MAGIT Exercises, Series 13

Exercise 1.

Let X be a compact topological space and let $f : E \to X$ be a topological real vector bundle. Show that there exists a continuous map $\phi : E \times_X E \to \mathbb{R}$ such that every $p \in X$ and for all $x, y \in E(p) = f^{-1}(p)$, we have $\phi(y, x) = \phi(x, y)$ and $\phi(x, x) > 0$ for all $x \in E(p) - 0$.

Exercise 2.

Let X be a compact (Hausdorff) topological space and let us fix a point $x \in X$. Let $E \to X$ be a topological (real) vector bundle and let $s : X \to E$ be a section such that $s(x) = 0$. Prove that there exists a finite number of sections $s_1, ..., s_n : X \to E$ and continuous functions $f_1, ..., f_n \in C^0(X)$ such that $s = \sum f_i s_i$ on X and $f_i(x) = 0$ for all i.

Exercise 3.

Let

$$1 \to G_1 \to G_2 \to G_3 \to 1$$

be a short exact sequence of topological groups. For a connected topological space X we denote by $G(X)$ the group of continuous functions on X with values in G. Prove that there exists

$$\delta_X : G_3(X) \to H^1(X, G_1)$$

such that

1. $X \to \delta_X$ is a natural transformation of functors $X \to G_3(X)$ and $X \to H^1(X, G_1)$,
2. we have the following exact sequence (of groups and pointed sets!):

$$1 \to G_1(X) \to G_2(X) \to G_3(X) \to H^1(X, G_1) \to H^1(X, G_2) \to H^1(X, G_3).$$

Exercise 4

Let $(X, \mathcal{R}_X)$ be a ringed space and let $\mathcal{A}$ be an $\mathcal{R}_X$-module. Prove that if $U, V \subset X$ are open subsets then we have an exact sequence:

$$... \to H^m(U \cup V, \mathcal{A}) \to H^m(U, \mathcal{A}) \oplus H^m(V, \mathcal{A}) \to H^m(U \cap V, \mathcal{A}) \to H^{m+1}(U \cup V, \mathcal{A}) \to ...$$

Exercise 5

Let X be a topological space and let $X = U_1 \cup U_2$ be an open covering of X such that $U_1 \cap U_2 \neq \emptyset$. Let F be a vector bundle on X such that $F|_{U_1}$ and $F|_{U_2}$ are trivial. Let $\mathcal{F}$ be the sheaf of sections of F.

1. Is F a trivial vector bundle?
2. Compute the first Čech cohomology group of $\mathcal{F}$ for the covering $X = U_1 \cup U_2$.

3. Given an example of X, F and a covering $X = U_1 \cup U_2$ such that the first Čech cohomology group of $\mathcal{F}$ for this covering is non-zero.

4. Is F a trivial vector bundle if $U_1 \cap U_2$ is connected?