Lecture 2

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1 Bounding the number of sections

The following proposition is elementary and its proof does not use any deep results.

PROPOSITION 1.1. *There exists an explicit function f depending only on r and C such that the following holds. Let X be a smooth projective surface and let H be a very ample divisor on X. If E is a rank r torsion free sheaf on X with*

$$\mu_{\max,H}(E) \leq C \cdot H^2$$

then we have

$$h^0(X,E) \leq f(r,C) \cdot H^2.$$

Proof. The proof is by induction on the rank r. For $r = 1$ we leave finding an appropriate bound as an exercise (see Exercise 1.0.2).

If $r = 2$ then let us take any section $\mathcal{O}_X \to E$ (if there are no sections there is nothing to prove) and its saturation $L \to E$. Let us set $L' = E/L$. By definition we have

$$\mu(L) \leq \mu_{\max}(E) \leq C \cdot H^2,$$

so

$$h^0(X,L) \leq f(1,C) \cdot H^2.$$

On the other hand

$$\mu(L') = 2\mu(E) - \mu(L) \leq 2\mu(E) \leq 2\mu_{\max}(E) \leq 2C \cdot H^2,$$

so

$$h^0(X,L') \le f(1,2C)\cdot H^2.$$

This implies

$$h^0(X,E) \le h^0(X,L) + h^0(X,L') \le (f(1,C)+f(1,2C))\cdot H^2.$$

Now let us assume that the proposition holds for sheaves of rank less than r (for some $r>2$) and let us consider a torsion free sheaf E of rank r.

If $h^0(X,E)\neq 0$ then let E' be the image of the evaluation map $H^0(X,E)\otimes \mathscr{O}_X \to E$. By definition $h^0(X,E') = h^0(X,E)$, so if $r' = rkE' < r$ then we get $h^0(X,E) \le f(r',C)\cdot H^2$.

So we can assume that $rkE' = r$. Replacing E with E' we can also assume that E is globally generated.

Since $r>2$ the cokernel of a general section $\mathscr{O}_X \to E$ is torsion free (see Exercise 1.0.3). Let us set $E' = E/\mathscr{O}_X$. Let $F'\subset E'$ be the maximal destabilizing subsheaf of E' and let $F\subset E$ be the preimage of F' in E. Let us set $r' = rkF'$. Then

$$\mu(F) = \frac{r'}{r'+1}\mu(F') \le \mu_{\max}(E) \le C\cdot H^2,$$

so

$$\mu_{\max}(E') \le \frac{r'+1}{r'} C\cdot H^2 \le 2C\cdot H^2.$$

By the induction assumption we have

$$h^0(X,E') \le f(r-1,2C)\cdot H^2,$$

so $h^0(X,E) \le (1+f(r-1,2C))\cdot H^2$.

Summing up, we see that as $f(r,C)$ we can take $f(1,C)+f(1,2C)$ for $r=2$ and $1+f(r-1,2C)$ for $r>2$. $\square$

The above proposition is a special case of much more general and precise result (see [La, Theorem 3.3]). However, the proof given here is completely elementary, whereas the one in [La] uses non-trivial results (e.g., Bogomolov's inequality that we will prove using the above proposition).

Exercise 1.0.1. Let E be a torsion free $\mathscr{O}_X$-module. Let G be an $\mathscr{O}_X$-submodule of E. Show that there exists a unique $\mathscr{O}_X$-submodule $G'\subset E$ such that the following conditions are satisfied:

1. G' contains G,

2. E/G is torsion free,

3. G' has the same rank as G.

Such G' is called a *saturation* of G.

Exercise 1.0.2. Find an explicit degree 2 polynomial g such that if L is a line bundle on a smooth projective surface X and $LH \leq C \cdot H^2$ for some $C \geq 0$ and very ample H then

$$h^0(X,L) \leq g(C) \cdot H^2.$$

Hint: There are a few ways to do that (even without assuming that H is very ample). The easiest way is to restrict to a smooth curve $C \in |H|$ and use the fact that $h^0(X, L(-([C]+1)H)) = 0$.

Exercise 1.0.3. Let E be a globally generated torsion free $\mathscr{O}_X$-module of rank $r > \dim X$ on a smooth projective variety X. Prove that the cokernel of a general section $\mathscr{O}_X \to E$ is torsion free.

Hint: Compare with [Ha, Chapter II, Exercise 8.2].

2 Spreading out

Let X be a smooth projective variety defined over a field k of characteristic 0. Then there exists a subring $R \subset k$ finitely generated as a $\mathbb{Z}$-algebra and a smooth projective scheme $\mathscr{X} \to S = \operatorname{Spec} R$ such that $\mathscr{X} \otimes_R k \simeq X$. For every closed point $s \in S$ the residue field $k(s)$ is finite.

Note that choice of R and $\mathscr{X}/R$ is highly non-unique. But for any $\mathscr{X}_1/R_1$ and $\mathscr{X}_2/R_2$ there exists $R_3 \supset R_1$, $R_3 \supset R_2$ and $\mathscr{X}_3/R_3$ such that $\mathscr{X}_i \otimes_{R_i} R_3 \simeq \mathscr{X}_3$ for $i = 1,2$. Moreover, if for all but finitely many primes p, the maps $R_i/pR_i \to R_3/pR_3$ are injective, so we usually need not to worry about the choice of $R \subset k$.

Exercise 2.0.1. Use spreading out to prove the following fact.

Let k be an algebraically closed field of characteristic zero. Let $\sigma : \mathbb{A}^n_k \to \mathbb{A}^n_k$ be an automorphism of order 2 (an involution). Then σ has a fixed point, i.e., there exists a k-point x of $\mathbb{A}^n_k$ such that $\sigma(x) = x$.

Hint: involutions of an affine space over a finite field of characteristic $\neq 2$ have a fixed point.

3 Bogomolov's inequality in characteristic zero

Let X be a smooth projective k-variety of dimension n. Let us recall the following:

PROPOSITION 3.1. *Assume that k has positive characteristic p. Let us fix a very ample divisor H such that $T_X(H)$ is globally generated. For any rank r torsion free $\mathcal{O}_X$-module E we have*

$$\mu_{\max,H}(F_X^*E) \le p\mu_{\max,H}(E) + (r-1)H^n.$$

Let X be a smooth projective surface defined over a field k and let H be an ample divisor on X. Let E be a rank r torsion free $\mathcal{O}_X$-module. We define a *discriminant* of E by the following formula:

$$\Delta(E) := \int_X \left(2rc_2(E) - (r-1)c_1^2(E)\right).$$

THEOREM 3.2. (Bogomolov, [Bo]) *Assume that* $\operatorname{char} k = 0$. *If E is slope H-semistable then* $\Delta(E) \ge 0$.

Proof. Since $\Delta(E) \ge \Delta(E^{**})$, replacing E with E^{**} we can assume that E is locally free.

Let us first assume that $\det E \simeq \mathcal{O}_X$. Without loss of generality we can assume that $T_X(H)$ is globally generated (we can always replace H by its large multiple). Let us consider a spreading out $\mathscr{X} \to S = \operatorname{Spec} R$, where $R \subset k$ is a finitely generated ring over $\mathbb{Z}$. By Proposition 1.1 and Proposition 4.1 we see that there exists a constant C such that for all closed points $s \in S$ we have

$$h^0(X_s, F_{X_s}^*E_s) \le C$$

and

$$h^2(X_s, F_{X_s}^*E_s) = h^0(X_s, F_{X_s}^*E_s^* \otimes \omega_{X_s}) \le C.$$

Therefore

$$\chi(X_s, F_{X_s}^*E_s) \le 2C.$$

But by the Riemann–Roch theorem we have

$$\chi(X_s, F_{X_s}^*E_s) = r\chi(X_s, \mathcal{O}_{X_s}) - \int_{X_s} c_2(F_{X_s}^*E_s) = r\chi(X, \mathcal{O}_X) - p_s^2 \cdot \int_X c_2(E),$$

where p_s is the characteristic of the residue field $k(s)$. Therefore taking $p_s \gg 0$ we get $\int_X c_2(E) \ge 0$, as required.

To prove the inequality in the general case we need the following covering lemma due to Bloch and Gieseker:

LEMMA 3.3. *Let X be a smooth projective variety defined over some algebraically closed field k and L be a line bundle on X. Let us fix a positive integer r. Then there exists a smooth projective variety Y, a finite surjective morphism $f : Y \to X$ and a line bundle M on Y such that $M^{\otimes r} \simeq f^* L$.*

Let us apply this lemma to $L = \det E$. Then $E' = f^* E \otimes M^{-1}$ is a slope $f^* H$-semistable vector bundle with $\det E' \simeq \mathscr{O}_Y$. Therefore $\Delta(E') \geq 0$ by the first part of the proof. But $\Delta(E) = \Delta(E')/\deg f$, so $\Delta(E) \geq 0$. □

Remark 3.4. The standard proof of Bogomolov's inequality uses semistability of symmetric powers and a restriction theorem (see, e.g., [HL, Theorem 3.4.1]). There is also another algebraic proof of Gieseker via positive characteristic (see [Gi, Theorem 3.2]), but using a weaker form of Proposition 1.1 that is proven in a quite complicated way by restriction to curves. Our proof is modeled on Gieseker's proof with a simplified proof of a stronger result.

4 Bogomolov's inequality in positive characteristic

Let X be a smooth projective variety defined over a field k and let H be an ample divisor on X. Assume that k has positive characteristic. If k has positive characteristic then we say that E is *strongly slope H-semistable* if all the Frobenius pull backs $(F_X^n)^* E$ of E for $n \geq 0$ are slope H-semistable.

THEOREM 4.1. *Let us assume that* $\dim X = 2$. *If E is strongly H-semistable then*

$$\Delta(E) \geq 0.$$

Proof. The proof follows a similar strategy as the proof of Theorem 4.2.

1. By taking the reflexivization we can reduce the statement to the case when E is locally free.

2. Taking a covering and twisting the pull back by an appropriate line bundle we can assume that $\det(E) \simeq \mathscr{O}_X$.

3. Apply Proposition 4.1 to sequences $\{(F_X^n)^* E\}_n$ and $\{(F_X^n)^* E^* \otimes \omega_X\}_n$.

4. Look at $\chi(X, (F_X^n)^* E)$ for large n. Use the above and Serre's duality to show that $\chi(X, (F_X^n)^* E)$ is uniformly bounded from the above.

5. Use the Riemann–Roch theorem for $\chi(X,(F_X^n)^*E)$ to complete the proof of the inequality.

□

Exercise 4.0.1. Using the above theorem show that there exists a constant $C = C(X,H)$ such that for every slope H-semistable E we have

$$\Delta(E) \geq C.$$

5 Counterexample to Kodaira's vanishing theorem in positive characteristic

There exists a smooth projective surface X defined over a field k of positive characteristic and an ample divisor H such that $H^1(X,\mathscr{O}_X(-H)) \neq 0$ (see [Ra]).

6 Counterexample to Bogomolov's inequality in positive characteristic

Let (X,H) be as in the previous subsection. Since $\mathrm{E}xt^1(\mathscr{O}_X(H),\mathscr{O}_X) \simeq H^1(X,\mathscr{O}_X(-H))$, there exists a non-trivial extension

$$(*) \qquad 0 \to \mathscr{O}_X \to E \to \mathscr{O}_X(H) \to 0.$$

Clearly E is locally free (in the following we do not distinguish between locally free $\mathscr{O}_X$-modules and the corresponding vector bundles on X).

CLAIM 6.1. *The vector bundle E is slope H-stable.*

Proof. Assume that E is not slope H-stable and let $L \subset E$ be a subsheaf which violates the stability condition. Without loss of generality we can assume that E/L is torsion free (we can always replace $L \subset E$ by the kernel of the map $E \to (E/L)/\mathrm{Torsion}$; this subsheaf also destabilizes E and has the required property). Since E/L has rank 1, we can write it as $I_Z(H-L)$ for some zero-dimensional scheme Z (here we slightly abuse notation writing L as a divisor). By assumption we have

$$L.H \geq \frac{1}{2}H^2 > 0.$$

Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{O}_X & \longrightarrow & E & \longrightarrow & \mathscr{O}_X(H) & \longrightarrow & 0. \\ & & & & \uparrow & \nearrow & & & \\ & & & & L & & & & \end{array}$$

Note that $\mathrm{Hom}(L, \mathscr{O}_X) = H^0(X, L^{-1}) = 0$ since $L^{-1}.H = -L.H < 0$. Therefore the map $L \to \mathscr{O}_X(H)$ is non-zero and hence it is given by an effective divisor D such that $L = \mathscr{O}_X(H - D)$. Note that if $D.H = 0$ then $D = 0$ and the sequence $(*)$ splits, a contradiction. Therefore $D.H > 0$. Computation of $c_2(E)$ by means of the short exact sequence

$$(**) \qquad 0 \to L \to E \to I_Z(H - L) \to 0.$$

leads to

$$c_2(E) = \deg Z + L.(H - L) \geq L.(H - L) = D.(H - D).$$

But $c_2(E) = 0$ (from $(*)$), so we have

$$D.H \leq D^2.$$

By the Hodge index theorem $D^2 \cdot H^2 \leq (D.H)^2$, so $(D.H) \cdot H^2 \leq (D.H)^2$ which gives $H^2 \leq D.H$. Since L destabilizes E we have

$$(H - 2D).H = (2L - H).H \geq 0,$$

and hence $H^2 \geq 2D.H$. But this implies that $D.H \leq 0$, a contradiction. □

Note that $c_1E = H$ and $c_2E = 0$, so

$$\Delta(E) = \int_X (4c_2E - c_1^2E) = -H^2 < 0.$$

Hence E does not satisfy Bogomolov's inequality.

CLAIM 6.2. *Let us assume that k has positive characteristic p. Then E is not strongly H-semistable.*

Proof. Pulling back $(*)$ we get

$$0 \to \mathscr{O}_X \to (F_X^n)^*E \to (F_X^n)^*\mathscr{O}_X(H) \simeq \mathscr{O}_X(p^nH) \to 0.$$

But $\mathcal{E}xt^1(\mathscr{O}_X(p^nH), \mathscr{O}_X) \simeq H^1(X, \mathscr{O}_X(-p^nH)) = 0$ for $n \gg 0$ by Serre's vanishing theorem. So the above sequence splits for large n, contradicting semistability of $(F_X^n)^*E$. □

The following exercise shows Mumford's proof of Kodaira's vanishing on complex projective surfaces.

Exercise 6.0.1. Use Bogomolov's inequality in characteristic 0 to prove the following version of Kodaira's vanishing theorem on surfaces:

Let H be a nef and big divisor on a smooth complex projective surface. Then $H^1(X, \mathcal{O}_X(-H)) = 0$.

Hint:

Prove that a non-zero class in $H^1(X, \mathcal{O}_X(-H))$ leads to a slope H-semistable vector bundle

$$0 \to \mathcal{O}_X \to E \to \mathcal{O}_X(H) \to 0$$

and use Bogomolov's inequality for E.

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