Lecture I

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1 Harder–Narasimhan filtration and its properties

Let X be a smooth projective variety of dimension n defined over some algebraically closed field k. Let L be an ample (or nef) line bundle on X. For any torsion free $\mathcal{O}_X$-module E on X we define its *slope* by

$$\mu_L(E) = \frac{c_1(E).c_1(L)^{n-1}}{rk E}.$$

We say that E is *slope L-semistable* if for every subsheaf $E' \subset E$ we have $\mu(E') \leq \mu(E)$. E is *slope L-stable* if for every subsheaf $E' \subset E$ of rank less than the rank of E we have $\mu_L(E') < \mu_L(E)$.

Let us recall basic facts about the Harder–Narasimhan filtration. In these notes semistability always means slope semistability with respect to some polarization (ample or sometimes nef).

Let X be a normal projective variety over a field k and let us fix a nef line bundle L. Let E be a torsion free $\mathcal{O}_X$-module. Then there exists a unique filtration $E_0 = 0 \subset E_1 \subset ... \subset E_m = E$ such that all the quotients $E^i = E_i/E_{i-1}$ are torsion free and slope L-semistable, and

$$\mu_L(E^1) > \mu_L(E^2) > ... > \mu_L(E^m).$$

This filtration is called the *Harder–Narasimhan filtration* of E.

The first term $E_1 = E^1$ in the filtraton is called the *maximal destabilizing subsheaf* and its slope is denoted by $\mu_{\max,L}(E)$. Similarly, the last quotient E^m in the filtration is called the *minimal destabilizing quotient* and its slope is denoted by $\mu_{\min,L}(E)$.

Let $E_0 = 0 \subset E_1 \subset ... \subset E_m = E$ be any filtration such that all the quotients E_i/E_{i-1} are torsion free. Let us set $r_i := rk\, E_i$ and $d_i := \deg_L E_i = c_1(L)^{\dim X-1}.c_1(E_i)$ for $i = 1,...,m$. By convention we set $(r_0, d_0) = (0,0)$. Then we can consider the polygonal curve in $\mathbb{R}^2$ obtained by joining the points (r_i, d_i) for $i = 0,...,m$. If $E_\bullet$ is the Harder–Narasimhan filtration then the corresponding curve is called the *Newton polygon* of E.

PROPOSITION 1.1. *Let $E_0 = 0 \subset E_1 \subset ... \subset E_m = E$ be any filtration such that all the quotients E_i/E_{i-1} are torsion free. Then the corresponding polygonal curve lies below the Newton polygon of E. Moreover, if the corresponding polygonal curve coincides with the Newton polygon of E then $E_\bullet$ is the Harder–Narasimhan filtration of E.*

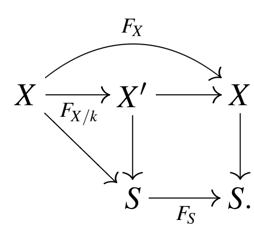
Exercise 1.0.1. Prove the above proposition.

LEMMA 1.2. *Let $E_0 = 0 \subset E_1 \subset ... \subset E_m = E$ be any filtration such that all the quotients E_i/E_{i-1} are torsion free and slope L-semistable. Prove that the maximal (minimal) slopes of the corresponding polygonal curve are not larger (respectively, smaller) than that of the Newton polygon of E.*

2 Semistability in positive characteristic

Let X be a scheme of characteristic $p > 0$ (i.e., $\mathscr{O}_X$ is a sheaf of $\mathbb{F}_p$-algebras). By $F_X : X \to X$ we denote the *absolute Frobenius morphism* of X, which is defined as identity on topological spaces and raising to the p-th power on the structure sheaves.

Let X be a scheme defined over a perfect field k of characteristic p. Let X' be the product of the structure morphism $X \to S = \operatorname{Spec} k$ over the absolute Frobenius morphism $F_S : S \to S$. Then $F_X : X \to X$ factors through a k-morphism $F_{X/k} : X \to X'$, which is called the *geometric Frobenius morphism.* So we have the following commutative diagram



In the following we often write F instead of $F_{X/k}$ if no confusion is likely to occur.

Example 2.1. *(Raynaud)* Below we give examples of semistable bundles with unstable Frobenius pull back.

Let C be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic p. Then $(F_C)_*\mathscr{O}_C$ is semistable but $F_C^*((F_C)_*\mathscr{O}_C)$ is not semistable.

Exercise 2.0.2. Let E be a vector bundle on C as above. Prove that $F_C^*((F_C)_*E)$ is not semistable.

Assume that $p \geq 3$ and consider the vector bundle B_1 defined as the cokernel of the canonical map $\mathscr{O}_C \to (F_C)_*\mathscr{O}_C$. Then B_1 is semistable but $F_C^*B_1$ is not semistable.

Exercise 2.0.3. Show that $F_C^*B_1$ is not semistable.

Hint: Compute the degree of B_1 and use $B_1 \subset (F_C)_*\omega_C$ coming from the Frobenius push down of the de Rham complex on C.

3 Modules with connections

Let X be a k-scheme and let E be an $\mathscr{O}_X$-module.

Definition 3.1. We say that $\nabla : E \to E \otimes_{\mathscr{O}_X} \Omega_{X/k}$ is a k-connection if it is a morphism of abelian sheaves such that

$$\nabla(fs) = f\nabla(s) + s \otimes df$$

for all open subsets $U \subset X$ and sections $f \in \mathscr{O}_X(U)$ and $s \in E(U)$.

It is important to note that ∇ is usually not a morphism of sheaves of $\mathscr{O}_X$-modules.

A k-connection can be extended to a morphism of abelian sheaves

$$\nabla = \nabla_i : E \otimes_{\mathscr{O}_X} \Omega^i_{X/k} \to E \otimes_{\mathscr{O}_X} \Omega^{i+1}_{X/k}$$

by

$$\nabla(s \otimes \omega) = s \otimes d\omega + (-1)^i \nabla(s) \wedge \omega,$$

where $\nabla(s) \wedge \omega$ is the image of $\nabla(s) \otimes \omega$ under the canonical map $E \otimes_{\mathscr{O}_X} \Omega_{X/k} \otimes_{\mathscr{O}_X} \Omega^i_{X/k} \to E \otimes_{\mathscr{O}_X} \Omega^{i+1}_{X/k}$ given by $s \otimes \omega_1 \otimes \omega_2 \to s \otimes (\omega_1 \wedge \omega_2)$. The composition

$$K = \nabla_1 \circ \nabla : E \to E \otimes_{\mathscr{O}_X} \Omega^2_{X/k}$$

is $\mathscr{O}_X$-linear and it is called the *curvature*.

Exercise 3.0.4. Check that

$$\nabla_{i+1} \circ \nabla_i (s \otimes \omega) = K(s) \otimes \omega.$$

If $K = 0$ then we say that the connection ∇ is *integrable*. In that case we get a complex of sheaves of k-modules

$$E \xrightarrow{\nabla} E \otimes_{\mathscr{O}_X} \Omega_X \longrightarrow \ldots \longrightarrow E \otimes_{\mathscr{O}_X} \Omega^i_{X/k} \xrightarrow{\nabla_i} E \otimes_{\mathscr{O}_X} \Omega^{i+1}_{X/k} \longrightarrow \ldots$$

called the *de Rham complex of* (E, ∇) and denoted by $(E \otimes_{\mathscr{O}_X} \Omega^\bullet_{X/k}, \nabla)$.

Example 3.2. Let X be a smooth complex variety and let (E, ∇) be an $\mathscr{O}_X$-module with a connection. Then E is locally free and $c_i(E) = 0 \in H^{2i}(X, \mathbb{Q})$ for $i > 0$.

These facts are no longer true in positive characteristic and in the next subsection we will see that there are many pairs (E, ∇) for which E is non-locally free or $c_i(E)$ are non-zero.

If $\nabla : E \to E \otimes_{\mathscr{O}_X} \Omega_{X/k}$ is a k-connection then it defines $T_{X/k} \to End_k E$ by sending $D \in T_{X/k} = \mathrm{Hom}(\Omega_{X/k}, \mathscr{O}_X)$ to the endomorphism

$$E \xrightarrow{\nabla} E \otimes_{\mathscr{O}_X} \Omega_{X/k} \xrightarrow{\mathrm{id} \otimes D} E \otimes_{\mathscr{O}_X} \mathscr{O}_X = E.$$

Now if k has characteristic p and D is a derivation then D^p is also a derivation. Therefore we can define a *p-curvature* $\psi : F_X^* T_X \to End\, E$ of (E, ∇) by

$$\psi(D) = (\nabla(D))^p - \nabla(D^p).$$

4 Cartier's inseparable descent

Let k be a field of characteristic $p > 0$ and let X be a smooth k-scheme. Let E be a quasi-coherent $\mathscr{O}_{X'}$-module. Then $F^*_{X/k} E = F^{-1}_{X/k} E \otimes_{F^{-1}_{X/k} \mathscr{O}_{X'}} \mathscr{O}_X$ has a canonical connection defined by $\nabla_{can}(e \otimes f) = e \otimes df$.

THEOREM 4.1. (Cartier) *There is an equivalence of categories between the category of quasi-coherent $\mathscr{O}_{X'}$-modules and the category of quasi-coherent $\mathscr{O}_X$-modules with integrable k-connection and vanishing p-curvature. This equivalence is given by the functors sending an $\mathscr{O}_{X'}$-module E to $(F^*_{X/k} E, \nabla_{can})$ and (V, ∇) to $V^\nabla = \{v \in V : \nabla(v) = 0\}$.*

Exercise 4.0.5. Let us consider a functor between the category of quasi-coherent $\mathcal{O}_X$-modules and the category of quasi-coherent $\mathcal{O}_X$-modules with integrable k-connection and vanishing p-curvature given by sending an $\mathcal{O}_X$-module E to (F_X^*E, ∇_{can}), where ∇_{can} is the canonical connection from Cartier's theorem for the pull back of E via $X' \to X$.

1. When is this functor an equivalence of categories?

2. Can you generalize Cartier's theorem to the relative set up?

5 Bounding semistability of Frobenius pull backs

Now let us assume that X and Y be smooth projective k-varieties of dimension n and let $f : Y \to X$ be a finite, separable map. If E is L-semistable on X then f^*E is f^*L-semistable.

On the other hand we saw that this fails if we do not assume that f is separable. Here we show how to bound instability of pull-back of sheaves by the Frobenius morphism.

PROPOSITION 5.1. *Let us assume that k has characteristic $p > 0$. Let us fix a very ample divisor H such that $T_X(H)$ is globally generated. For any rank r torsion free $\mathcal{O}_X$-module E we have*

$$\mu_{\max,H}(F_X^*E) \leq p\mu_{\max,H}(E) + (r-1)H^n.$$

Proof. Let us first assume that E is H-semistable and let $E_0 = 0 \subset E_1 \subset ... \subset E_m = F_X^*E$ be the Harder–Narasimhan filtration of F_X^*E. Note that none of the subsheaves $E_i \subset F_X^*E$ descends to a subsheaf of E, as it would contradict semistability of E. Therefore by Cartier's theorem these subsheaves are not preserved by the canonical connection ∇_{can} of F_X^*E. It follows that the induced $\mathcal{O}_X$-linear maps

$$E_i \to F_X^*E/E_i \otimes \Omega_X$$

are nonzero. But $T_X(H)$ is globally generated and therefore we can embed Ω_X into $\mathcal{O}_X(H)^{\oplus N}$ for some N. So for every $i = 1, ..., m-1$ we have non-zero maps

$$E_i \to F_X^*E/E_i \otimes \mathcal{O}_X(H)^{\oplus N},$$

whose existence implies

$$\mu(E_i/E_{i-1}) \leq \mu(E_{i+1}/E_i) + H^n.$$

Summing these inequalities for $i = 1, ..., m-1$ we get

$$\mu_{\max}(F_X^*E) \leq \mu_{\min}(F_X^*E) + (m-1)H^n \leq \mu(F_X^*E) + (r-1)H^n = p\mu(E) + (r-1)H^n,$$

which proves the required inequality.

Now if E is not H-semistable then we can apply the obtained inequality to all the quotients of the Harder–Narasimhan filtration of E. Since $\mu_{\max,H}(F_X^*E)$ is bounded from the above by the maximum of slopes of any filtration with semistable quotients, we immediately get the required inequality. □

Exercise 5.0.6. Use the above proposition to show that for any torsion free $\mathscr{O}_X$-module E and any very ample divisor H

$$-\infty < \liminf_{n\to\infty} \frac{\mu_{\min,H}((F_X^n)^*E)}{p^n} \leq \limsup_{n\to\infty} \frac{\mu_{\max,H}((F_X^n)^*E)}{p^n} < +\infty$$

Exercise 5.0.7. Generalize the above exercise to show that the assertion holds also if H is assumed to be nef instead of very ample.