MAGIT Exercises, Series 1

Exercise 1.

Show that natural injection $\mathbb{Q} \to \mathbb{R}$ is an epimorphism in the category of topological Hausdorff spaces. Is it an epimorphism in the category of all topological spaces?

Exercise 2.

Let $F : Set \to Gr$ be the functor sending X to the free group generated by X and let $G : Gr \to Set$ be the forgetful functor. Show that for every set X and group H we have an isomorphism

$$Mor_{Gr}(F(X), H) \simeq Mor_{Set}(X, G(H)),$$

which is natural in both X and H (and make sense of this statement!).

Exercise 3.

Give a proof of Yoneda's lemma.

Exercise 4.

Let $\hat{\mathcal{C}}$ be the category of functors from $\mathcal{C}^{op}$ to Set. Show that $F : \mathcal{C}^{op} \to Set$ is representable by $A \in Ob\,\mathcal{C}$ if and only if for every $B \in Ob\,\mathcal{C}$ we have a natural isomorphism

$$Mor_{\hat{\mathcal{C}}}(h_B, F) \simeq Mor_{\mathcal{C}}(B, A).$$

Exercise 5.

Let $\hat{\mathcal{C}}$ be the category of functors from $\mathcal{C}^{op}$ to Set. We say that $F : \mathcal{C}^{op} \to Set$ is *corepresentable* by $A \in Ob\,\mathcal{C}$ if and only if for every $B \in Ob\,\mathcal{C}$ we have a natural isomorphism

$$Mor_{\hat{\mathcal{C}}}(F, h_B) \simeq Mor_{\mathcal{C}}(A, B).$$

Show that if F is representable then it is corepresentable and give an example of a functor that is corepresentable but which is NOT representable.