MAGIT Exercises, Series 2

Exercise 1.

Show that Z is the equalizer of f and g:

$$Z \xrightarrow{h} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

then h is a monomorphism.

Exercise 2.

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors and $\sigma : FG \to \mathrm{id}_{\mathcal{D}}$ and $\tau : \mathrm{id}_{\mathcal{C}} \to GF$ natural transformations of functors. Assume that

$$F \xrightarrow{F\tau} FGF \xrightarrow{\sigma F} F$$

is the identity of F and

$$G \xrightarrow{\tau G} GFG \xrightarrow{G\sigma} G$$

is the identity of G. Show that F and G are adjoint to each other.

Exercise 3.

Find the left adjoint functor to the forgetful functors:

1. $R - Mod \to Ab$, where R is a fixed ring,
2. $Top \to Set$,
3. from the category of associative k-algebras to the category of k-vector spaces, where k is a fixed field.

Exercise 4.

Let p be a fixed prime and let us consider

1. Let I be the category associated to the partially ordered set $\mathbb{Z}_{\geq 0}$. Define the I-diagram M in the category Ab by sending n to $\mathbb{Z}/p^n\mathbb{Z}$ and $\mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ by sending $x \to p^{n-m}x$ for $n \geq m$. Show that the colimit $colim_I M$ is the subgroup of $\mathbb{Q}/\mathbb{Z}$ consisting of classes of a/p^n for some integers a and $n \geq 0$.
2. Let I be the category associated to a poset given by the divisibility in $\mathbb{Z}_{>0}$: $m \geq n$ if and only if n divides m. Define the I-diagram M in the category Ab by sending n to $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ by sending $x \to \frac{m}{n}x$ for $m \geq n$. Show that the colimit $colim_I M$ is isomorphic to $\mathbb{Q}/\mathbb{Z}$.

Exercise 5.

Let G-Set be the category of sets with an action of a group G (and G-equivariant maps as morphisms). Let $H \subset G$ be a subgroup. Check if the forgetful functor G-Set $\to$ H-Set has a left adjoint functor. Does it have a right-adjoint functor?