MAGIT Exercises, Series 5

Exercise 1.

Show that every $\mathbb{Z}$-module has an injective resolution.

Exercise 2.

Let $f : X \to Y$ be a continuous map of topological spaces and for each presheaf $\mathcal{F}$ on X we define a presheaf $f_*\mathcal{F}$ by

$$f_*\mathcal{F}(V) := \mathcal{F}(f^{-1}(V)).$$

For a presheaf $\mathcal{G}$ on Y we define a presheaf $f^{-1}\mathcal{G}$ by

$$f^{-1}\mathcal{G}(U) := \lim_{V \supset f(U)} \mathcal{G}(V).$$

Let $PSh(X)$ denote the category of presheaves of abelian groups on X (and analogously for Y). Prove that $f_* : PSh(X) \to PSh(Y)$ and $f^{-1} : PSh(Y) \to PSh(X)$ extend naturally to functors and that these functors are adjoint.

Exercise 3.

In the notation of the previous exercise:

1. Is f_* always left exact? Is f_* always right exact?
2. Is f^{-1} always left exact? Is f^{-1} always right exact?

Exercise 4.

Let k be a fixed field and let $\mathcal{C}$ be the category of k-vector spaces with filtrations. By definition an object of $\mathcal{C}$ is a pair $(V, F^\bullet)$, where V is a k-vector space and $F^\bullet$ is a decreasing filtration $... \subset F^{n+1}V \subset F^nV \subset F^{n-1}V \subset ...$, where $F^nV \subset V$ are k-vector subspaces. A morphism $(V, F^\bullet) \to (W, F^\bullet)$ is a k-linear map $f : V \to W$ such that for all n we have $f(F^nV) \subset F^nW$. Check that $\mathcal{C}$ is an additive category. Does it contain kernels and cokernels? Does it contain images and coimages? Is it abelian?

Exercise 5.

Let $\mathcal{A}$ be an abelian category which has enough injectives and let $\{X_s\}_{s\in S}$ be a set of objects of $\mathcal{A}$. Show that there exists an abelian subcategory $\mathcal{B} \subset \mathcal{A}$ such that

1. the class of objects of $\mathcal{B}$ forms a set containing all X_s,
2. $\mathcal{B}$ has enough injectives,
3. an object of $\mathcal{B}$ is injective if and only if it is injective in $\mathcal{A}$.