§1. Introduction.

For a connected space $M$, let $F(M,d)$ be the space of ordered configurations of $d$ distinct points in $M$, which is defined by

$$F(M,d) = \{(x_1, \cdots, x_d) \in M^d : x_i \neq x_j \text{ if } i \neq j \}.$$ 

Let $\Sigma_d$ be the symmetric group of $d$ letters $\{1, 2, \cdots, d\}$. $\Sigma_d$ acts on $F(M,d)$ freely in the usual manner. The orbit space

$$C_d(M) = F(M,d)/\Sigma_d$$

is called the space of configurations of $d$ distinct points in $M$. In this paper we shall assume that $M$ is an open manifold, i.e. each component is non-compact and without boundary. Adding a point near one of the ends of $M$ gives (up to homotopy) a stabilization map

$$j_d : C_d(M) \rightarrow C_{d+1}(M).$$

The following is well-known:

**Theorem 0 ([Se]).**

If $M$ is an open manifold, then the stabilization map $j_d : C_d(M) \rightarrow C_{d+1}(M)$ is a homology equivalence up to dimension $[d/2]$. □

(We shall call a map $f : X \rightarrow Y$ a homology equivalence up to dimension $m$ if the induced homomorphism

$$f_* : H_i(X,\mathbb{Z}) \rightarrow H_i(Y,\mathbb{Z})$$

is bijective when $i < m$ and surjective when $i = m$.)

**Remarks.** Various special cases of this result were known earlier. For example:

1. Let $M = \mathbb{R}^q$ ($q > 2$). Then $\lim_{q \to \infty} C_d(\mathbb{R}^q) = K(\Sigma_d, 1)$. The homology stabilization of the maps $K(\Sigma_d, 1) \rightarrow K(\Sigma_{d+1}, 1)$ follows from work of Nakaoka ([Na]). This also follows from Theorem 0.

2. Let $M = \mathbb{R}^2$. Then $C_d(M) = K(\text{Br}_d, 1)$. The statement of Theorem 0 in this case was proved by Arnold ([A]).
Let \( \tilde{C}_d(M) = F(M,d)/A_d \), where \( A_d \subset \Sigma_d \) is the alternating group of \( d \) letters \( \{1, \cdots , d\} \). We shall call \( \tilde{C}_d(M) \) the space of oriented configurations of \( d \) distinct points in \( M \). There is a non-trivial double covering \( \tilde{C}_d(M) \to C_d(M) \). Adding a point near an end of \( M \) gives a stabilization map

\[
\tilde{j}_d : \tilde{C}_d(M) \to \tilde{C}_{d+1}(M).
\]

In this note we shall determine the homological stability dimension for the spaces \( \tilde{C}_d(M) \), when \( M \) is obtained from a compact Riemann surface by removing a finite number of points.

More precisely, we shall prove:

**Theorem 1.** Let \( M \) be a compact Riemann surface, and let

\[
M' = M \setminus \{n \text{ points}\}
\]

where \( n \geq 1 \). Then the stabilization map

\[
\tilde{j}_d : \tilde{C}_d(M') \to \tilde{C}_{d+1}(M')
\]

is a homology equivalence up to dimension \( [(d - 1)/3] \). Moreover, this bound is the best possible.

We shall give a proof in the next section, based on the calculations of [BCT] and [BCM]. First we make some remarks and pose a question:

**Remarks.** (1) It seems somewhat surprising that the answer is (about) \( d/3 \), not \( d/2 \) as in the un-oriented case.

(2) An analogous argument proves a similar result for McDuff’s configuration space \( C^n_\pm(M) \) of “positive and negative particles” ([McD]). An application of this will be given in [GKY].

**Question.** Is Theorem 1 true for any open manifold?

§2. PROOF OF THEOREM 1.

Without loss of generality we shall from now on assume that

\[
M' = C - \{l \text{ points}\}
\]

and write \( C_n \) for \( C_n(M') \) and \( \tilde{C}_n \) for \( \tilde{C}_n(M') \). We shall only consider the case \( l \geq 1 \). The case \( l = 0 \) can be dealt with in a similar way.

We shall show that

\[
(*) \quad H_q(\tilde{C}_d, F) \to H_q(\tilde{C}_{d+1}, F)
\]
is bijective for $q < n(d)$ and surjective for $q = n(d)$ if $F = \mathbb{Z}/p$ ($p$ is any prime) or $F = \mathbb{Q}$, where

$$n(d) = \begin{cases} \frac{d}{2} & \text{if } F \neq \mathbb{Z}/3 \\ \frac{(d-1)}{3} & \text{if } F = \mathbb{Z}/3 \end{cases}$$

Theorem 1 follows from this and the universal coefficient theorem. (The case $F = \mathbb{Z}/2$ is trivial. Indeed, since $\tilde{C}_d \to C_d$ is a double covering and the stabilization map $C_d \to C_{d+1}$ is a homology equivalence up to dimension $[d/2]$, the result follows from the Gysin exact sequence.)

We shall make use of the following well known fact (cf. [B]):

**Lemma 2.** Let $G$ be a group and $H \subset G$ a subgroup of $G$ of index 2. Let $F$ be any field of characteristic not equal to 2. Then there is a natural additive isomorphism

$$H_q(H, F) \cong H_q(G, F) \oplus H_q(G, F(-1))$$

for any $q \geq 1$, where $F(-1)$ denotes the field $F$ with the $G$-module structure given by

$$g \cdot f = \begin{cases} -f & g \notin H \\ f & g \in H \end{cases}$$

for $f \in F$ and $g \in G$. □

Let us take $G = \pi_1(C_d)$ and $H = \pi_1(\tilde{C}_d)$. Since $\tilde{C}_d \to C_d$ is a double covering, $H$ can be identified with a subgroup of $G$ of index 2. We have $\tilde{C}_d \cong K(H, 1)$, $C_d \cong K(G, 1)$ and we can identify the covering map with the map $K(H, 1) \to K(G, 1)$ induced by the inclusion $H \subset G$. We can thus apply Lemma 2 to obtain:

**Lemma 3.** If $F = \mathbb{Z}/p$ ($p$ any odd prime) or $F = \mathbb{Q}$, then there is a natural additive isomorphism

$$H_q(\tilde{C}_d, F) \cong H_q(C_d, F) \oplus H_q(C_d, F(-1))$$

for any $q \geq 1$ □

Now, since $C_d \to C_{d+1}$ is a homology equivalence up to dimension $[d/2]$, Theorem 1 follows directly from the following result:

**Lemma 4.** Let $q$ and $d$ be positive integers such that $1 \leq q \leq [d/2]$ and $(q, d) \neq (1, 2)$.

1. If $F = \mathbb{Z}/p$ ($p$ prime, $p \geq 7$) or $F = \mathbb{Q}$, then
   $$H_q(C_d, F(-1)) = 0$$

2. If $F = \mathbb{Z}/5$ and $(q, d) \neq (3, 6)$, then
   $$H_q(C_d, \mathbb{Z}/5(-1)) = 0$$

3. If $F = \mathbb{Z}/3$ and $d \geq 3q + 2$, then
   $$H_q(C_d, \mathbb{Z}/3(-1)) = 0$$
Proof. Let 1 \leq q \leq \lfloor d/2 \rfloor.

By (8.4) of [BCM], if \( n \) is sufficiently large, then

\[
H_q(C_d, \mathbf{F}(-1)) \cong H_{q+(2n+1)d}(\Omega^2 S^{2n+3} \times (\Omega S^{2n+3})^l, \mathbf{F})
\]

Note that

\[
H_j((\Omega S^{2n+3})^l, \mathbf{F}) \cong \left\{ \begin{array}{ll} F^{m(\beta)} & \text{if } j = (2n+2)\beta, \beta \geq 0 \\ 0 & \text{otherwise} \end{array} \right.
\]

and there is a stable splitting ([CMM], [S])

\[
\Omega^2 S^{2n+3} \cong \bigvee \Sigma \geq 1 \Sigma^{2n} D_\alpha
\]

where we take

\[
m(\beta) = \left( \frac{\beta + l - 1}{l - 1} \right) \quad \text{and} \quad D_\alpha = F(C, \alpha) \land \Sigma_\alpha (\land^{\alpha} S^1).
\]

Since \( D_\alpha \) has the homotopy type of a CW complex of dimension \( 2\alpha - 1 \), \( H_j(D_\alpha, \mathbb{Z}/p) = 0 \) for any \( j \geq 2\alpha \).

Applying the K"unneth formula one can show that

\((**)
\[
H_q(C_d, \mathbf{F}(-1)) \cong \oplus_{\alpha=1}^d \check{H}_{q+2\alpha-d}(D_\alpha, \mathbf{F})^{\times(\beta - \alpha)}
\]

From now on we shall only consider the case \( \mathbf{F} = \mathbb{Z}/p \) (\( p \) an odd prime). The case \( \mathbf{F} = \mathbb{Q} \) can be dealt with analogously.

The following is well known:

**Lemma 5.** Let \( p \geq 3 \) be any odd prime.

1. There is a multiplicative isomorphism

\[
(a) \quad H_\ast(\Omega^2 S^3, \mathbb{Z}/p) = \mathbb{Z}/p[x_1, x_2, \cdots] \otimes E[y_0, y_1, y_2, \cdots]
\]

where \( \deg(x_i) = 2p^i - 2 \) and \( \deg(y_i) = 2p^i - 1 \).

2. There is an additive isomorphism

\[
(b) \quad \check{H}_\ast(D_\alpha, \mathbb{Z}/p) = \oplus_{J=(\epsilon_0, m_1, \epsilon_1, \cdots) \in J} \mathbb{Z}/p\{ \prod_{j \geq 1} x_j^{m_j} \cdot \prod_{j \geq 0} y_j^{\epsilon_j} \}
\]

where we take:

\[
J = \{ J = (\epsilon_0, m_1, \epsilon_1, \cdots) : \epsilon_j \in \{ 0, 1 \}, m_j \geq 0, w(J) = \alpha \}
\]

and

\[
w(J) = \epsilon_0 + \sum_{j \geq 1} p^j (m_j + \epsilon_j).
\]

□
¿From Lemma 5

(c) \[ \dim \mathbf{Z}_p \tilde{H}_{q+2\alpha-d}(D_\alpha, \mathbf{Z}/p) = \text{card}(\mathcal{F}) \]

where

\[ \mathcal{F} = \{ J = (\epsilon_0, m_1, \epsilon_1, \cdots) \neq (0, 0, \cdots) : \epsilon_j \in \{0, 1\}, m_j \geq 0, D(J) = q + 2\alpha - d, w(J) = \alpha \} \]

and

\[ D(J) = \epsilon_0 + \sum_{j \geq 1} \{ 2(p^j - 1)m_j + (2p^j - 1)\epsilon_j \}. \]

Here \( \text{card}(S) \) denotes the cardinality of a finite set \( S \).

Note that for \( J = (\epsilon_0, m_1, \epsilon_1, \cdots) \), if \( w(J) = \alpha \), then

\[ D(J) = q + 2\alpha - d \Leftrightarrow H(J) = \epsilon_0 + \sum_{j \geq 1} (2m_j + \epsilon_j) = d - q \]

Hence

(d) \( \mathcal{F} = \{ J = (\epsilon_0, m_1, \epsilon_1, \cdots) \neq (0, 0, \cdots) : \epsilon_j \in \{0, 1\}, m_j \geq 0, w(J) = \alpha, H(J) = d-q \} \).

By (c) and (d) it suffices to show:

**Claim.** Let \( 1 \leq q \leq \lfloor d/2 \rfloor, 1 \leq \alpha \leq d \) and \( (q, d) \neq (1, 2) \).

(1) If \( p \geq 7 \) is an odd prime or \( p = 5 \) and \( (q, d) \neq (3, 6) \), then \( \mathcal{F} = \emptyset \).

(2) If \( p = 3 \) and \( d \geq 3q + 2 \), \( \mathcal{F} = \emptyset \).

**Proof of Claim.** (1) Assume that \( p \geq 5 \) is a prime and \( J = (\epsilon_0, m_1, \epsilon_1, \cdots) \in \mathcal{F} \).

Since \( 1 \leq q \leq \lfloor d/2 \rfloor \leq d/2 \),

\[ \epsilon_0 + \sum_{j \geq 1} (2m_j + \epsilon_j) = H(J) = d - q \geq d/2 \geq \alpha/2 = \{ \epsilon_0 + \sum_{j \geq 1} p^j(m_j + \epsilon_j) \}/2. \]

Hence

(e) \[ \epsilon_0 + \sum_{j \geq 1} \{(4 - p^j)m_j + (2 - p^j)\epsilon_j \} \geq 0 \]

Since \( J \neq (0, 0, \cdots) \), one can deduce from (e) that

\[ J = (\epsilon_0, m_1, \epsilon_1, m_2, \epsilon_2, \cdots) = \begin{cases} (1, 0, 0, 0, 0, 0, \cdots) & \text{if } p \geq 7 \\ (1, 0, 0, 0, 0, 0, 0, \cdots) \text{ or } (1, 1, 0, 0, 0, 0, 0, \cdots) & \text{if } p = 5 \end{cases} \]

Hence

\[ (q, d) = \begin{cases} (1, 2) & p \geq 7 \\ (1, 2), (3, 6) & p = 5 \end{cases} \]
This is a contradiction.

(2) Assume \( d \geq 3q + 2 \) and \( p = 3 \). Then

\[
\alpha - d = w(J) - (q + H(J)) \\
= \{\epsilon_0 + \sum_{j \geq 1} 3^j (m_j + \epsilon_j)\} - \{\epsilon_0 + \sum_{j \geq 1} (2m_j + \epsilon_j)\} - q \\
= \sum_{j \geq 1} \{(3^j - 2)m_j + (3^j - 1)\epsilon_j\} - q \\
\geq \frac{1}{2} \sum_{j \geq 1} (2m_j + \epsilon_j) - q \\
= \frac{1}{2} (d - q - \epsilon_0) - q \quad \text{(by } H(J) = d - q) \\
= \frac{1}{2} (d - 3q - \epsilon_0) \\
\geq \frac{1}{2} \{(3q + 2) - 3q - 1\} = \frac{1}{2} > 0
\]

Hence \( \alpha = w(J) > d \), which is a contradiction. \( \square \)

This completes the proof of Theorem 2. \( \square \)
References


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