Lecture 3

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1 Higgs sheaves and modules with connections

Let X be a smooth projective variety defined over some field k and let us fix a polarization H.

Definition 1.1. A *Higgs sheaf* (E, θ) is a pair consisting of a quasi-coherent $\mathscr{O}_X$-module E and an $\mathscr{O}_X$-linear map $\theta : E \to E \otimes_{\mathscr{O}_X} \Omega_X$ satisfying the integrability condition $\theta \wedge \theta = 0$.

If E is coherent and torsion free, we say that (E, θ) is *slope semistable* if the inequality $\mu(E') \leq \mu(E)$ is satisfied for every Higgs subsheaf (E', θ') of (E, θ).

A *system of Hodge sheaves* is a Higgs sheaf (E, θ) with decomposition $E = \bigoplus E^j$ such that $\theta : E^j \to E^{j-1} \otimes_{\mathscr{O}_X} \Omega_X$.

As above, a system of Hodge sheaves (E, θ) is *slope semistable* if the inequality $\mu(E') \leq \mu(E)$ is satisfied for every subsystem of Hodge sheaves $(E', \theta') \subset (E, \theta)$. One can check that then (E, θ) is slope semistable as a Higgs sheaf.

Similarly as above we can define notion of semistability for a coherent $\mathscr{O}_X$-module with an (integrable) connection. Namely, we say that (V, ∇) is *slope semistable* if the inequality $\mu(V') \leq \mu(V)$ is satisfied for every subsheaf $V' \subset V$ preserved by ∇ (i.e., such that $\nabla(V') \subset V' \otimes \Omega_{X/k}$).

Exercise 1.0.1. Assume that k has characteristic zero. Prove that every coherent $\mathscr{O}_X$-module with an integrable connection is locally free.

Exercise 1.0.2. Assume that k has characteristic zero. Prove that every coherent $\mathscr{O}_X$-module with an integrable connection is slope semistable. Show examples when this fails in positive characteristic.

2 Simpson's correspondence

Let X be a smooth complex projective variety and let us fix a point $x \in X(\mathbb{C})$. Let X^{an} denote the complex manifold underlying X and let

$$\rho : \pi_1(X^{an}, x) \to \mathrm{GL}(n, \mathbb{C})$$

be a representation of the topological fundamental group of X^{an}. This correspond to a local system of complex vector spaces L on X^{an} (i.e., L is a locally constant sheaf of $\mathbb{C}$-vector spaces on X in the analytic topology). Then

$$(V^{an} = L \otimes_{\mathbb{C}} \mathscr{O}_{X^{an}}, \nabla^{an} := 1_L \otimes d_{\mathscr{O}_{X^{an}}} : V^{an} \to V^{an} \otimes_{\mathscr{O}_{X^{an}}} \Omega_{X^{an}/\mathbb{C}})$$

is an analytic vector bundle (more precisely, a locally free $\mathscr{O}_{X^{an}}$-module of finite rank) with an analytic connection. By GAGA there exists an algebraic vector bundle (V, ∇) with an algebraic integrable connection, whose analitification is isomorphic to (V^{an}, ∇^{an}).

If ρ is irreducible then (V, ∇) does not contain any $\mathscr{O}_X$-submodules preserved by ∇. Simpson's correspondence says that such modules with integrable connections correspond to stable Higgs bundles with vanishing rational Chern classes.

In the simplest case of unitary representations we recover the correspondence between irreducible unitary representations of $\pi_1(X^{an}, x)$ and stable vector bundles with vanishing Chern classes (a result due to Narasimhan–Seshadri in the curve case and Donaldson, Uhlenbeck–Yau in higher dimensions).

The plan for today's lecture is to generalize this correspondence to positive characteristic for analogues of variations of complex Hodge structure (and elements of the nilpotent cone on the side of Higgs bundles).

3 Witt ring

A ring R of characteristic p is called *perfect* if the endomorphism $x \to x^p$ of R is an automorphism.

Let R be a complete DVR (discrete valuation ring) with the residue field $k = R/m$ and uniformizer π. We normalize the valuation v of R so that $v(\pi) = 1$. Assume that R has characteristic zero but char $k = p$. Then $e = v(p)$ is called the *absolute ramification index* of R.

We say that R is *absolutely unramified* if $e = 1$, i.e., if p is a local uniformizer of R.

Let us recall that for every perfect field k of characteristic p there exists a unique (up to an isomorphism) complete DVR of characteristic zero, which is absolutely unramified and has k as its residue field. This ring is called the *ring of Witt vectors* and it is denoted by $W(k)$. The *ring of Witt verctors of length n* is defined by $W_n(k) := W(k)/p^n$.

For example, if $k = \mathbb{F}_p$ then $W(\mathbb{F}_p) = \mathbb{Z}_p$ is the ring of p-adic integers and $W_n(\mathbb{F}_p) = \mathbb{Z}/p^n$.

The ring $W_n(k)$ of Witt vectors of length n is characterized uniquely up to an isomorphism by the following properties:

1. $W_n(k)$ is flat over $\mathbb{Z}/p^n$,
2. $W_n(k)/pW_n(k) \simeq k$.

We also have $W(k) = \varprojlim W_n(k)$, where the homomorphisms $W_{n+1}(k) \to W_n(k)$ come from $\mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n$.

The Frobenius automorphism of k defines by functoriality automorphisms $\sigma_n : W_n(k) \to W_n(k)$ and $\sigma : W(k) \to W(k)$.

In the following we will need only $W_2(k)$ that can be defined explicitly by some simple formulas. In fact, we define $W_2(R)$ for any commutative ring R in the following way. As a set $W_2(R)$ is just a set R^2 of pairs of elements from R. We define addition and multiplication of $a = (a_0, a_1)$ and $b = (b_0, b_1)$ by the following formulas:

$$a + b := \left(a_0 + b_0, a_1 + b_1 - \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} a_0^i b_0^{p-1} \right)$$

and

$$a \cdot b := (a_0 b_0, a_1 b_0^p + a_0^p b_1 + p a_1 b_1).$$

To explain these formulas we interpret a pair (a_0, a_1) as $a_0^p + pa_1$ and we note the following formulas for polynomials with integral coefficients:

$$(x_0^p + px_1) + (y_0^p + py_1) = (x_0 + y_0)^p + p \left(x_1 + y_1 - \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} x_0^i y_0^{p-1} \right)$$

and

$$(x_0^p + px_1)(y_0^p + py_1) = x_0^p y_0^p + p(x_1 y_0^p + x_0^p y_1 + p x_1 y_1).$$

THEOREM 3.1. *Let R be a complete DVR of characteristic zero with perfect residue field k of characteristic $p > 0$. Then there exists a unique homomorphism $\psi : W(k) \to R$ such that the following diagram is commutative:*

$$\begin{array}{ccc} W(k) & \xrightarrow{\psi} & R. \\ \downarrow & \swarrow & \\ k & & \end{array}$$

Moreover, ψ is injective and R is a free $W(k)$-module of rank e and $\{1, \pi, ..., \pi^{e-1}\}$ is a basis of R as a $W(k)$-module.

A basic example of a complete DVR with absolute ramification index e is given by

$$R = W(k)[[T]]/f \cdot W(k)[[]T],$$

where $f(T) = T^e + \sum_{i=0}^{e-1} a_i T^i$ is an Eisenstein polynomial, i.e., all a_i are divisible by p and a_0/p is a unit.

4 Lifting

Let X be a flat S-scheme and let $S \to \tilde{S}$ be a closed embedding of schemes. We say that X is *liftable to* $\tilde{S}$ if there exists a flat morphism $\tilde{X} \to \tilde{S}$ such that its base change via $S \to \tilde{S}$ is isomorphic to X.

Now let Let X be a scheme defined over a perfect field k of characteristic p. We say that X is *liftable to characteristic zero*, if there exists an embedding of $S = \operatorname{Spec} k$ into a scheme $\tilde{S}$ of characteristic 0 such that X is liftable to $\tilde{S}$.

In the following we are particularly interested in schemes that can be lifted to $W_2(k)$, i.e., X/k is liftable to $\tilde{S} := \operatorname{Spec} W_2(k)$. Schemes that satisfy such condition behave quite similar to schemes in characteristic zero. However, see the following non-trivial examples:

Example 4.1. (W. Lang, Liedtke–Satriano,...) There exist examples of smooth k-schemes X that lift to characteristic zero but do not lift to $W_2(k)$.

Example 4.2. (R. Vakil) For any $n > 0$ there exist schemes X/k that lift to $W_n(k)$ but do not lift to $W_{n+1}(k)$.

On a positive side we have the following remark.

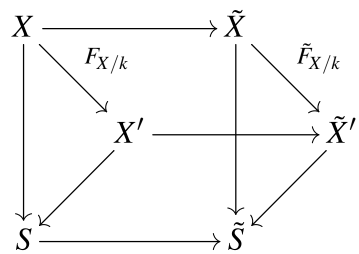
We say that X/k is *liftable to characteristic zero with a smooth base* if there exists a smooth and separated $\mathbb{Z}$-scheme S, a flat morphism $\mathscr{X} \to S$ and a point $s : \operatorname{Spec} k \to S$ such that the base change of $\mathscr{X} \to S$ by s is isomorphic to X.

Exercise 4.0.1. Let k be a perfect field of characteristic p. If a k-scheme X is liftable to characteristic zero with a smooth base then it is liftable to $W(k)$.

5 Results of Deligne and Illusie

Let k be a perfect field of characteristic $p > 0$ and let us set $S = \operatorname{Spec} k$ and $\tilde{S} = \operatorname{Spec} W_2(k)$. Let X be a smooth k-variety liftable to $\tilde{X}/\tilde{S}$. Then X' has a natural lifting $\tilde{X}'$ to $\tilde{S}$, which is defined as the base change of $\tilde{X} \to \tilde{S}$ via $\tilde{S} \to \tilde{S}$ coming from $\sigma_2 : W_2(k) \to W_2(k)$.

Let us assume that $F_{X/k} : X \to X'$ has a lifting $\tilde{F}_{X/k} : \tilde{X} \to \tilde{X}'$ so that we have a commutative diagram



Note that the map

$$\tilde{F}^* : \tilde{F}^*_{X/k}\Omega^1_{\tilde{X}'/\tilde{S}} \to \Omega^1_{\tilde{X}/\tilde{S}}$$

vanishes after pulling back to X. So we can define

$$\zeta = p^{-1}\tilde{F}^* : \Omega^1_{X'/S} \to F_*\Omega^1_{X/S}$$

such that $d\zeta = 0$, i.e., we have a map of complexes of $\mathscr{O}_{X'}$-modules

$$\varphi^1_{(\tilde{X},\tilde{F})} : \Omega^1_{X'/S}[-1] \to F_*\Omega^\bullet_{X/S}.$$

This map depends not only on the choice of $\tilde{X}$ but also on the choice of the Frobenius lifting $\tilde{F}$. In the following an important role will be played by the study of differences between maps corresponding to various choices of the Frobenius lift $\tilde{F}$ while keeping the same choice of $\tilde{X}$.

If X is smooth with a fixed lift $\tilde{X}$ over $W_2(k)$ then we can find a covering $\{\tilde{U}_\alpha\}_{\alpha\in I}$ of $\tilde{X}$ such that for each $\alpha \in I$ there exists $\tilde{F}_\alpha : \tilde{U}_\alpha \to \tilde{U}'_\alpha$ lifting the Frobenius morphism $F_\alpha : U_\alpha \to U'_\alpha$. In [DI] Deligne and Illusie used this to prove that $\varphi^1_{(\tilde{X},\tilde{F})}$ can be "glued" to a map

$$\varphi^1_{\tilde{X}} : \Omega^1_{X'/S}[-1] \to F_*\Omega^\bullet_{X/S}$$

in the derived category $D(X')$ of coherent sheaves on X', without assuming existence of the global lift of the Frobenius.

The map $\varphi^1_{(\tilde{X},\tilde{F})}$ can be prolonged to a quasi-isomorphism of complexes of $\mathscr{O}_{X'}$-modules

$$\varphi_{(\tilde{X},\tilde{F})}:\bigoplus_{i\geq 0}\Omega^i_{X'/S}[-i]\to F_*\Omega^\bullet_{X'/S}$$

Exercise 5.0.1. Show that the Frobenius push-forward of the de Rham complex $F_*\Omega^\bullet_{X/S}$ is a complex of $\mathscr{O}_{X'}$-modules.

6 Ogus–Vologodsky's non-abelian Hodge theory in positive characteristic

Let $\mathrm{MIC}_{p-1}(X/k)$ be the category of quasi-coherent $\mathscr{O}_X$-modules V with integrable connection ∇ such that the p-curvature $\psi: T_{X/k}\to End_k V$ is nilpotent of level $\leq p-1$, i.e., for all open subsets $U\subset X$ and for all derivations $D\in T_{X/k}(U)$ we have $\psi(D)^p=0$.

Now let $\mathrm{HIG}_{p-1}(X/k)$ be the category of Higgs $\mathscr{O}_X$-modules with a nilpotent Higgs field of level $\leq p-1$, i.e., we require $\theta^p=0$.

THEOREM 6.1. *Let X/k be a smooth variety defined over a field k of positive characteristic with a fixed lifting $\tilde{X}/W_2(k)$. Then the Cartier transform*

$$C_{\mathscr{X}/\mathscr{S}}:\mathrm{MIC}_{p-1}(X/k)\to \mathrm{HIG}_{p-1}(X'/k)$$

defines an equivalence of categories with quasi-inverse

$$C^{-1}_{\mathscr{X}/\mathscr{S}}:\mathrm{HIG}_{p-1}(X/k)\to \mathrm{MIC}_{p-1}(X'/k).$$

Let us now recall Cartier's theorem:

THEOREM 6.2. *There is an equivalence of categories between the category of quasi-coherent $\mathscr{O}_{X'}$-modules and the category of quasi-coherent $\mathscr{O}_X$-modules with integrable k-connection and vanishing p-curvature. This equivalence is given by the functors sending an $\mathscr{O}_{X'}$-module E to $(F^*_{X/k}E,\nabla_{can})$ and (V,∇) to V^∇.*

Ogus–Vologodsky's correspondence is a generalization of Cartier's theorem that extends the above equivalence to categories whose objects are extensions of objects from both categories (up to some level). More precisely, if $(E,\theta)\in$

$\mathrm{HIG}_{p-1}(X'/k)$ then we consider the filtration $N^p = 0 \subset N^{p-1} \subset ... \subset N^0 = (E,\theta)$ of length p given by $N^i = \ker\theta^i$. This filtration is a filtration by Higgs submodules with quotients $N_i = N^i/N^{i-1}$ that have zero Higgs field. Ogus–Vologodsky's correspondence tells us how to take extensions of $(F^*_{X/k}N_i, \nabla_{can}) = C^{-1}_{\tilde{X}/W_2(k)}(N_i, 0)$ and construct $(V,\nabla) = C^{-1}_{\tilde{X}/W_2(k)}(E,\theta)$ so that we can extend Cartier's equivalence to larger categories.

Note that this interpretation proves the following lemma:

LEMMA 6.3. *Let $(E,\theta) \in \mathrm{HIG}_{p-1}(X'/k)$ and assume E is coherent as an $\mathscr{O}_{X'}$-module. Then*

$$[C^{-1}_{\tilde{X}/W_2(k)}(E)] = F^*_{X/k}[E],$$

where $[\cdot]$ denotes the class of a coherent $\mathscr{O}_X$-module in Grothendieck's K-group $K_0(X)$.

An important difference between Cartier's and Ogus–Vologodsky's theorem is that the equivalence in Ogus–Vologodsky's theorem depends on lifting to $W_2(k)$ and for different liftings we obtain different equivalences of the same categories.

7 Inverse Cartier and Cartier transforms (after Lan–Sheng–Zuo)

In this section we construct inverse Cartier and Cartier transforms from Ogus–Vologodsky's theorem.

7.1 Inverse Cartier transform

Assume that X is smooth and there exists a global lifting $\tilde{X}$ of X over $W_2(k)$.

Let $(E,\theta) \in \mathrm{HIG}_{p-1}(X'/k)$. We want to construct $(V,\nabla) = C^{-1}_{\tilde{X}/W_2(k)}(E,\theta) \in \mathrm{MIC}_{p-1}(X/k)$. Let us take a $W_2(k)$-lifting $\tilde{X}'$ of X'/k defined by $\tilde{X}$ by base change. Let us cover take a covering $\{\tilde{U}_\alpha\}_{\alpha\in I}$ of $\tilde{X}$ such that for each $\alpha \in I$ there exists $\tilde{F}_\alpha : \tilde{U}_\alpha \to \tilde{U}'_\alpha$ lifting the Frobenius morphism $F_\alpha : U_\alpha \to U'_\alpha$. By previous arguments the lifting $\tilde{F}_\alpha$ allows us to construct

$$\zeta'_\alpha = p^{-1}\tilde{F}^*_\alpha : F^*_\alpha\Omega^1_{U'_\alpha/k} \to \Omega^1_{U_\alpha/k}.$$

Therefore over each U_α we can define $(V_\alpha, \nabla_\alpha)$ by setting

$$V_\alpha := F_\alpha^*(E|_{U_\alpha})$$

and

$$\nabla_\alpha := \nabla_{can} + \zeta'_\alpha(F_\alpha^*\theta|_{U'_\alpha}).$$

Now we need to glue $(V_\alpha, \nabla_\alpha)$ and (V_β, ∇_β) over $U_{\alpha\beta} = U_\alpha \cap U_\beta$. To do so we need the following lemma due to Deligne and Illusie:

LEMMA 7.1. *There exist $\mathscr{O}_{U'_{\alpha\beta}}$-linear maps $h_{\alpha\beta} : \Omega_{U'_{\alpha\beta}} \to (F_{U'_{\alpha\beta}})_*\mathscr{O}_{\alpha\beta}$ such that*

1. *for all α, β we have*
$$\zeta_\alpha - \zeta_\beta = dh_{\alpha\beta},$$
2. *for all α, β, γ we have over $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$*
$$h_{\alpha\beta} + h_{\beta\gamma} = h_{\alpha\gamma}.$$

Let $h'_{\alpha\beta} : F^*_{U'_{\alpha\beta}}\Omega_{U'_{\alpha\beta}} \to \mathscr{O}_{\alpha\beta}$ be adjoint to $h_{\alpha\beta}$. Now we define gluing maps

$$g_{\alpha\beta} : V_\alpha|_{U_{\alpha\beta}} \to V_\beta|_{U_{\alpha\beta}}$$

using

$$h'_{\alpha\beta}(F^*\theta|_{U_{\alpha\beta}}) : F^*E|_{U_{\alpha\beta}} \to F^*E|_{U_{\alpha\beta}} \otimes F^*\Omega_{U'_{\alpha\beta}} \to F^*E|_{U_{\alpha\beta}}$$

by setting

$$g_{\alpha\beta} := \exp(h_{\alpha\beta}(F^*\theta|_{U_{\alpha\beta}})) = \sum_{i=0}^{p-1} \frac{(h_{\alpha\beta}(F^*\theta|_{U_{\alpha\beta}}))^i}{i!}.$$

Claim: $g_{\alpha\beta}$ allow us to glue $(V_\alpha, \nabla_\alpha)$ and (V_β, ∇_β) over $U_{\alpha\beta}$ to a global object $(V, \nabla) \in \mathrm{MIC}_{p-1}(X/k)$.

7.2 Cartier transform

Let $(V, \nabla) \in \mathrm{MIC}_{p-1}(X/k)$ and let $\psi : V \to V \otimes F^*\Omega_{X'/k}$ be the p-curvature of (V, ∇). Then we define $(E, \theta) = C(V, \nabla) \in \mathrm{HIG}_{p-1}(X'/k)$ by defining $(\tilde{V}, \tilde{\psi} : V \to V \otimes F^*\Omega_{X'/k})$ and then showing that it descends (by Cartier's descend) to X'.

First we define $(\tilde{V},\tilde{\nabla})$ that corresponds to (F^*E,∇_{can}), so it allows us to recover E. Again we do that locally: over each U_α we set

$$\tilde{V}_\alpha := V|_{U_\alpha}$$

and

$$\tilde{\nabla}_\alpha := \nabla_\alpha + \zeta'_\alpha(\psi|_{U_\alpha}).$$

These sheaves glue to $(\tilde{V},\tilde{\nabla})$ and the gluing is described by

$$\tilde{g}_{\alpha\beta} := \exp(h_{\alpha\beta}(\psi|_{U_{\alpha\beta}})).$$

The p-curvature ψ induces $\tilde{\psi}: V \to V \otimes F^*\Omega_{X'/k}$ parallel with respect to $\tilde{\nabla}$, so $(\tilde{V},\tilde{\psi})$ descends to (E,θ).

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