]^{+}_{\mathrm{B}}$ modalities. Thus in the description of the algorithm and in the following proofs it will be convenient to assume that the input formula does not have subformulas of the form $[\{i\}]^{+}_{\mathrm{B}}\psi$.

Algorithm 2: MarkNodes

repeat
  if *s is an unmarked state* **then**
    if *s is blatantly inconsistent* **then**
      Mark s `unsatisfiable`;
    else if *s has a successor marked* `unsatisfiable` **then**
      Mark s `unsatisfiable`;
    else if *s has an unmarked* $\mathrm{i_i}$*-successor* **then**
      Mark s `unsatisfiable`;
    else if *s has an unmarked* $\mathrm{b_i}$*-successor and no other* $\mathrm{b_i}$*-successor of s is marked* `satisfiable` **then**
      Mark s `unsatisfiable`;
    else if *there is no formula of the form* $\neg[i]_\mathrm{O}\psi$ *in* $L(s)$ *for which a successor could not be created* **then**
      Mark s `satisfiable`;
  end
  else if *s is an unmarked internal node* **then**
    if $L(s)$ *is blatantly inconsistent or all its successors are marked* `unsatisfiable` **then**
      Mark s `unsatisfiable`;
    else if *s has at least one successor marked* `satisfiable` **then**
      Mark s `satisfiable`;
  end
until *no new node marked* ;

In what follows relations of $\mathrm{B_i}$-successor, $\mathrm{G_i}$-successor and $\mathrm{I_i}$-successor between states will be used and are defined as follows. Let s and t be a subsequent states on a path within a pre-tableau. If a successor node of s on the path containing t is a $\mathrm{b_i}$-, $\mathrm{g_i}$- or $\mathrm{i_i}$-successor, then t is a $\mathrm{B_i}$-, $\mathrm{G_i}$- or $\mathrm{I_i}$-successor (respectively) of s. Thus the relation of $\mathrm{O_i}$-successor is a relation between states, while the relation of $\mathrm{o_i}$-successor is a relation between a state and a node (which may or may not be a state). Notions of $\mathrm{o_i}$- and $\mathrm{O_i}$-ancestor are defined analogically. The following lemmas will be useful in proving that Algorithm 1 stops and in assessing the size of pre-tableau created by Algorithm 1.

Lemma 2. *Let s and t be states of the pre-tableau constructed by Algorithm 1, such that t is a $\mathrm{B_i}$-successor of s. Then the following hold for* $\mathrm{O} \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$*:*

1. $L^{[i]_\mathrm{O}}(s) = L^{[i]_\mathrm{O}}(t)$.
2. $\neg[i]_\mathrm{O}\xi \in L(s)$ *and* $L^{\neg[i]_\mathrm{O}}(s,\xi) = L^{\neg[i]_\mathrm{O}}(t,\xi)$*, for any* $\neg[i]_\mathrm{O}\xi \in L(t)$.

Proof. Notice that if s has a $\mathrm{B_i}$-successor, then it is not blatantly inconsistent.

For point 1, let $\psi \in L^{[i]_\mathrm{O}}(s)$. Then it is either $[i]_\mathrm{O}\psi \in L(s)$ (and consequently $[i]_\mathrm{O}\psi \in L(t)$) or $\mathrm{O} = \mathrm{B}$, ψ is of the form $[i]_\mathrm{B}\xi$ $\psi \in L(s)$ and, consequently, $\psi \in L(t)$. Thus $\psi \in L^{[i]_\mathrm{O}}(t)$.

On the other hand, let $\psi \in L^{[i]_\mathrm{O}}(t)$. Then either $[i]_\mathrm{O}\psi \in L(t)$ or $\mathrm{O} = \mathrm{B}$, ψ is of the form $[i]_\mathrm{B}\xi$ and $\psi \in L(t)$. Suppose that the first case holds. Since $L(s)$ is a fully expanded tableau, so either $[i]_\mathrm{O}\psi \in L(s)$, $\neg[i]_\mathrm{O}\psi \in L(s)$ or ψ is of the form

$[G]^+_{\mathrm{O}}\xi$ with $i \in G$ and then $[i]_{\mathrm{O}}\psi \in L(s)$. Because the second possibility leads to blatant inconsistency of $L(t)$ (as by Algorithm 1 it implies that $\neg[i]_{\mathrm{O}}\psi \in L(t)$), it must be that the first or the third possibility holds and thus $\psi \in L^{[i]_{\mathrm{O}}}(s)$. The second case can be shown by similar arguments, as either $[i]_{\mathrm{B}}\xi \in L(s)$ or $\neg[i]_{\mathrm{B}}\xi \in L(s)$.

For point 2, let $\neg[i]_{\mathrm{O}}\xi \in L(t)$. Then, by the fact that $L(s)$ is a fully expanded tableau, it is either $\neg[i]_{\mathrm{O}}\xi \in L(s)$ or $[i]_{\mathrm{O}}\xi \in L(s)$. As the second case leads to blatant inconsistency of $L(t)$, it must be the first one that holds.

$L^{\neg[i]_{\mathrm{O}}}(s,\xi) = L^{\neg[i]_{\mathrm{O}}}(t,\xi)$ can be shown by similar arguments to those used to show point 1 (notice that $L^{\neg[i]_{\mathrm{O}}}(v,\xi) = \{\sim\xi\} \cup L^{[i]_{\mathrm{O}}}(v)$ or (in case of $\mathrm{O} = \mathrm{G}$) $L^{\neg[i]_{\mathrm{O}}}(v,\xi) = \{\sim\xi\} \cup L^{[i]_{\mathrm{O}}}(v) \cup L^{[i]_{\mathrm{I}}}(v)$). □

Lemma 3. *The maximal state height of a state of the pre-tableau constructed by Algorithm 1 for the input φ with modal context restricted by $\boldsymbol{R}$ is $\leq |\mathcal{A}|\cdot(\mathrm{dep}(\varphi)+2\cdot|\mathcal{A}|-1)$ and the maximal height of a node of the pre-tableau constructed by Algorithm 1 is $|\varphi|\cdot|\mathcal{A}|\cdot(\mathrm{dep}(\varphi)+2\cdot|\mathcal{A}|-1)$.*

Proof. Any sequence of executions of steps leading to a fully expanded tableau has length $\leq |\varphi|$. Thus on the path connecting any subsequent states s and t, there can be at most $|\varphi|-1$ internal nodes.

If s and t are states such that t is a $\mathrm{G_i}$-successor of s then $\mathrm{dep}(L(t)) < \mathrm{dep}(L(s))$.

Now consider any sequence of states $s_0,\ldots,s_m$, such that each s_j is an $\mathrm{I_{i_j}}$-successor of s_{j-1}. Take any $i \in \mathcal{A}$. Observe that each time states s and t, such that t is an $\mathrm{I_i}$-successor of s, occur in the above sequence, the only formulas that do not change modal depth between these states are formulas of the form $[G]^+_{\mathrm{I}}\psi$ or $\neg[G]^+_{\mathrm{I}}\psi$ with $i \in G$ as well as $\psi \in L(s)$ such that a formula of the form $[G]^+_{\mathrm{I}}\psi \in L(s)$ with $i \in G$. Take any state u that is a (possibly indirect) successor of t and such that the new formula of the form $[G]^+_{\mathrm{I}}\psi$ or $\neg[G]^+_{\mathrm{I}}\psi$ is added to $L(u)$.

This means that any formula in $L(s)$ having this formula as a subformula must be of the form $[i]_{\mathrm{I}}\xi$ or $\neg[i]_{\mathrm{I}}\xi$ (due to the modal context restriction $\mathbf{R}$). Thus after $\mathrm{dep}(L(s_{j_0}))$ occurrences of $\mathrm{I_i}$-successor in the sequence no new formula of the form $[G]^+_{\mathrm{I}}\psi$ or $\neg[G]^+_{\mathrm{I}}\psi$ with $i \in G$ can be added to the label of any state. Now let s be a state such that it is a $\mathrm{dep}(L(s_{j_0}))$'th $\mathrm{I_i}$-successor in the sequence and let t be a (not necessarily immediate) successor of s which is an $\mathrm{I_i}$-successor. Since for any formula in $L(s)$ that is of the form $[G]^+_{\mathrm{I}}\psi$ or $\neg[G]^+_{\mathrm{I}}\psi$ it must be that $i \in G$ and there can not be a formula of the form $[i]_{\mathrm{I}}\psi$ in $L(s)$ (due to the modal context restriction $\mathbf{R}$), so $L(t)$ can contain at most one formula of the form $\neg[G]^+_{\mathrm{I}}\psi$ with $i \in G$. Let u be the first (but not necessarily immediate) successor of t in the sequence which is an $\mathrm{I_i}$-successor and let G' be a set such that $k \in G'$ iff there is an $\mathrm{I_k}$-successor in the sequence occurring between t and u. Then the only formulas of the form $[G]^+_{\mathrm{I}}\psi$ or $\neg[G]^+_{\mathrm{I}}\psi$ in $L(u) \setminus L(t)$ are those for which $G' \nsubseteq G$. Notice that if $L(t) \cap [G]^+_{\mathrm{I}} = L(s) \cap [G]^+_{\mathrm{I}}$, with $i \in G$, then $L^{[i]_{\mathrm{I}}}(t) = L^{[i]_{\mathrm{I}}}(u)$ and $L^{\neg[i]_{\mathrm{I}}}(t,\xi) = L^{\neg[i]_{\mathrm{I}}}(u,\xi)$ (recall that there can be at most

Algorithm 3: FormPropositionalTableau

Input: a state s and a formula ψ

if ψ *is of the form* $\neg\neg\xi$ **then**
  Create a successor t of s and set $L(t) := L(s) \cup \{\xi\}$;
end
else if ψ *is of the form* $\xi_1 \wedge \xi_2$ **then**
  Create a successor t of s and set $L(t) := L(s) \cup \{\xi_1, \xi_2\}$;
end
else if ψ *is of the form* $\neg(\xi_1 \wedge \xi_2)$ **then**
  Create two successors t_1 and t_2 of s and set $L(t_1) := L(s) \cup \{\sim\xi_1\}$ and $L(t_2) := L(s) \cup \{\sim\xi_2\}$;
end

one formula of the form $\neg[G]^+_{\mathrm{I}}\psi \in L(t) \cap L(u)$ with $i \in G$). Thus there can be at most $2 \cdot (|\mathcal{A}| - 1) + 1 = 2 \cdot |\mathcal{A}| - 1$ $\mathrm{I_i}$-successors following t in the sequence (one of them connected to "dropping" the formula of the form $\neg[G]^+_{\mathrm{I}}\psi$ within the sequence), and so there can be at most $\mathrm{dep}(L(s_{i_0})) + 2 \cdot |\mathcal{A}| - 1$ $\mathrm{I_i}$-successors in the whole sequence. This shows that the length of the sequence must be $\leq |\mathcal{A}| \cdot (\mathrm{dep}(L(s_{i_0})) + 2 \cdot |\mathcal{A}| - 1)$.

If t is a $\mathrm{B_i}$-successor of s then, by Lemma 2, t cannot have any $\mathrm{B_i}$-, $\mathrm{G_i}$- nor $\mathrm{I_i}$-successors. Thus, for any successor node u of t, $\mathrm{dep}(L(s)) < \mathrm{dep}(L(u))$.

For the sequence of states $s_0, \ldots, s_m$, such that each s_j is an $\mathrm{B_{i_j}}$-successor of s_{j-1} and $i_j = i_k$ implies $j = k$, similar restriction on the length of the sequence apply as in the case of successors of type I.

All above arguments show that the maximal state height of a state of the pre-tableau must be $\leq |\mathcal{A}| \cdot (\mathrm{dep}(\varphi) + 2 \cdot |\mathcal{A}| - 1)$ and the maximal height of a node of the pre-tableau constructed by Algorithm 1 must be $\leq |\varphi| \cdot |\mathcal{A}| \cdot (\mathrm{dep}(\varphi) + 2 \cdot |\mathcal{A}| - 1)$. □

The following is an immediate consequence of Lemma 3.

Corollary 1. *For any formula φ Algorithm 1 terminates.*

What remains to be shown is that Algorithm 1 is valid.

Lemma 4. *A formula φ with modal context restricted by $\boldsymbol{R}$ is* TeamLog *satisfiable iff Algorithm 1 returns* `satisfiable` *on the input φ.*

Proof. For the right to left implication we will show how to construct a TeamLog tableau $\mathcal{T} = (W, \{B_i : i \in \mathcal{A}\}, \{G_i : i \in \mathcal{A}\}, \{I_i : i \in \mathcal{A}\}, L)$ for φ based on the pre-tableau constructed by Algorithm 1 if it returns `satisfiable` on the input φ. Before the construction, the following adjustment has to be made to the set of states. As long as no new state is marked `satisfiable`, find an unmarked state v that is left unmarked because of some formula of the form $\neg[i]_{\mathrm{O}}\psi \in L(v)$ and some state t marked `satisfiable`, with the label $L(t)$ that prevented creation of a successor of v for the formula $\neg[i]_{\mathrm{O}}\psi \in L(v)$; mark v as `satisfiable`. From now on treat t as an $\mathrm{o_i}$ successor of v. When no new

Algorithm 4: FormFullyExpandedTableau

Input: a state s and a formula ψ

if *there is* $\xi \in \mathrm{Sub}(\psi)$ *such that* $\{\xi, \neg\xi\} \cap L(s) = \varnothing$ **then**
  Create two successors t_1 and t_2 of s and set $L(t_1) := L(s) \cup \{\xi\}$ and $L(t_2) := L(s) \cup \{\sim\xi\}$;
end
else if ψ *is of the form* $[G]^+_{\mathrm{O}}\xi$ **then**
  Create a successor t of s and set $L(t) := L(s) \cup \bigcup_{i\in G} \{[i]_{\mathrm{O}}\xi, [i]_{\mathrm{O}}[G]^+_{\mathrm{O}}\xi\}$;
end
else if ψ *is of the form* $\neg[G]^+_{\mathrm{O}}\xi$ **then**
  foreach $i \in G$ **do**
    Create a successor t of s and set $L(t) := L(s) \cup \{\neg[i]_{\mathrm{O}}\xi, \neg[i]_{\mathrm{O}}[G]^+_{\mathrm{O}}\xi\}$;
  end
end

state can be marked, the tableau can be created as follows. As the set of states W take the set of all states in the pre-tableau that are marked `satisfiable`. Each $O_i = \{(x, y) \in W \times W : y \text{ is an } \mathrm{O}_i\text{-successor of } x\}$. The labelling function L is created on the basis of the labelling function of the pre-tableau. To satisfy condition **I2**, labels of states must be extended, so that if there is a formula of the form $\neg[G]^+_{\mathrm{O}}\psi$ in the label of some state s and the condition is not satisfied, then the successor leave state accessible with a sequence of $\mathrm{O_{i_1}}, \ldots, \mathrm{O_{i_m}}$ successor relations, with $i_j \in G$ for all $1 \le j \le m$ and such that its label contains ψ must be found (notice that if s is marked `satisfiable` then such state must exist). Then the formula $\neg[i_1]_{\mathrm{O}} \ldots [i_m]_{\mathrm{O}}\psi$ must be added to the label of s.

It is easy to check that $\mathcal{T}$ is a TeamLog tableau. Moreover, since $\varphi \in L(root)$ and *root* is marked `satisfiable`, so $\mathcal{T}$ is a TeamLog tableau for φ. Thus, by Lemma 1, φ is satisfiable.

For the left to right implication, we will show that for any node v that is marked `unsatisfiable` $\bigwedge L(v)$ is unsatisfiable. The proof is by induction on the maximal length of paths from a node to one of its successor leaves.

In most cases the proof is analogical to the similar proofs for the tableau method (e.g. see [5]), where usually it is shown that if a node is not marked `satisfiable`, then it is unsatisfiable. In the proof we will concentrate on the new cases related to iterated modalities.

Let v be an internal node with successors created during fully expanded tableau formation. Let u be a successor of v, created for a formula $[G]^+_{\mathrm{O}}\xi \in L(u)$ and suppose that $L(v)$ is satisfiable and is marked `unsatisfiable`. Thus u must be marked `unsatisfiable` as well, and there is an interpretation $(\mathcal{M}, s)$ such that $(\mathcal{M}, s) \models \bigwedge L(v)$. Thus $(\mathcal{M}, s) \models [G]^+_{\mathrm{O}}\xi$ and for any t such that $(s, t) \in O_G$ it must be that $(\mathcal{M}, t) \models \xi$ and $(\mathcal{M}, t) \models [G]^+_{\mathrm{O}}\xi$. Thus $(\mathcal{M}, s) \models [j]_{\mathrm{O}}\xi$ and $(\mathcal{M}, s) \models [j]_{\mathrm{O}}[G]^+_{\mathrm{O}}\xi$, so $(\mathcal{M}, s)$ is also an interpretation for $L(u)$, which contradicts the induction hypothesis as u is marked `unsatisfiable`.

Algorithm 5: CreateSuccessors

Input: a state s

if $\neg[i]_{\mathrm{B}}\psi \in L(s)$ **then**

If there is no $\mathrm{B_i}$ ancestor state t of s, such that $L^{\neg[i]_{\mathrm{B}}}(t,\psi) = L^{\neg[i]_{\mathrm{B}}}(s,\psi)$ and there is no ancestor state t of s, such that $L(t) = L^{\neg[i]_{\mathrm{B}}}(s,\psi)$, then create a successor u of s (called $\mathrm{b_i}$-successor) with $L(u) = L^{\neg[i]_{\mathrm{B}}}(s,\psi)$;

end

else if $[i]_{\mathrm{B}}\psi \in L(s)$ *and there are no formulas of the form* $\neg[i]_{\mathrm{B}}\xi \in L(s)$ **then**

If there is no ancestor state t of s, such that $L^{[i]_{\mathrm{B}}}(t) = L^{[i]_{\mathrm{B}}}(s)$ and there is no ancestor state t of s, such that $L(t) = L^{[i]_{\mathrm{B}}}(s)$, then create a successor u of s (called $\mathrm{b_i}$-successor) with $L(u) = L^{[i]_{\mathrm{B}}}(s)$;

end

else if $\neg[i]_{\mathrm{O}}\psi \in L(s)$ *and* $\mathrm{O} \in \{\mathrm{G}, \mathrm{I}\}$ **then**

If there is no ancestor state t of s, such that $L(t) = L^{\neg[i]_{\mathrm{O}}}(s,\psi)$, then create a successor u of s (called $\mathrm{o_i}$-successor) with $L(u) = L^{\neg[i]_{\mathrm{O}}}(s,\psi)$;

end

else if $[i]_{\mathrm{I}}\psi \in L(s)$ *and there are no formulas of the form* $\neg[i]_{\mathrm{I}}\xi \in L(s)$ **then**

If there is no ancestor state t of s, such that $L(t) = L^{[i]_{\mathrm{I}}}(s)$, then create a successor u of s (called $\mathrm{i_i}$-successor) with $L(u) = L^{[i]_{\mathrm{I}}}(s)$;

end

Now let U be a set of all successors of v created for a formula $\neg[G]^+_{\mathrm{O}}\xi \in L(u)$ and suppose that $L(v)$ is satisfiable and v is marked `unsatisfiable`. Then all $u \in U$ are marked `unsatisfiable` and there is an interpretation $(\mathcal{M}, s)$ such that $(\mathcal{M}, s) \models \bigwedge L(v)$. Thus $(\mathcal{M}, s) \models \neg[G]^+_{\mathrm{O}}\xi$ and there is a sequence of states $s_0, \ldots, s_m$ in $\mathcal{M}$ such that $s = s_0$, for each $1 \leq i \leq m$, $(s_{i-1}, s_i) \in O_{j_i}$ with $j_i \in G$ and $(\mathcal{M}, s_0) \models \neg\xi$. Assume that $m = 1$. Then $(\mathcal{M}, s) \models \neg[j_1]_{\mathrm{O}}\xi$ and there is $u \in U$ such that $L(u)$ is satisfiable, as $(\mathcal{M}, s)$ is an interpretation for it. Thus we have contradiction with the induction hypothesis, as u is marked `unsatisfiable`. Now assume that $m > 1$. Take u such that $\neg[i_1]_{\mathrm{O}}[G]^+_{\mathrm{O}} \in L(u)$. Then $(\mathcal{M}, s)$ is an interpretation for $L(u)$, as $(\mathcal{M}, s_1) \models [G]^+_{\mathrm{O}}\psi$. Thus we have a contradiction with the induction hypothesis again.

Now let v be a state and assume that v has an $\mathrm{i_i}$-successor u that is left unmarked (and for this reason v is marked `unsatisfiable`). We will show that if u is not marked, then $L(u)$ is unsatisfiable. Take any state that is a successor of u and there is no state on the path from u to that state. Since u is left unmarked, so no node on that path (including that state) can be marked `satisfiable`.

Moreover, since u is satisfiable, one of these states must be satisfiable as well (and as such left unmarked, as otherwise it would have to be marked `unsatisfiable`, which would contradict the induction hypothesis). Take any such state and let it be denoted by t. Then there must be a formula of the form $\neg[G]^+_{\mathrm{O}}\psi$ such that $\neg[G]^+_{\mathrm{O}}\psi \in L(t)$ and a successor could not be created for it. Thus there must be an ancestor s of t such that $L(s) = L^{\neg[i]_{\mathrm{O}}}(s,\psi)$. Observe that it must be that $\mathrm{O} = \mathrm{I}$ and all successors of states on the path from s to t must be of the type I (as otherwise formulas of infinite modal depth would be needed to have $\neg[G]^+_{\mathrm{O}}$ in $L(t)$ (for it must be a subformula of a formula in $L(s)$ that will allow it for being in $L(t)$). Let G'' be a set of all j such that there is an $\mathrm{i_j}$-successor on the path between t and u. Observe that this implies that for any $\xi \in L(t) \setminus \{\neg\psi, \neg[i]_{\mathrm{I}}\psi\}$, ξ must be of the form $[G']^+_{\mathrm{I}}\zeta$ or must be a subformula of a formula of such form, with $G'' \cup \{i\} \subseteq G'$. For assume $\xi \in L(t) \setminus \{\neg\psi, \neg[i]_{\mathrm{I}}\psi\}$. Then there must be $\eta \in L(s)$ such that ξ is its subformula or $\xi = \eta$. Let η be a maximal such formula (with respect to the relation of being a subformula). Then it must be that $[i]_{\mathrm{I}}\eta \in L(t)$ and either η is a subformula of some formula in $L(s)$ or it is of the form $[G']^+_{\mathrm{I}}\zeta$ (with $[G']^+_{\mathrm{I}}\zeta \in L(t)$ and $G'' \cup \{i_1\} \subseteq G'$). Since η is assumed to be maximal, the second case must hold.

For $\neg[i]_{\mathrm{I}}\psi$ three cases are possible.

(i). there is a formula $[G']^+_{\mathrm{I}}\zeta \in L(t)$ with $[i]_{\mathrm{I}}\psi$ being its subformula and $G'' \cup \{i\} \subseteq G'$ as in argumentation above, or

(ii). there is a formula of the form $\neg[G]^+_{\mathrm{I}}\psi \in L(t)$ with $i \in G$, and t was created during fully expanded tableau formation for that formula, or

(iii). $\psi = \neg[G]^+_{\mathrm{I}}\xi \in L(t)$ with $i \in G$ and t was created during fully expanded tableau formation for ψ.

The case (i) is impossible, as it violates the modal context restriction **R** (notice that $i \in G \cap G'$). Similarly with the case (ii), as otherwise $\neg[G]^+_{\mathrm{I}}\psi$ would have to be a subformula of a formula of the form $[G']^+_{\mathrm{I}}\zeta$ and this would violate the modal context restriction **R**. For the case (iii), assume that t is satisfiable. Then its direct predecessor, t' such that $L(t) = L(t') \cup \{\neg[i]_{\mathrm{I}}[G]^+_{\mathrm{I}}\xi\}$ must be satisfiable as well and there exists an interpretation $(\mathcal{M}, w)$ such that $(\mathcal{M}, w) \models \bigwedge L(t')$. Thus $(\mathcal{M}, w) \models \neg[G]^+_{\mathrm{I}}\xi$ and there is a sequence of states $w_0, \dots, w_m$ in $\mathcal{M}$ such that $w = w_0$, for each $1 \leq j \leq m$, $(w_{j-1}, w_j) \in O_{k_j}$ with $k_j \in G$ and $(\mathcal{M}, w_m) \models \neg\xi$. We will show that there must be a successor of the state v, such that there is no state on the path from it to v, the state is satisfiable and it is marked `satisfiable`. The proof is by induction on m. If $m = 1$, then t' has a successor created for the formula $\neg[k_1]_{\mathrm{I}}\xi$. By above discussion this state must be marked, and can not be marked `unsatisfiable` (as it would contradict the induction hypothesis of the main proof). Thus it is marked `satisfiable`. Now assume that $m > 1$ and consider a successor t'' of t' created for the formula $\neg[k_1]_{\mathrm{I}}[G]^+_{\mathrm{I}}\xi$. If t'' is not marked `satisfiable`, then it must be unmarked and there must be be an ancestor s of t'' such that $L(s) = L^{\neg[i]_{\mathrm{I}}}(t'', [G]^+_{\mathrm{O}}\xi)$. Since $(\mathcal{M}, w) \models \bigwedge L(t')$, so $(\mathcal{M}, w) \models \bigwedge L(t'')$ and so $(\mathcal{M}, w_1) \models L^{\neg[i]_{\mathrm{I}}}(t'', [G]^+_{\mathrm{I}}\xi)$, and thus $(\mathcal{M}, w_1) \models L(s)$. Since for any formula $\eta \in L(t'') \setminus \{\neg[G]^+_{\mathrm{I}}\xi, \neg[k_1]_{\mathrm{I}}[G]^+_{\mathrm{I}}\xi\}$ there is a formula of the form $[G']^+_{\mathrm{O}}\zeta \in L(t'') \cap L(s)$ such that η is this formula or its subformula, and $\neg[G]^+_{\mathrm{O}}\xi \in L(s)$, so it is easy to see that $L(v) = L(s)$. Moreover there must be a successor s' of s such that there is no state on the path from s to s', $(\mathcal{M}, w_1) \models L(s')$, $\neg[G]^+_{\mathrm{I}}\xi \in L(s')$ and successors of s' are created during fully expanded tableau formation for $\neg[G]^+_{\mathrm{I}}\xi$. Then either a successor s'' of s' created for a formula $\neg[k_2]_{\mathrm{I}}[G]^+_{\mathrm{I}}\xi$ is marked `satisfiable` or, by induction hypothesis, there is a successor s''' of s that is marked `satisfiable`. In either case there is a corresponding successor v' of v with $L(v') = L(s'')$ or $L(v') = L(s''')$ and so it must be marked `satisfiable` as well. Thus if v is unmarked, then $L(v)$ must be unsatisfiable.

Now let v be a state and assume that v has all $\mathrm{b_i}$-successors unmarked (and for this reason v is marked `unsatisfiable`). This case is similar to the case of an unmarked $\mathrm{i_i}$-successor and we will omit the analysis of this case here. We just want to remark that the differences come from the fact that states left unmarked for one $\mathrm{b_i}$-successor, say u, of v may be made so because of another $\mathrm{b_i}$-successor, say u, of v. But then for any formula of the form $\neg[j]_{\mathrm{O}}\psi \in L(u)$ it is $\neg[j]_{\mathrm{O}}\psi \in L(t)$ and for any formula of the form $\neg[i]_{\mathrm{O}}\psi \in L(u)$ such that successor for it could not be created because of a successor of $L(v)$, it must be $\neg[i]_{\mathrm{O}}\psi \in L(v)$ and $\neg[i]_{\mathrm{O}}\psi \in L(t)$.

Observe, by the discussion above, that *root* node of a pre-tableau can never be left unmarked. Thus if *root* node in a pre-tableau is not marked `satisfiable`, then it is marked `unsatisfiable` and φ must be unsatisfiable. Thus if it is satisfiable, the *root* node must be marked `satisfiable`. □

Since the algorithm is valid, it shows that if a TeamLog formula with modal context restricted by $\mathbf{R}$ is satisfiable if is satisfied in a tree-like model with maximal height of states depending on the modal depth of the input formula, as stated in the following corollary.

Corollary 2. *The formula φ with modal context restricted by $\boldsymbol{R}$ is* TeamLog *satisfiable iff there exists a* TeamLog *tableau for φ in which the number of states is $\leq |\varphi|^{|\mathcal{A}|\cdot(\mathrm{dep}(\varphi)+2\cdot|\mathcal{A}|-1)}$.*

Proof. The right to left implication is obvious. For the left to right implication notice that from the proof of Lemma 4 we know that any state in the tableau constructed on the basis of the pre-tableau constructed by Algorithm 1 has $\leq |\varphi|$ successors. Thus, by Lemma 3, the tableau has $\leq |\varphi|^{|\mathcal{A}|\cdot(\mathrm{dep}(\varphi)+2\cdot|\mathcal{A}|-1)}$ states. □

We also have the following theorem.

Theorem 1. *The satisfiability problem for* TeamLog *with the modal context restriction $\boldsymbol{R}$ is PSPACE-complete*

Proof. Since the maximal height of a node of the pre-tableau constructed by Algorithm 1 for a formula φ with modal context restricted by $\mathbf{R}$ is $\leq |\varphi| \cdot |\mathcal{A}| \cdot (\mathrm{dep}(\varphi) + 2 \cdot |\mathcal{A}| - 1)$ and the algorithm is deterministic, it can be run on a deterministic Turing machine by depth-first search using polynomial space. Thus the TeamLog satisfiability problem with the modal context restriction $\mathbf{R}$ is in PSPACE. On the other hand, the problem of $\text{TeamLog}^{\mathrm{ind}}$ satisfiability, which is a subset of restricted TeamLog, is PSPACE-hard (as shown in [6]), so the problem is PSPACE-complete in the case of restricted TeamLog. □

Remark 1. We would like to remark here that if TeamLog formulas are not restricted with $\mathbf{R}$, then the satisfiability problem is EXPTIME-complete. Moreover the proof of this fact uses a TeamLog formula of modal depth 2 and modal context such that the modality $[i]_{\mathrm{O}}$ is in scope of the modality $[G]^+_{\mathrm{O}}$ with $i \in G$. In this sense our result can be seen as the minimal modal context restriction that moves the satisfiability problem to a new complexity class.

Remark 2. In logical frameworks for CPS, like TeamLog, important notions need to be defined using formulas of modal depth greater than 1 and incorporating different iterated modalities. For example the notion of collective intention, which is fundamental for defining cooperative teams of agents is defined in terms of mutual intention and collective belief: $[G]_{\mathrm{CINT}}\varphi \leftrightarrow [G]^+_{\mathrm{I}}\varphi \wedge [G]^+_{\mathrm{B}}[G]^+_{\mathrm{I}}\varphi$. Our result shows that such combinations of modalities do not lead to EXPTIME-hardness and with bounding modal depth by some constant we can have fragment of the framework which is NPTIME-complete.

By Corollary 2 we can see that if we restrict the modal depth of TeamLog formulas whose modal context is restricted by **R**, then the satisfiability problem will be NPTIME-complete.

Theorem 2. *For any fixed k, if the modal depth of formulas is bounded by k, then the satisfiability problem for* TeamLog *with modal context restriction* ***R*** *is NPTIME-complete.*

Proof. Let φ be a formula with modal context restricted by **R**. By Corollary 2 the size of the tableau for a satisfiable formula φ is bounded by $\mathrm{O}(|\varphi|^k)$. This means that the satisfiability of φ with bounded modal depth can be checked by non-deterministic Algorithm 6.

Algorithm 6: DecideSatisfiabilityNonDeterministic

Input: a formula φ
Output: a decision whether φ is satisfiable or not

Guess a tableau $\mathcal{T}$ satisfying φ;
if $\mathcal{T}$ *is a tableau for* φ **then**
    return `satisfiable`;

Since the tableau $\mathcal{T}$ constructed by Algorithm 6 is of polynomial size, so checking if it is a tableau for φ can be realized in polynomial time. This shows that satisfiability of φ can be checked in NPTIME. The problem is also NPTIME-complete, as the satisfiability problem for propositional logic is NPTIME-hard. □

4 Discussion and Conclusions

As shown in [6] the satisfiability problem for Dunin-Kęplicz and Verbrugge theory of teamwork is decidable although intractable: for the individual fragment of the theory it is PSPACE-complete, while for the group fragment it is EXPTIME-complete. Moreover reduction of modal depth of formulas moves the satisfiability problem to NPTIME, in the case of individual fragment, and leaves it in EXPTIME in the of group fragment, if modal depth is allowed to be > 1.

In this paper we finished the study of restrictions of modal depth of the TeamLog framework by finding modal context restriction that makes group fragment of the framework PSPACE-complete. Combining this restriction with bounding modal depth makes the satisfiability problem of the group fragment of the framework NPTIME-complete. This is an important finding that shows that definitions of crucial notions, like common intention, using formulas of modal depth > 1 and incorporating different iterated modalities are not responsible for "high" computational complexity EXPTIME.

In the next step we plan to investigate how the complexity of the satisfiability of the framework would be affected if we restricted the structure of the formulas. We plan to look, first of all, at modal horn formulas, which are particularly interesting from the perspective of AI applications. There are results the may bring positive outcomes in the case of the considered theory of teamwork (see [9]). Combining these results with our findings on modal depth and modal context restrictions can bring reductions that would make the satisfiability problem tractable. In particular the restriction leading to NPTIME-completeness, presented here, may lead to finding a restriction of the language that would lead to PTIME-completeness, which would be very significant achievement, when compared with EXPTIME-completeness of unrestricted language.

The methods used in this paper can be adapted to other multiagent theories built upon multi-modal logic (e.g. KARO [10]).

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