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Computer Aided Geometric Design 22 (2005) 632–658
COMPUTER AIDED GEOMETRIC DESIGN
www.elsevier.com/locate/cagd

Curvature formulas for implicit curves and surfaces

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Available online 21 July 2005

Abstract

Curvature formulas for implicit curves and surfaces are derived from the classical curvature formulas in Differential Geometry for parametric curves and surfaces. These closed formulas include curvature for implicit planar curves, curvature and torsion for implicit space curves, and mean and Gaussian curvature for implicit surfaces. Some extensions of these curvature formulas to higher dimensions are also provided.

Keywords: Curvature; Torsion; Gaussian curvature; Mean curvature; Gradient; Hessian; Implicit curves; Implicit surface

1. Introduction

Curvature formulas for parametrically defined curves and surfaces are well-known both in the classical literature on Differential Geometry (Spivak, 1975; Stoker, 1969; Struik, 1950) and in the contemporary literature on Geometric Modeling (Farin, 2002; Hoschek and Lasser, 1993).

Curvature formulas for implicitly defined curves and surfaces are more scattered and harder to locate. In the classical geometry literature, a curvature formula for implicit planar curves is presented in (Fulton, 1974); an algorithm, but no explicit formulas, for finding the curvature and torsion of implicitly defined space curves is provided in (Willmore, 1959). Mean and Gaussian curvature formulas for implicit surfaces can be found in (Spivak, 1975, vol. 3), but, somewhat surprisingly, almost nowhere else in Differential Geometry texts in the English language. German geometry papers and texts with curvature formulas for implicit surfaces seem to be more common (Dombrowski, 1968; Gromoll et al., 1975; Knoblauch, 1888, 1913), but these references remain largely inaccessible to most English speaking researchers in Geometric Modeling.

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0167-8396/$ – see front matter
doi:10.1016/j.cagd.2005.06.005

Nevertheless, curvature formulas for implicitly defined curves and surfaces are important in Geometric Modeling applications, and many of these formulas do appear scattered throughout the Geometric Modeling literature. Curvature formulas for implicit curves and surfaces in normal form appear in (Hartmenn, 1999). A curvature formula for arbitrary implicit planar curves appears in (Bajaj and Kim, 1991; Blinn, 1997); mean and Gaussian curvature formulas for arbitrary implicitly defined surfaces are furnished by (Belyaev et al., 1998; Turkiyyah et al., 1997). To derive these curvature formulas for implicit surfaces, (Belyaev et al., 1998) refer to (Turkiyyah et al., 1997) who in turn refer to (Spivak, 1975, vol. 3). Procedures for finding curvature and torsion formulas for implicit space curves as well as mean and Gaussian curvature formulas for implicit surfaces are given in (Patrikalakis and Maekawa, 2002), but explicit closed formulas are not provided.

Curvature formulas for implicit curves and surfaces also appear in some recent texts on level set methods (Osher and Fedkiw, 2003; Sethian, 1999). A curvature formula for implicit planar curves is presented in both of these texts; (Osher and Fedkiw, 2003) also contains explicit formulas for the mean and Gaussian curvature of implicit surfaces.

The purpose of this paper is to provide a service to the Geometric Modeling community by collecting in one easily accessible place curvature formulas for implicitly defined curves and surfaces. In order to better understand the relationships between curvature formulas in different dimensions and different co-dimensions, we shall develop, in each case, not just one closed formula, but several equivalent expressions. We shall also provide a bridge between the parametric and implicit formulations by deriving these curvature formulas for implicit curves and surfaces from the more commonly known curvature formulas for parametric curves and surfaces.

In Section 2, we review the classical curvature formulas for parametric curves and surfaces. We use these formulas in Section 3 to derive curvature formulas for implicit planar curves and in Section 4 to derive mean and Gaussian curvature formulas for implicit surfaces. Section 5 is devoted to deriving curvature and torsion formulas for implicit space curves. In Section 6 we collect all our curvature formulas for implicit curves and surfaces together in one easily accessible location. Readers interested only in the formulas, but not their derivations, can skip directly to Section 6. We close in Section 7 with a few open questions for future research.

2. Curvature formulas for parametric curves and surfaces

For planar curves, curvature has several equivalent definitions:

(i) amount of deviation of the curve from the tangent line;
(ii) rate of change of the tangent direction;
(iii) reciprocal of the radius of the osculating circle;
(iv) element of area of circular image/element of arclength.

For surfaces, curvature is more complicated. In analogy with curves, curvature for surfaces should capture the deviation of the surface from the tangent plane. But, unlike planar curves, there is more than one way to measure this deviation. The shape of the osculating paraboloid gives a rough measure of how the surface deviates locally from the tangent plane. More precise information is provided by the mean and Gaussian curvatures.

Gaussian curvature

(i) product of the principal curvatures;
(ii) element of area of spherical image/element of surface area.

Mean curvature

(iii) average of the principal curvatures;
(iv) rate of change of surface area under small deformations in the normal direction.

From these first principles, explicit curvature formulas can be derived for parametric curves and surfaces. We shall review these formulas in the following two subsections.

2.1. Curvature formulas for parametric curves

Consider a parametric curve $P(s)$ in 3-dimensions parametrized by arc length. Let $P(t)$ be any other parametrization of P, and let P', P'', and P''' denote the first, second, and third derivatives of P with respect to t. The unit tangent vector of P is given by

$$T = \frac{\mathrm{d}P}{\mathrm{d}s} = \frac{P'}{|P'|}.$$

Therefore, in 3-dimensions, the curvature

$$k = \left|\frac{\mathrm{d}T}{\mathrm{d}s}\right| = \frac{|P' \times P''|}{|P'|}.$$

In 2-dimensions, this curvature formula reduces to

$$k = \left|\frac{\mathrm{d}T}{\mathrm{d}s}\right| = \frac{|\operatorname{Det}(P'\ P'')|}{|P'|^3}.$$

Torsion measures deviation from the osculating plane. We shall see shortly that the torsion

$$\tau = \frac{\operatorname{Det}\left(T\ \frac{\mathrm{d}T}{\mathrm{d}s}\ \frac{\mathrm{d}^2T}{\mathrm{d}s^2}\right)}{k^2} = \frac{\operatorname{Det}(P'\ P''\ P''')}{|P' \times P''|^2}.$$

The Frenet equations (see below) express the derivatives of the tangent (T), normal (N), and binormal $(T \times N)$ in terms of the tangent, normal, binormal, curvature and torsion. We can use the Frenet equations to generate explicit formulas for the curvature and torsion in terms of the tangent and normal vectors and their derivatives (see Table 1). We shall make use of these explicit formulas in Sections 3 and 5, when we develop closed formulas for the curvature and torsion of implicit curves.

2.2. Curvature formulas for parametric surfaces

Let $P(s,t)$ be a parametric surface, and let P_s and P_t denote the partial derivatives of P with respect to s and t. (Higher order derivatives will be denoted in the usual way by repeating s and t the appropriate

Table 1
Explicit formulas for the curvature and torsion in terms of the tangent and normal vectors and their derivatives. These formulas follow easily from the Frenet equations

Frenet equations (Stoker, 1969)	Curvature formulas	Torsion formulas
$\frac{dT}{ds} = kN$	$k = \frac{dT}{ds} \cdot N$	$\tau = \frac{dN}{ds} \cdot (T \times N)$
$\frac{dN}{ds} = -kT + \tau(T \times N)$	$k = -\frac{dN}{ds} \cdot T$	$\tau = -\frac{d(T \times N)}{ds} \cdot N$
$\frac{d(T \times N)}{ds} = -\tau N$	$k = \lvert \frac{dT}{ds} \times T \rvert$	$\tau = \mathrm{Det}\big(T \ N \ \frac{dN}{ds}\big)$
	$k = \lvert \frac{dN}{ds} \times N \rvert$ if $\tau = 0$	$\tau = \frac{\mathrm{Det}\big(T \ \frac{dT}{ds} \ \frac{d^2T}{ds^2}\big)}{k^2}$

number of times.) The normal vector to the surface is perpendicular to the tangent vectors P_s and P_t. Therefore, the unit normal is given by

$$N = N(s,t) = \frac{P_s \times P_t}{|P_s \times P_t|}.$$

Mean and Gaussian curvature for a parametric surface are usually defined in terms of the first and second fundamental forms of the surface. These forms are given by the following matrices (Stoker, 1969):

First fundamental form:

$$I = \begin{pmatrix} P_s \bullet P_s & P_s \bullet P_t \\ P_t \bullet P_s & P_t \bullet P_t \end{pmatrix};$$

Second fundamental form:

$$II = \begin{pmatrix} P_{ss} \bullet N & P_{st} \bullet N \\ P_{ts} \bullet N & P_{tt} \bullet N \end{pmatrix} = -\begin{pmatrix} P_s \bullet N_s & P_s \bullet N_t \\ P_t \bullet N_s & P_t \bullet N_t \end{pmatrix}.$$

Notice that in our two matrices for the second fundamental form $P_{ss} \bullet N = -P_s \bullet N_s$ because $P_s \bullet N = 0$. Differentiating $P_s \bullet N = 0$ with respect to s yields $P_{ss} \bullet N + P_s \bullet N_s = 0$. The other equalities in these matrices for the second fundamental form can be established in a similar manner.

In terms of the first and second fundamental forms, the mean and Gaussian curvatures are computed by the expressions given below. In the formula for the mean curvature, II^* denotes the adjoint of II—that is,

$$II^* = \begin{pmatrix} P_{tt} \bullet N & -P_{ts} \bullet N \\ -P_{st} \bullet N & P_{ss} \bullet N \end{pmatrix} = \begin{pmatrix} -P_t \bullet N_t & P_t \bullet N_s \\ P_s \bullet N_t & -P_s \bullet N_s \end{pmatrix}.$$

Gaussian curvature:

$$K_G = \frac{\mathrm{Det}(II)}{\mathrm{Det}(I)}.$$

Mean curvature:

$$K_M = \frac{\mathrm{Trace}(I * II^*)}{2\,\mathrm{Det}(I)}.$$

Though these curvature formulas are the classical formulas found in most standard books on Differential Geometry (Stoker, 1969), we need to massage these formulas slightly in order to use them effectively when we study implicit surfaces.

Lemma 2.1.

(i) $\mathrm{Det}(I) = |P_s \times P_t|^2$.
(ii) $\mathrm{Det}(II) = (P_s \times P_t) \bullet (N_s \times N_t)$.

Proof. (i) This result follows by expanding the determinant and invoking the vector identity:

$$(a \bullet a)(b \bullet b) - (a \bullet b)^2 = |a \times b|^2.$$

(ii) This result follows by expanding the determinant and invoking the vector identity:

$$(a \bullet c)(b \bullet d) - (a \bullet d)(b \bullet c) = (a \times b) \bullet (c \times d). \qquad \square$$

Corollary 2.2.

$$K_G = \frac{(P_s \times P_t) \bullet (N_s \times N_t)}{|P_s \times P_t|^2}.$$

Lemma 2.3. $\mathrm{Trace}(I * II^*) = (P_s \times P_t) \bullet (P_t \times N_s) - (P_s \times P_t) \bullet (P_s \times N_t)$.

Proof. This result follows by computing the trace and then twice invoking the vector identity:

$$(a \bullet c)(b \bullet d) - (a \bullet d)(b \bullet c) = (a \times b) \bullet (c \times d). \qquad \square$$

Corollary 2.4.

$$K_M = \frac{(P_s \times P_t) \bullet ((P_t \times N_s) - (P_s \times N_t))}{2|P_s \times P_t|^2}.$$

3. Curvature formulas for implicit planar curves

Curvature is a second order effect—only the first and second derivatives appear in the curvature formula for parametric curves. Therefore, for implicit planar curves $F(x, y) = 0$, the curvature should depend only on the gradient ∇F and the hessian $H(F)$. We shall adopt the following notation:

$$\nabla F = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{pmatrix} = (F_x \; F_y),$$

$$H(F) = \begin{pmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{pmatrix} = \begin{pmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{pmatrix} = \nabla(\nabla F).$$

Here ∇ applied to a row vector means take the gradient of each component and write these component gradients in a matrix as consecutive column vectors.

Since the gradient of $F(x, y)$ is perpendicular to the level curves $F(x, y) = c$, the gradient ∇F is parallel to the normal of $F(x, y) = 0$. Therefore we have the following formulas:

Planar implicit curves:

(1) Implicit curve: $F(x, y) = 0$.

(2) Normal: $\nabla F = (F_x, F_y)$.
(3) Unit normal:

$$N(F) = \frac{\nabla F}{|\nabla F|} = \frac{(F_x, F_y)}{\sqrt{F_x^2 + F_y^2}}.$$

(4) Tangent: $\mathrm{Tan}(F) = k \times \nabla F = (-F_y, F_x)$.
(5) Unit tangent:

$$T(F) = \frac{\mathrm{Tan}(F)}{|\mathrm{Tan}(F)|} = \frac{(-F_y, F_x)}{\sqrt{F_x^2 + F_y^2}}.$$

Proposition 3.1 (Curvature formula for implicit planar curves).

$$k = -\frac{T(F) * H(F) * T(F)^{\mathrm{T}}}{|\nabla F|} = -\frac{(-F_y\ \ F_x) * \left(\begin{smallmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{smallmatrix}\right) * \left(\begin{smallmatrix} -F_y \\ F_x \end{smallmatrix}\right)}{(F_x^2 + F_y^2)^{3/2}}. \tag{3.1}$$

Proof. From the Frenet equations,

$$k = -\frac{\mathrm{d}N}{\mathrm{d}s} \bullet T.$$

Hence by the chain rule,

$$k = -\left(\frac{\partial N}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial N}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s}\right) \bullet T = -T * \nabla N * T^{\mathrm{T}}.$$

But by the quotient rule,

$$\nabla N = \nabla\left(\frac{\nabla F}{|\nabla F|}\right) = \frac{|\nabla F|\nabla(\nabla F) - \nabla(|\nabla F|)^{\mathrm{T}} * \nabla F}{|\nabla F|^2}.$$

Moreover

$$\nabla(\nabla F) = H(F),$$
$$\nabla F * T^{\mathrm{T}} = \nabla F \bullet T = 0.$$

Therefore we conclude that

$$k = -\frac{T(F) * H(F) * T(F)^{\mathrm{T}}}{|\nabla F|}. \qquad \square$$

A word about invariance is in order here. The curve $F(x, y) = 0$ is identical to the curve $cF(x, y) = 0$ for any constant $c \neq 0$. Therefore we would expect that the curvature of $F(x, y) = 0$ should be the same as the curvature of $cF(x, y) = 0$. If $c > 0$, then replacing F by cF on the right hand side of Eq. (3.1) introduces a factor of c^3 in both the numerator and the denominator, so the curvature k is unchanged. However, replacing F by $-F$ on the right hand side of Eq. (3.1) changes the sign of the numerator, but not the sign of the denominator, thus changing the sign of k. Replacing F by $-F$ also changes the direction of the unit normal $N(F) = \nabla F/|\nabla F|$. Therefore the curvature vector,

$$\vec{\mathbf{k}} = kN(F) = k\nabla F/|\nabla F|$$

is invariant, but the sign of the scalar curvature k depends on the choice of the direction of the unit normal, which, in turn, depends on the sign of F. This sign dependence shows up in the parametric setting as well. By the Frenet equations $\frac{dT}{ds} = kN$. The derivative $\frac{dT}{ds}$ is an invariant; the signs of k and N are not invariants, but rather are mutually dependent. Nevertheless, we shall continue to focus on expressions for the scalar curvature k rather than the curvature vector $\vec{\mathbf{k}}$, since it is expressions for the scalar curvature that we plan to extend to curvature formulas for implicit space curves and implicit surfaces.

Example 3.1 (*Circles*). We can check our curvature formula on the circles

$$F(x, y) \equiv x^2 + y^2 - R^2 = 0.$$

By Proposition 3.1 to compute the curvature, we need to compute

$$k = -\frac{(-F_y\ F_x) * \begin{pmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{pmatrix} * \begin{pmatrix} -F_y \\ F_x \end{pmatrix}}{(F_x^2 + F_y^2)^{3/2}}.$$

Substituting for the derivatives of F, we find that

$$k = -\frac{(-2y\ 2x) * \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} * \begin{pmatrix} -2y \\ 2x \end{pmatrix}}{((2x)^2 + (2y)^2)^{3/2}} = -\frac{8(x^2 + y^2)}{(4x^2 + 4y^2)^{3/2}} = -\frac{8R^2}{8R^3} = -\frac{1}{R}.$$

Notice that the curvature here is negative, but notice too that $\nabla F = (2x, 2y)$ is the outward pointing normal. Thus, as one would expect, the curvature vector $\vec{\mathbf{k}} = k\nabla F / |\nabla F|$ points into the circle and has magnitude $|k| = 1/R$.

Eq. (3.1) allows us to calculate the curvature of implicit planar curves, but it is difficult to see how this formula can be extended either to implicit space curves or to implicit surfaces. Implicit space curves are defined by the intersection of two implicit surfaces: $\{F_1(x, y, z) = 0\} \cap \{F_2(x, y, z) = 0\}$. Thus for implicit space curves we have two hessians to consider: $H(F_1)$ and $H(F_2)$. How then are we to replace the hessian $H(F)$ in Eq. (3.1)? For implicit surfaces $F(x, y, z) = 0$, we have analogues of the gradient and the hessian, but no analogue of the tangent vector $\mathrm{Tan}(F)$, so once again it is unclear how to generalize Eq. (3.1). To overcome these shortcomings, we shall seek alternative ways to package Eq. (3.1)—that is, we shall seek equivalent expressions for the curvature for implicit planar curves that can be extended either to implicit space curves or to implicit surfaces.

To find new expressions for the curvature, we can proceed in the following fashion. In Table 1, we presented four formulas for the curvature in terms of the unit tangent, the unit normal, and their derivatives with respect to arclength. Using these formulas and proceeding as in Proposition 3.1 applying the chain rule and the quotient rule, we arrive at the following results.

Alternative curvature formulas

Curvature formula—$k = -\frac{dN}{ds} \bullet T$

$$k = -\frac{\mathrm{Tan}(F) * H(F) * \mathrm{Tan}(F)^{\mathrm{T}}}{|\nabla F|^3} = -\frac{(-F_y\ F_x) * \begin{pmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{pmatrix} * \begin{pmatrix} -F_y \\ F_x \end{pmatrix}}{(F_x^2 + F_y^2)^{3/2}}. \tag{3.1}$$

Curvature formula—$k = \frac{\mathrm{d}T}{\mathrm{d}s} \bullet N$

$$k = \frac{\mathrm{Tan}(F) * \nabla(\mathrm{Tan}(F)) * \nabla F^{\mathrm{T}}}{|\nabla F|^3} = -\frac{(-F_y\ F_x) * \left(\begin{smallmatrix} -F_{xy} & F_{xx} \\ -F_{yy} & F_{xy} \end{smallmatrix}\right) * \left(\begin{smallmatrix} F_x \\ F_y \end{smallmatrix}\right)}{(F_x^2 + F_y^2)^{3/2}}. \tag{3.2}$$

Curvature formula—$k = |\frac{\mathrm{d}N}{\mathrm{d}s} \times N|$ $(\tau = 0)$

$$k = \frac{|(\mathrm{Tan}(F) * H(F)) \times \nabla F|}{|\nabla F|^3} = \frac{\left|\left((-F_y\ F_x) * \left(\begin{smallmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{smallmatrix}\right)\right) \times (F_x\ F_y)\right|}{(F_x^2 + F_y^2)^{3/2}}. \tag{3.3}$$

Curvature formula—$k = |\frac{\mathrm{d}T}{\mathrm{d}s} \times T|$

$$k = \frac{|(\mathrm{Tan}(F) * \nabla(\mathrm{Tan}(F))) \times \mathrm{Tan}(F)|}{|\mathrm{Tan}(F)|^3} = \frac{\left|\left((-F_y\ F_x) * \left(\begin{smallmatrix} -F_{xy} & F_{xx} \\ -F_{yy} & F_{xy} \end{smallmatrix}\right)\right) \times (-F_y\ F_x)\right|}{(F_x^2 + F_y^2)^{3/2}}. \tag{3.4}$$

Note that in the plane, the cross products that appear in Eqs. (3.3) and (3.4) are really scalars—the determinants of the two factors. Thus if we want the signed curvature, we should compute these determinants and ignore the absolute value.

We can easily check the validity of Eqs. (3.2)–(3.4) by expanding the right hand sides and observing that these expressions are each the same as the right hand side of Eq. (3.1). Eq. (3.4) is particularly interesting because the right hand side depends only on $\mathrm{Tan}(F)$. For implicit space curves, the tangent direction is known, since if the space curve is given by the intersection of two implicit surfaces $\{F_1(x, y, z) = 0\} \cap \{F_2(x, y, z) = 0\}$, then the tangent is parallel to $\nabla F_1 \times \nabla F_2$. Thus we expect that Eq. (3.4) for the curvature of implicit planar curves will readily extend to implicit space curves; we shall have more to say about this extension in Section 5.

What about curvature formulas for implicit surfaces? For implicit surfaces we want to consider both the mean and the Gaussian curvature. Therefore we need two different expressions for the curvature of an implicit planar curve that readily extend, but in different ways, to implicit surfaces. None of the expressions in Eqs. (3.1)–(3.4) will do, since these expressions all depend on $\mathrm{Tan}(F)$, and for implicit surfaces there is no analogue of $\mathrm{Tan}(F)$. For implicit surfaces, we need formulas that depend only on the gradient and the hessian. Therefore, we must take another approach.

One device for developing new expressions for the curvature is to exploit the adjoint operator (see Table 2).

Adopting the notation in Table 2, it is easy to verify the following identities:

Table 2
The adjoint operator $*$ for constants, 2-dimensional row and column vectors, and 2×2 matrices

Constants	k	$k^* = k$
Row vectors	$r = (r_1, r_2)$	$r^* = (-r_2, r_1)^{\mathrm{T}}$
Column vectors	$c = (c_1, c_2)^{\mathrm{T}}$	$c^* = -(-c_2, c_1)$
Matrices	$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$	$M^* = \begin{pmatrix} m_{22} & -m_{21} \\ -m_{12} & m_{11} \end{pmatrix}$

$$c^* * r^* = r * c,$$
$$(M * c)^* = c^* * M^*,$$
$$(r * M)^* = M^* * r^*.$$

Therefore for any 2-dimensional row and column vectors r, c and any 2×2 matrix M

$$r * M * c = (r * M * c)^* = c^* * M^* * r^*.$$

The adjoint $H^*(F)$ of the hessian is given by

$$H^*(F) = \begin{pmatrix} \frac{\partial^2 F}{\partial y^2} & -\frac{\partial^2 F}{\partial y \partial x} \\ -\frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial x^2} \end{pmatrix} = \begin{pmatrix} F_{yy} & -F_{yx} \\ -F_{xy} & F_{xx} \end{pmatrix} = -\nabla\big(\mathrm{Tan}(F)\big).$$

Moreover, by construction, the adjoint of the gradient is the tangent and the adjoint of the tangent is the negative of the gradient—that is,

$$\nabla^*(F) = \mathrm{Tan}(F)^{\mathrm{T}},$$
$$\mathrm{Tan}^*(F) = -\nabla(F)^{\mathrm{T}}.$$

Therefore, applying the adjoint operator to the curvature formula

$$k = -\frac{\mathrm{Tan}(F) * H(F) * \mathrm{Tan}(F)^{\mathrm{T}}}{|\nabla F|^3}$$

generates the following adjoint hessian formula for the curvature of an implicit planar curve:

Adjoint hessian formula

$$k = -\frac{\nabla F * H^*(F) * \nabla F^{\mathrm{T}}}{|\nabla F|^3} = -\frac{(F_x \ F_y) * \begin{pmatrix} F_{yy} & -F_{yx} \\ -F_{xy} & F_{xx} \end{pmatrix} * \begin{pmatrix} F_x \\ F_y \end{pmatrix}}{(F_x^2 + F_y^2)^{3/2}}. \tag{3.5}$$

Again it is easy to verify directly that the right hand side of Eq. (3.5) agrees with the right hand side of Eq. (3.1), so Eq. (3.5) is indeed yet another valid expression for the curvature of an implicit planar curve. Moreover, this expression in terms of the gradient and the adjoint of the hessian readily extends to implicit surfaces, so we can expect this adjoint hessian formula to represent the curvature of an implicit surface. We shall return to this topic again in Section 4.

The adjoint hessian formula may extend to one of the two curvatures—mean or Gaussian—for an implicit surface, but we need still another equivalent curvature expression if we hope to represent the other curvature for implicit surfaces. Also we might like to work directly with the hessian rather than with the adjoint of the hessian. Fortunately there is another equivalent formulation for the curvature of an implicit planar curve that uses only the gradient and the hessian.

To eliminate $H^*(F)$ from Eq. (3.5), simply observe that

$$H(F) + H^*(F) = \mathrm{Trace}\big(H(F)\big) I,$$

where I is the identity matrix. Now substituting for $H^*(F)$ in Eq. (3.5) leads to the following expression for the curvature:

Hessian formula

$$k = \frac{\nabla F * H(F) * \nabla F^{\mathrm{T}} - |\nabla F|^2\, \mathrm{Trace}(H(F))}{|\nabla F|^3}. \tag{3.6}$$

We close this section with two additional alternative expressions for the curvature of an implicit planar curve that will show up again when we study curvature for implicit surfaces. As usual the validity of these formulas can be verified by direct computation of the right hand sides.

Determinant computation

$$k = \frac{\mathrm{Det}\begin{pmatrix} H(F) & \nabla F^{\mathrm{T}} \\ \nabla F & 0 \end{pmatrix}}{|\nabla F|^3}. \tag{3.7}$$

Divergence formula

$$k = -\nabla \bullet N(F) = -\nabla \bullet \left(\frac{\nabla F}{|\nabla F|} \right). \tag{3.8}$$

The divergence of the unit normal is often taken as the definition of the curvature for an implicit planar curve. We have not used this definition here because we wanted to develop curvature formulas for implicit curves directly from known curvature formulas for parametric curves. Also, although of theoretical interest, this divergence formula is less practical as a computational tool than many of the other expressions for the curvature developed in this section.

4. Curvature formulas for implicit surfaces

We expect curvature formulas for implicit surfaces $F(x, y, z) = 0$, just like curvature formulas for implicit curves $F(x, y) = 0$, to depend only on the gradient ∇F, the hessian $H(F)$, and the adjoint of the hessian $H^*(F)$. As with curves, we shall adopt the following notation for surfaces:

$$\nabla F = \left(\tfrac{\partial F}{\partial x} \ \tfrac{\partial F}{\partial y} \ \tfrac{\partial F}{\partial z} \right) = (F_x \ F_y \ F_z),$$

$$H(F) = \begin{pmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial x \partial z} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} & \frac{\partial^2 F}{\partial y \partial z} \\ \frac{\partial^2 F}{\partial z \partial x} & \frac{\partial^2 F}{\partial z \partial y} & \frac{\partial^2 F}{\partial z^2} \end{pmatrix} = \begin{pmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{pmatrix} = \nabla(\nabla F),$$

$$H^*(F) = \begin{pmatrix} \mathrm{Cofactor}(F_{xx}) & \mathrm{Cofactor}(F_{xy}) & \mathrm{Cofactor}(F_{xz}) \\ \mathrm{Cofactor}(F_{yx}) & \mathrm{Cofactor}(F_{yy}) & \mathrm{Cofactor}(F_{yz}) \\ \mathrm{Cofactor}(F_{zx}) & \mathrm{Cofactor}(F_{zy}) & \mathrm{Cofactor}(F_{zz}) \end{pmatrix}$$

$$= \begin{pmatrix} F_{yy}F_{zz} - F_{yz}F_{zy} & F_{yz}F_{zx} - F_{yx}F_{zz} & F_{yx}F_{zy} - F_{yy}F_{zx} \\ F_{xz}F_{zy} - F_{xy}F_{zz} & F_{xx}F_{zz} - F_{xz}F_{zx} & F_{xy}F_{zx} - F_{xx}F_{zy} \\ F_{xy}F_{yz} - F_{xz}F_{yy} & F_{yx}F_{xz} - F_{xx}F_{yz} & F_{xx}F_{yy} - F_{xy}F_{yx} \end{pmatrix}.$$

Here again ∇ applied to a row vector means take the gradient of each component and write these component gradients in a matrix as consecutive column vectors. As with curves, the gradient ∇F is parallel to the normal of the surface $F(x, y, z) = 0$. Therefore, the unit normal is given by

$$N(F) = \frac{\nabla F}{|\nabla F|} = \frac{(F_x\ F_y\ F_z)}{\sqrt{F_x^2 + F_y^2 + F_z^2}}.$$

With this notation in hand, we are now ready to develop curvature formulas for implicit surfaces.

The following curvature formulas for implicit surfaces appear in (Spivak, 1975, vol. 3, p. 204) (Spivak gives only the expressions on the far right); see also (Knoblauch, 1913, pp. 89–94):

Gaussian curvature

$$K_G = \frac{\nabla F * H^*(F) * \nabla F^{\mathrm{T}}}{|\nabla F|^4} = -\frac{\begin{vmatrix} H(F) & \nabla F^{\mathrm{T}} \\ \nabla F & 0 \end{vmatrix}}{|\nabla F|^4}. \tag{4.1}$$

Mean curvature

$$K_M = \frac{\nabla F * H(F) * \nabla F^{\mathrm{T}} - |\nabla F|^2 \operatorname{Trace}(H)}{2|\nabla F|^3} = \frac{-\operatorname{coeff}(\lambda) \text{ in} \begin{vmatrix} H(F)-\lambda I & \nabla F^{\mathrm{T}} \\ \nabla F & 0 \end{vmatrix}}{2|\nabla F|^3}. \tag{4.2}$$

Notice that Eq. (4.1) for Gaussian curvature is an extension to surfaces of the adjoint hessian formula (Eq. (3.5)) for the curvature of implicit planar curves. The normalizing factor $|\nabla F|^4$ that appears in the denominator insures that $cF(x, y, z) = 0$ has the same curvature as $F(x, y, z) = 0$. In addition, Eq. (4.2) for the mean curvature is an extension to implicit surfaces of the hessian formula (Eq. (3.6)) for the curvature of implicit planar curves. The factor of two in the denominator occurs because the mean curvature is the average of the two principal curvatures.

The principal curvatures k_1, k_2 can be computed from the mean and Gaussian curvatures K_M, K_G by the standard formula

$$k_1, k_2 = K_M \pm \sqrt{K_M^2 - K_G}. \tag{4.3}$$

One can also verify that

$$k_1, k_2 = \frac{-\operatorname{roots}\left\{\begin{vmatrix} H(F)-\lambda I & \nabla F^{\mathrm{T}} \\ \nabla F & 0 \end{vmatrix}\right\}}{|\nabla F|} \tag{4.4}$$

by using Eqs. (4.1) and (4.2) to demonstrate that Eq. (4.4) is equivalent to Eq. (4.3).

We shall now derive Eqs. (4.1) and (4.2) for the Gaussian and mean curvatures of implicit surfaces from the corresponding curvature formulas for parametric surfaces.

Theorem 4.1.

$$K_G = \frac{\nabla F * H^*(F) * \nabla F}{|\nabla F|^4}.$$

Proof. We shall only consider regular points on the implicit surface—points where $\nabla F \neq 0$—since the Gaussian curvature in not defined at points on the surface that are not regular. By the implicit function theorem, if $\nabla F \neq 0$, then the surface has a local parametrization $P(s,t)$. Let

$$N = N(s,t) = \frac{P_s \times P_t}{|P_s \times P_t|};$$

then N is the unit normal to the surface. Therefore by Corollary 2.2

$$K_G = \frac{(P_s \times P_t) \bullet (N_s \times N_t)}{|P_s \times P_t|^2}.$$

We shall now show how to transform the right hand side of this equation into the right hand side of Eq. (4.1). Since the unit normal to the surface is also given by the formula

$$N = \frac{\nabla F}{|\nabla F|},$$

it follows by the chain rule that if $u = s, t$, then

$$N_u = \frac{P_u * H(F)}{|\nabla F|} + \text{term parallel to } \nabla F.$$

Hence

$$N_s \times N_t = \frac{(P_s * H(F)) \times (P_s * H(F))}{|\nabla F|^2} + \text{terms perpendicular to } \nabla F.$$

But for all 3-dimensional vectors a, b and all 3×3 matrices M,

$$(a * M) \times (b * M) = (a \times b) * M^*,$$

where M^* denotes the adjoint of M. Therefore

$$N_s \times N_t = \frac{(P_s \times P_t) * H^*(F)}{|\nabla F|^2} + \text{terms perpendicular to } \nabla F.$$

Now since ∇F and $P_s \times P_t$ are both parallel to the surface normal, there is a constant λ such that

$$P_s \times P_t = \lambda \nabla F.$$

Thus,

$$(P_s \times P_t) \bullet (N_s \times N_t) = \frac{\lambda^2 \nabla F * H^*(F) * \nabla F^{\mathrm{T}}}{|\nabla F|^2},$$

$$|P_s \times P_t|^2 = \lambda^2 |\nabla F|^2.$$

Therefore

$$K_G = \frac{(P_s \times P_t) \bullet (N_s \times N_t)}{|P_s \times P_t|^2} = \frac{\nabla F * H^*(F) * \nabla F^{\mathrm{T}}}{|\nabla F|^4}. \qquad \square$$

Corollary 4.2.

$$K_G = -\frac{\begin{vmatrix} H(F) & \nabla F^{\mathrm{T}} \\ \nabla F & 0 \end{vmatrix}}{|\nabla F|^4}.$$

Proof. This corollary follows by expanding the right-hand side and verifying that the result gives the same expression as the expansion of the right-hand side in the curvature formula in Theorem 4.1. □

Theorem 4.3.

$$K_M = \frac{\nabla F * H(F) * \nabla F^{\mathrm{T}} - |\nabla F|^2 \operatorname{Trace}(H)}{2|\nabla F|^3}.$$

Proof. Again we shall only consider regular points on the implicit surface—points where $\nabla F \neq 0$—since the mean curvature in not defined at points on the surface that are not regular. By the implicit function theorem, if $\nabla F \neq 0$, then the surface has a local parametrization $P(s,t)$. Let

$$N = N(s,t) = \frac{P_s \times P_t}{|P_s \times P_t|};$$

then N is the unit normal to the surface. Therefore by Corollary 2.4

$$K_M = \frac{(P_s \times P_t) \bullet ((P_t \times N_s) - (P_s \times N_t))}{2|P_s \times P_t|^2}.$$

We shall now show how to transform the right-hand side of this equation into the right-hand side of Eq. (4.2). Since the unit normal to the surface is also given by the formula

$$N = \frac{\nabla F}{|\nabla F|},$$

it follows by the chain rule exactly as in the proof of Theorem 4.1 that if $u = s, t$, then

$$N_u = \frac{P_u * H(F)}{|\nabla F|} + \text{term parallel to } \nabla F.$$

Hence

$$P_t \times N_s = \frac{P_t \times (P_s * H(F))}{|\nabla F|} + \text{terms perpendicular to } \nabla F,$$

$$P_s \times N_t = \frac{P_s \times (P_t * H(F))}{|\nabla F|} + \text{terms perpendicular to } \nabla F,$$

so

$$P_t \times N_s - P_s \times N_t = \frac{P_t \times (P_s * H(F)) - P_s \times (P_t * H(F))}{|\nabla F|} + \text{terms perpendicular to } \nabla F.$$

But for all 3-dimensional vectors a, b and all symmetric 3×3 matrices M,

$$b \times (a * M) - a \times (b * M) = (a \times b) * M - \operatorname{Trace}(M)(a \times b).$$

(Since this identity is not well known, and is perhaps even new, we verified this identity in *Mathematica* using symbolic computation.) Therefore

$$P_t \times N_s - P_s \times N_t = \frac{(P_s \times P_t) * H(F) - \operatorname{Trace}(H(F))(P_s \times P_t)}{|\nabla F|} + \text{terms perpendicular to } \nabla F.$$

Now since ∇F and $P_s \times P_t$ are both parallel to the surface normal, there is a constant λ such that

$$P_s \times P_t = \lambda \nabla F.$$

Thus

$$(P_s \times P_t) \bullet \big((P_t \times N_s) - (P_s \times N_t)\big) = \frac{\lambda^2 \nabla F * H(F) * \nabla F^{\mathrm{T}} - \lambda^2 \operatorname{Trace}(H(F))|\nabla F|^2}{|\nabla F|},$$

$$|P_s \times P_t|^2 = \lambda^2 |\nabla F|^2.$$

Therefore

$$K_M = \frac{(P_s \times P_t) \bullet ((P_s \times N_t) - (P_t \times N_s))}{2|P_s \times P_t|^2} = \frac{\nabla F * H(F) * \nabla F^{\mathrm{T}} - |\nabla F|^2 \operatorname{Trace}(H)}{2|\nabla F|^3}. \qquad \square$$

Corollary 4.4.

$$K_M = \frac{-\operatorname{coeff}(\lambda) \text{ in} \begin{vmatrix} H(F) - \lambda I & \nabla F^{\mathrm{T}} \\ \nabla F & 0 \end{vmatrix}}{2|\nabla F|^3}.$$

Proof. This corollary follows by expanding the right hand side and verifying that the result gives the same expression as the expansion of the right hand side in the curvature formula in Theorem 4.3. $\square$

Corollary 4.5 (Divergence formula for mean curvature).

$$K_M = -\nabla \bullet N(F) = -\nabla \bullet \left(\frac{\nabla F}{|\nabla F|}\right).$$

Proof. Again this corollary follows by expanding the right hand side and verifying that the result gives the same expression as the expansion of the right hand side in the curvature formula in Theorem 4.3. $\square$

The divergence of the unit normal is often taken as the definition of the mean curvature for implicit surfaces. This divergence formula mimics the corresponding divergence formula for the curvature of an implicit curve. We have not used this definition here because, just as in the curve case, we wanted to develop the mean curvature formula for implicit surfaces directly from the well-known formula for the mean curvature of parametric surfaces. Moreover, although of theoretical interest, this divergence formula is less practical as a computational tool than Eq. (4.2).

Before computing some examples, let us pause here for a moment and comment upon the invariance of these curvature formulas. The surface $F(x, y, z) = 0$ is identical to the surface $cF(x, y, z) = 0$ for any constant $c \neq 0$. Therefore, naively, we would expect that the curvature of $F(x, y, z) = 0$ should be the same as the curvature of $cF(x, y, z) = 0$. For implicit curves, we saw that this invariance does not quite hold if $c < 0$. What about curvature invariance for implicit surfaces?

For Gaussian curvature, this invariance does indeed hold. If $c \neq 0$, then replacing F by cF on the right hand side of Eq. (4.1) introduces a factor of c^4 in both the numerator and the denominator, so the Gaussian curvature K_G is unchanged.

The mean curvature, however, behaves more like the curvature of implicit planar curves. If $c > 0$, then replacing F by cF on the right hand side of Eq. (4.2) introduces a factor of c^3 in both the numerator and the denominator, so the mean curvature K_M is unchanged. However, replacing F by $-F$ on the right hand side of Eq. (4.2) changes the sign of the numerator, but not the sign of the denominator, thus changing the sign of K_M. Of course, replacing F by $-F$ also changes the direction of the unit normal $N(F) = \nabla F/|\nabla F|$. Therefore the mean curvature vector,

$$\vec{\mathbf{K}}_M = K_M N(F) = K_M \nabla F/|\nabla F|$$

is invariant, but the sign of the mean curvature K_M depends on the choice of the direction of the unit normal, which, in turn, depends on the sign of F. This sign dependence is exactly the same sign dependence we observed in the curvature formula for implicit planar curves. In this way the mean curvature, more than the Gaussian curvature, resembles the curvature of implicit planar curves. The divergence formula is yet another way that the mean curvature more closely mimics the behavior of the curvature of implicit planar curves.

Example 4.1 (*Spheres*). We can check our curvature formulas on the spheres

$$F(x, y, z) \equiv x^2 + y^2 + z^2 - R^2 = 0.$$

To compute the mean and Gaussian curvatures, we need

$$\nabla F = (F_x \ F_y \ F_z) = (2x \ 2y \ 2z),$$

$$H(F) = \begin{pmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

$$H^*(F) = \begin{pmatrix} \mathrm{Cofactor}(F_{xx}) & \mathrm{Cofactor}(F_{xy}) & \mathrm{Cofactor}(F_{xz}) \\ \mathrm{Cofactor}(F_{yx}) & \mathrm{Cofactor}(F_{yy}) & \mathrm{Cofactor}(F_{yz}) \\ \mathrm{Cofactor}(F_{zx}) & \mathrm{Cofactor}(F_{zy}) & \mathrm{Cofactor}(F_{zz}) \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

$$\mathrm{Trace}\big(H(F)\big) = 6.$$

Gaussian curvature:

$$K_G = \frac{\nabla F * H^*(F) * \nabla F^{\mathrm{T}}}{|\nabla F|^4},$$

$$K_G = \frac{(2x \ 2y \ 2z)\left(\begin{smallmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2x \\ 2y \\ 2z \end{smallmatrix}\right)}{(4x^2 + 4y^2 + 4z^2)^2} = \frac{16(x^2 + y^2 + z^2)}{16(x^2 + y^2 + z^2)^2} = \frac{1}{R^2}.$$

Mean curvature:

$$K_M = \frac{\nabla F * H(F) * \nabla F^{\mathrm{T}} - |\nabla F|^2 \,\mathrm{Trace}(H)}{2|\nabla F|^3},$$

$$K_M = \frac{(2x \ 2y \ 2z)\left(\begin{smallmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2x \\ 2y \\ 2z \end{smallmatrix}\right) - 6(4x^2 + 4y^2 + 4z^2)}{2(4x^2 + 4y^2 + 4z^2)^{3/2}} = \frac{8R^2 - 24R^2}{2(8R^3)} = -\frac{1}{R}.$$

Notice that the mean curvature here is negative, but notice too that $\nabla F = (2x, 2y, 2z)$ is the outward pointing normal. Thus, as one would expect, the mean curvature vector $\vec{\mathbf{K}}_M = K_M \nabla F/|\nabla F|$ points into the sphere and has magnitude $|K_M| = 1/R$.

Example 4.2 (*Explicit functions*). Surfaces defined by explicit functions are special cases both of parametric and of implicit surfaces. Therefore, we can check our curvature formulas on the explicitly defined surfaces:

$$F(x, y, z) \equiv z - f(x, y, z) = 0.$$

To compute the mean and Gaussian curvatures, we first calculate

$$\nabla F = (f_x \ f_y \ 1),$$

$$H(F) = \begin{pmatrix} f_{xx} & f_{xy} & 0 \\ f_{yx} & f_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$H^*(F) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \operatorname{Det}(H(F)) \end{pmatrix},$$

$$\operatorname{Trace}\big(H(F)\big) = \operatorname{Trace}\big(H(f)\big).$$

Gaussian curvature:

$$K_G = \frac{\nabla F * H^*(F) * \nabla F^{\mathrm{T}}}{|\nabla F|^4} = \frac{\operatorname{Det}(H(f))}{(|\nabla f|^2 + 1)^2}.$$

Mean curvature:

$$K_M = \frac{\nabla F * H(F) * \nabla F^{\mathrm{T}} - |\nabla F|^2 \operatorname{Trace}(H)}{2|\nabla F|^3},$$

$$K_M = \frac{\nabla f * H(f) * \nabla f^{\mathrm{T}} - (|\nabla f|^2 + 1) \operatorname{Trace}(H(f))}{2(|\nabla f|^2 + 1)^{3/2}}.$$

The reader can easily verify that we get exactly these same expressions for the mean and Gaussian curvatures using the classical formulas presented in Section 2.2 for the mean and Gaussian curvature of a parametric surface.

Our formulas for Gaussian and mean curvature readily extend to implicit hypersurfaces in higher dimensions. If $F(x_1, \ldots, x_{n+1}) = 0$ represents an implicit n-dimensional hypersurface lying in an $(n+1)$-dimensional space, then we have the following general formulas for the Gaussian and mean curvatures (Dombrowski, 1968, pp. 167, 168; Gromoll et al., 1975, pp. 1091–1111):

$$(K_G)_n(F) = (-1)^n \frac{\nabla F * H^*(F) * \nabla F^{\mathrm{T}}}{|\nabla F|^{n+2}},$$

$$(K_M)_n(F) = \frac{\nabla F * H(F) * \nabla F^{\mathrm{T}} - |\nabla F|^2 \operatorname{Trace}(H(F))}{n|\nabla F|^3}.$$

Notice that the only essential differences between mean and Gaussian curvature formulas for surfaces in 3-dimensions and mean and Gaussian curvature formulas for hypersurfaces in $(n + 1)$-dimensions are the normalizations in the denominators.

Example 4.3 (*Hyperspheres*). We can check these curvature formulas on the hyperspheres S^n in R^{n+1}

$$S^n\colon\ x_1^2+\cdots+x_{n+1}^2-R^2=0.$$

Here we easily compute

$$H(S^n)=\mathrm{Diag}(2),$$
$$H^*(S^n)=\mathrm{Diag}(2^n),$$
$$\mathrm{Trace}\big(H(S^n)\big)=2(n+1).$$

Therefore

$$(K_G)_n(S^n)=(-1)^n\frac{2^{n+2}R^2}{(2R)^{n+2}}=\frac{(-1)^n}{R^n},$$
$$(K_M)_n(S^n)=\frac{8R^2-(8n+8)R^2}{n(2R)^3}=-\frac{1}{R}.$$

5. Curvature and torsion formulas for implicit space curves and beyond

Implicit space curves are defined by the intersection of two implicit surfaces:

$$\big\{F(x,y,z)=0\big\}\cap\big\{G(x,y,z)=0\big\}.$$

Since the gradients ∇F and ∇G are normal to their respective surfaces, their cross product is tangent to the intersection curve. As with planar curves, we shall adopt the following notation for space curves:

Curve tangent:

$$\mathrm{Tan}(F,G)=\nabla F\times\nabla G. \tag{5.1}$$

Unit tangent:

$$T(F,G)=\frac{\nabla F\times\nabla G}{|\nabla F\times\nabla G|}. \tag{5.2}$$

5.1. Curvature for implicit space curves

For implicit planar curves, we showed in Section 3 (Eq. (3.4)) that the curvature is given by

$$k=\frac{|(\mathrm{Tan}(F)*\nabla(\mathrm{Tan}(F)))\times\mathrm{Tan}(F)|}{|\,\mathrm{Tan}(F)|^3}. \tag{3.4}$$

Since we derived this curvature formula from the Frenet equations, this same curvature formula is also valid for space curves. Substituting Eq. (5.1) for $\mathrm{Tan}(F)$, we arrive at the following curvature formula for implicit space curves:

$$k=\frac{|((\nabla F\times\nabla G)*\nabla(\nabla F\times\nabla G))\times(\nabla F\times\nabla G)|}{|\nabla F\times\nabla G|^3}. \tag{5.3}$$

Example 5.1 (*Spheres* $\cap$ *Cylinders*). We can check this curvature formula on circles generated by spheres intersecting cylinders tangentially from the inside. Consider the following surfaces:

Spheres	Cylinders
$F(x,y,z) \equiv x^2+y^2+z^2-R^2=0$,	$G(x,y,z) \equiv x^2+y^2-R^2=0$,
$\nabla F = (2x\ 2y\ 2z)$,	$\nabla G = (2x\ 2y\ 0)$.

The tangents to the intersection curve are given by

$$\operatorname{Tan}(F,G) = \nabla F \times \nabla G = (-4yz\ 4xz\ 0).$$

Hence

$$|\nabla F \times \nabla G|^3 = \big(16z^2(x^2+y^2)\big)^{3/2} = 64z^3R^3,$$

$$\nabla(\nabla F \times \nabla G) = \begin{pmatrix} 0 & 4z & 0 \\ -4z & 0 & 0 \\ -4y & 4x & 0 \end{pmatrix},$$

$$(\nabla F \times \nabla G) * \nabla(\nabla F \times \nabla G) = (-4yz\ \ 4xz\ \ 0) * \begin{pmatrix} 0 & 4z & 0 \\ -4z & 0 & 0 \\ -4y & 4x & 0 \end{pmatrix} = (-16xz^2\ \ -16yz^2\ \ 0),$$

$$\begin{aligned}\big|\big((\nabla F \times \nabla G) * \nabla(\nabla F \times \nabla G)\big) \times (\nabla F \times \nabla G)\big| &= \big|(-16xz^2\ \ -16yz^2\ \ 0) \times (-4yz\ \ 4xz\ \ 0)\big| \\ &= \big|\big(0\ \ 0\ \ -64z^3(x^2+y^2)\big)\big| = 64z^3R^2.\end{aligned}$$

Therefore, as expected,

$$k = \frac{|((\nabla F \times \nabla G) * \nabla(\nabla F \times \nabla G)) \times (\nabla F \times \nabla G)|}{|\nabla F \times \nabla G|^3} = \frac{64z^3R^2}{64z^3R^3} = \frac{1}{R}.$$

We would like to extend our curvature formula—Eq. (5.3)—for implicit space curves to a curvature formula for implicit curves in $(n+1)$-dimensions—that is, to curves generated by the intersection of n implicit equations:

$$\big\{F_1(x_1,\dots,x_{n+1})=0\big\} \cap \dots \cap \big\{F_n(x_1,\dots,x_{n+1})\big\}.$$

However, in order to generalize Eq. (5.3), we first need to generalize the cross product from 3-dimensions to $(n+1)$-dimensions. Actually there are two ways to extend the cross product from 3-dimensions to $(n+1)$-dimensions and we shall require both techniques.

The first cross product in Eq. (5.3) is used to compute the tangent to the intersection curve. Recall that the gradients

$$\nabla F_1 = (F_{1x_1},\dots,F_{1x_{n+1}}), \quad \dots, \quad \nabla F_n = (F_{nx_1},\dots,F_{nx_{n+1}})$$

are normal to their respective hypersurfaces. Therefore the tangent to the intersection curve is a vector perpendicular to each of these gradient vectors.

The extension of the cross product to $(n+1)$-dimensions that generates a vector perpendicular to a collection of n vectors is given by a determinant. Let $e = (e_1, \dots, e_{n+1})$ be the canonical basis for $\mathbf{R}^{n+1}$, where e_i is the vector that has a one in the ith position and a zero everywhere else. Then

$$\operatorname{Tan}(F_1, \dots, F_n) = \operatorname{Det}\begin{pmatrix} e \\ \nabla F_1 \\ \vdots \\ \nabla F_n \end{pmatrix} = \operatorname{Det}\begin{pmatrix} e_1 & \dots & e_{n+1} \\ F_{1x_1} & \dots & F_{1x_{n+1}} \\ \vdots & \vdots & \vdots \\ F_{nx_1} & \dots & F_{nx_{n+1}} \end{pmatrix}.$$

This expression for $\operatorname{Tan}(F_1, \dots, F_n)$ is perpendicular to $\nabla F_1, \dots, \nabla F_n$ because taking the dot product of $\operatorname{Tan}(F_1, \dots, , F_n)$ with ∇F_i is equivalent to substituting ∇F_i for e, and the determinant of a matrix with two identical rows is identically zero.

The other cross product in Eq. (5.3)—corresponding to the cross product in Eq. (3.1)—is the cross product of two, rather than n, vectors. To generalize this cross product from vectors in 3-dimensions to vectors in $(n+1)$-dimensions, we introduce the wedge product (Northcott, 1984). The wedge product of two vectors in an $(n+1)$-dimensional space spanned by $e_1, \dots, e_{n+1}$ is a vector in a space of dimension $\binom{n+1}{2}$ spanned by a new collection of vectors denoted by $\{e_i \wedge e_j\}$, where $i < j$. Let $u = u_1\mathbf{e}_1 + \dots + u_{n+1}\mathbf{e}_{n+1}$ and $v = v_1\mathbf{e}_1 + \dots + v_{n+1}\mathbf{e}_{n+1}$. Then we define

$$u \wedge v = (u_1\mathbf{e}_1 + \dots + u_{n+1}\mathbf{e}_{n+1}) \wedge (v_1\mathbf{e}_1 + \dots + v_{n+1}\mathbf{e}_{n+1}) = \sum_{i<j} \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix} (\mathbf{e}_i \wedge \mathbf{e}_j).$$

Notice that the wedge product, just like the cross product, is anti-commutative and distributes through addition.

Actually to extend Eq. (5.3), we need only compute the magnitude of the wedge product. The magnitude of the wedge product is given by the formula

$$|u \wedge v|^2 = \left|(u_1\mathbf{e}_1 + \dots + u_{n+1}\mathbf{e}_{n+1}) \wedge (v_1\mathbf{e}_1 + \dots + v_{n+1}\mathbf{e}_{n+1})\right|^2 = \sum_{i<j} \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix}^2.$$

Notice that

$$\sum_{i<j} \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix}^2 = \left(\sum_i u_i^2\right)\left(\sum_j v_j^2\right) - \left(\sum_k u_k v_k\right)^2.$$

Therefore, much like the cross product, wedge product satisfies the identity

$$|u \wedge v|^2 = |u|^2|v|^2 - (u \bullet v)^2.$$

Using this identity, we can avoid altogether the computation of $u \wedge v$. Thus in the curvature formula we present below, the wedge product is used only as a device to compress the notation.

The curvature formula for a curve defined by the intersection of n implicit hypersurfaces $F_1(x_1, \dots, x_{n+1}) = 0, \dots, F_n(x_1, \dots, x_{n+1}) = 0$ now becomes

$$k = \frac{|(\operatorname{Tan}(F_1, \dots, F_n) * \nabla(\operatorname{Tan}(F_1, \dots, F_n))) \wedge \operatorname{Tan}(F_1, \dots, F_n)|}{|\operatorname{Tan}(F_1, \dots, F_n)|^3}, \tag{5.4}$$

where $\nabla(\operatorname{Tan}(F_1, \dots, F_n))$ is interpreted in the usual fashion to mean that we take the gradient of each component of $\operatorname{Tan}(F_1, \dots, F_n)$ and write these gradients in a matrix as consecutive column vectors.

Example 5.2 (3-*sphere* $\cap$ 2-*sphere* $\cap$ *cylinder*). We can check this curvature formula on circles generated by 3-spheres intersecting 2-spheres intersecting cylinders tangentially. Consider the following surfaces:

3-*Spheres*:

$$F_1(x, y, z, w) \equiv x^2 + y^2 + z^2 + w^2 - R^2 = 0,$$
$$\nabla F_1 = (2x \;\; 2y \;\; 2z \;\; 2w).$$

2-*Spheres*:

$$F_2(x, y, z, w) \equiv x^2 + y^2 + z^2 - R^2 = 0,$$
$$\nabla F_2 = (2x \;\; 2y \;\; 2z \;\; 0).$$

Cylinders:

$$F_3(x, y, z, w) \equiv x^2 + y^2 - R^2 = 0,$$
$$\nabla F_3 = (2x \;\; 2y \;\; 0 \;\; 0).$$

The tangents to the intersection curve are given by

$$\operatorname{Tan}(F_1, F_2, F_3) = \operatorname{Det}\begin{pmatrix} e \\ \nabla F_1 \\ \nabla F_2 \\ \nabla F_3 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ 2x & 2y & 2z & 2w \\ 2x & 2y & 2z & 0 \\ 2x & 2y & 0 & 0 \end{pmatrix} = -(8yzw)e_1 + (8xzw)e_2.$$

Hence

$$\left|\operatorname{Tan}(F_1, F_2, F_3)\right|^3 = 512z^3w^3R^3,$$

$$\nabla\big(\operatorname{Tan}(F_1, F_2, F_3)\big) = \begin{pmatrix} 0 & 8zw & 0 & 0 \\ -8zw & 0 & 0 & 0 \\ -8yw & 8xw & 0 & 0 \\ -8yz & 8xz & 0 & 0 \end{pmatrix},$$

$$\begin{aligned}\operatorname{Tan}(F_1, F_2, F_3) * \nabla\big(\operatorname{Tan}(F_1, F_2, F_3)\big) &= (-8yzw \;\; 8xzw \;\; 0 \;\; 0) * \begin{pmatrix} 0 & 8zw & 0 & 0 \\ -8zw & 0 & 0 & 0 \\ -8yw & 8xw & 0 & 0 \\ -8yz & 8xz & 0 & 0 \end{pmatrix} \\ &= -(64z^2w^2x)e_1 - (64z^2w^2y)e_2,\end{aligned}$$

$$\begin{aligned}&\big(\operatorname{Tan}(F_1, F_2, F_3) * \nabla\big(\operatorname{Tan}(F_1, F_2, F_3)\big)\big) \wedge \operatorname{Tan}(F_1, F_2, F_3) \\ &\quad = \big(-(64z^2w^2x)e_1 - (64z^2w^2y)e_2\big) \wedge \big(-(8yzw)e_1 + (8xzw)e_2\big) \\ &\quad = -(512z^3w^3R^2)(e_1 \wedge e_2).\end{aligned}$$

Therefore, as expected,

$$k = \frac{|(\operatorname{Tan}(F_1, F_2, F_3) * \nabla(\operatorname{Tan}(F_1, F_2, F_3))) \wedge \operatorname{Tan}(F_1, F_2, F_3)|}{|\operatorname{Tan}(F_1, F_2, F_3)|^3} = \frac{512z^3w^3R^2}{512z^3w^3R^3} = \frac{1}{R}.$$

5.2. Torsion for implicit space curves

To develop a closed form expression for the torsion of an implicit space curve, we begin, as usual, with a classical formula for the torsion of a parametric curve. In Section 2.1, Table 1 provides four formulas for the torsion of a space curve in terms of the unit tangent, the unit normal, and their derivatives. For implicit space curves, however, we know only the unit tangent (Eq. (5.2)); therefore only the last formula in Table 1 is readily applied. Thus starting with

$$\tau = \frac{\mathrm{Det}\big(T\ \frac{\mathrm{d}T}{\mathrm{d}s}\ \frac{\mathrm{d}^2T}{\mathrm{d}s^2}\big)}{k^2},$$
$$T(F,G) = \frac{\nabla F \times \nabla G}{|\nabla F \times \nabla G|},$$

we shall now derive a closed expression for the torsion of an implicit space curve.

Theorem 5.1 (Torsion of an implicit space curve).

$$\tau = \frac{\mathrm{Det}(T^*\ T^{**}\ T^{***})}{|T^* \times T^{**}|^2}, \tag{5.5}$$

where

$$T^* = \nabla F \times \nabla G,$$
$$T^{**} = T^* * \nabla T^* = (\nabla F \times \nabla G) * \nabla(\nabla F \times \nabla G),$$
$$\begin{aligned} T^{***} &= T^* * \nabla(\nabla T^*) * T^{*\mathrm{T}} + T^* * \nabla T^* * \nabla T^* \\ &= (\nabla F \times \nabla G) * \nabla\big(\nabla(\nabla F \times \nabla G)\big) * (\nabla F \times \nabla G)^{\mathrm{T}} \\ &\quad + (\nabla F \times \nabla G) * \nabla(\nabla F \times \nabla G) * \nabla(\nabla F \times \nabla G). \end{aligned}$$

Here ∇ applied to a matrix such as ∇T^* means apply ∇ to each column vector of the matrix to generate a list of three consecutive matrices.

Proof. We start with the formula

$$\tau = \frac{\mathrm{Det}\big(T\ \frac{\mathrm{d}T}{\mathrm{d}s}\ \frac{\mathrm{d}^2T}{\mathrm{d}s^2}\big)}{k^2}.$$

To compute the determinant in the numerator, observe that

$$T = \frac{T^*}{|T^*|}.$$

Moreover, by the chain rule,

$$\frac{\mathrm{d}T^*}{\mathrm{d}s} = T * \nabla T^* = \frac{T^* * \nabla T^*}{|T^*|}.$$

Therefore, by the quotient rule,

$$\frac{\mathrm{d}T}{\mathrm{d}s} = \frac{T^* * \nabla T^*}{|T^*|^2} - \frac{\big(\frac{\mathrm{d}|T^*|}{\mathrm{d}s}\big)T^*}{|T^*|^2} = \frac{T^{**}}{|T^*|^2} + \text{term parallel to } T^*.$$

Differentiating one more time by the chain rule and the quotient rule, we find that

$$\begin{aligned}\frac{\mathrm{d}^2T}{\mathrm{d}s^2} &= \frac{T^* * \nabla(\nabla T^*) * T^{*\mathrm{T}}}{|T^*|^3} + \frac{T^* * \nabla T^* * \nabla T^*}{|T^*|^3} \\ &\quad + \text{terms parallel to } \frac{T^* * \nabla T^*}{|T^*|} + \text{terms parallel to } T^* \\ &= \frac{T^{***}}{|T^*|^3} + \text{terms parallel to } \frac{T^* * \nabla T^*}{|T^*|} + \text{terms parallel to } T^*.\end{aligned}$$

Therefore

$$\tau = \frac{\mathrm{Det}\left(T \ \frac{\mathrm{d}T}{\mathrm{d}s} \ \frac{\mathrm{d}^2T}{\mathrm{d}s^2}\right)}{k^2} = \frac{\mathrm{Det}(T^* \ T^{**} \ T^{**})}{k^2|T^*|^6}.$$

But by Eq. (5.3),

$$k^2 = \frac{|((\nabla F \times \nabla G) * \nabla(\nabla F \times \nabla G)) \times (\nabla F \times \nabla G)|^2}{|\nabla F \times \nabla G|^6} = \frac{|T^{**} \times T^*|^2}{|T^*|^6}.$$

Hence

$$\tau = \frac{\mathrm{Det}(T^* \ \ T^{**} \ \ T^{***})}{|T^* \times T^{**}|^2}. \qquad \square$$

Example 5.3 (*Explicit functions of one variable*). Space curves defined by the intersection of explicit functions of a single variable are special cases of both parametric and implicit curves. Therefore, we can check our torsion formula on the intersection of two explicitly defined surfaces:

Equations	Gradients
$F(x,y,z) \equiv y - f(x) = 0,$	$\nabla F = (f' \ 1 \ 0),$
$G(x,y,z) \equiv z - g(x) = 0,$	$\nabla G = (g' \ 0 \ 1).$

Now we have the following matrices:

$$T^* = \nabla F \times \nabla G = (1 \ \ -f' \ \ -g'),$$

$$\nabla T^* = \begin{pmatrix} 0 & -f'' & -g'' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\nabla(\nabla T^*) = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -f''' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -g''' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right),$$

$$\nabla T^* * \nabla T^* = 0.$$

To compute the numerator of the torsion, observe that

$$T^{**} = T^* * \nabla T^* = (1 \ \ -f' \ \ -g') * \begin{pmatrix} 0 & -f'' & -g'' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (0 \ \ -f'' \ g''),$$

$$T^{***} = T^* * \nabla(\nabla T^*) * T^{*\mathrm{T}}$$

$$= (1\ \ -f'\ \ -g') * \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -f''' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -g''' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) * \begin{pmatrix} 1 \\ -f' \\ -g' \end{pmatrix}$$

$$= (0\ \ -f'''\ \ -g''').$$

Hence

$$\operatorname{Det}(T^*\ T^{**}\ T^{***}) = \operatorname{Det}\begin{pmatrix} 1 & 0 & 0 \\ -f' & -f'' & -f''' \\ -g' & -g'' & -g''' \end{pmatrix} = f''g''' - f'''g''.$$

Similarly, for the denominator we have

$$\begin{aligned} |T^* \times T^{**}|^2 &= \left|(1\ \ -f'\ \ -g') \times (0\ \ -f''\ \ -g'')\right|^2 = |f''g' - f'g''\ \ g''\ \ -f''|^2 \\ &= (f''g' - f'g'')^2 + (g'')^2 + (f'')^2. \end{aligned}$$

Therefore

$$\tau = \frac{\operatorname{Det}(T^*\ T^{**}\ T^{***})}{|T^* \times T^{**}|} = \frac{f''g''' - f'''g''}{(f''g' - f'g'')^2 + (g'')^2 + (f'')^2}.$$

The reader can easily verify that in this case we get exactly the same expression for the torsion using the classical equation given in Section 2.1 for the torsion of a parametric curve:

$$\tau = \frac{\operatorname{Det}(P'\ P''\ P''')}{|P' \times P''|^2}.$$

6. Summary

Here we collect in one location for easy reference all of our curvature formulas for implicit curves and surfaces.

6.1. Curvature formulas for implicit planar curves

Notation

Implicit curve: $F(x, y) = 0$.
Normal: $\nabla F = (F_x, F_y)$.
Tangent: $\operatorname{Tan}(F) = k \times \nabla F = (-F_y, F_x)$.
Hessian:

$$H(F) = \begin{pmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{pmatrix} = \nabla(\nabla F).$$

Adjoint of the hessian:

$$H^*(F) = \begin{pmatrix} F_{yy} & -F_{yx} \\ -F_{xy} & F_{xx} \end{pmatrix} = -\nabla\big(\operatorname{Tan}(F)\big).$$

Curvature formulas

$$k = -\frac{\mathrm{Tan}(F) * H(F) * \mathrm{Tan}(F)^{\mathrm{T}}}{|\nabla F|^3}, \qquad k = \frac{\mathrm{Tan}(F) * \nabla(\mathrm{Tan}(F)) * \nabla F^{\mathrm{T}}}{|\nabla F|^3},$$

$$k = \frac{\mathrm{Det}((\mathrm{Tan}(F) * H(F))^{\mathrm{T}}\ \nabla F^{\mathrm{T}})}{|\nabla F|^3}, \qquad k = \frac{\mathrm{Det}((\mathrm{Tan}(F) * \nabla(\mathrm{Tan}(F)))^{\mathrm{T}}\ (\mathrm{Tan}(F))^{\mathrm{T}})}{|\mathrm{Tan}(F)|^3},$$

$$k = -\frac{\nabla F * H^*(F) * \nabla F^{\mathrm{T}}}{|\nabla F|^3}, \qquad k = \frac{\nabla F * H(F) * \nabla F^{\mathrm{T}} - |\nabla F|^2\,\mathrm{Trace}(H(F))}{|\nabla F|^3},$$

$$k = \frac{\begin{vmatrix} H(F) & \nabla F \\ \nabla F & 0 \end{vmatrix}}{|\nabla F|^3}, \qquad k = -\nabla \bullet \left(\frac{\nabla F}{|\nabla F|}\right).$$

6.2. Curvature formulas in co-dimension 1

Notation

Implicit hypersurface: $F(x_1, \ldots, x_{n+1}) = 0$.
Gradient: $\nabla F = (F_{x_1}, \ldots, F_{x_{n+1}})$.
Hessian:

$$H(F) = \begin{pmatrix} F_{x_1x_1} & \ldots & F_{x_1x_{n+1}} \\ \vdots & \ddots & \vdots \\ F_{x_{n+1}x_1} & \ldots & F_{x_{n+1}x_{n+1}} \end{pmatrix} = \nabla(\nabla F).$$

Adjoint of the hessian:

$$H^*(F) = \begin{pmatrix} \mathrm{Cofactor}(F_{x_1x_1}) & \ldots & \mathrm{Cofactor}(F_{x_1x_{n+1}}) \\ \vdots & \ddots & \vdots \\ \mathrm{Cofactor}(F_{x_{n+1}x_1}) & \ldots & \mathrm{Cofactor}(F_{x_{n+1}x_{n+1}}) \end{pmatrix}.$$

Gaussian curvature

$$k = -\frac{\nabla F * H^*(F) * \nabla F^{\mathrm{T}}}{|\nabla F|^3},$$

$$K_G = \frac{\nabla F * H^*(F) * \nabla F^{\mathrm{T}}}{|\nabla F|^4},$$

$$(K_G)_n(F) = (-1)^n \frac{\nabla F * H^*(F) * \nabla F^{\mathrm{T}}}{|\nabla F|^{n+2}},$$

$$(K_G)_n = (-1)^{n-1} \frac{\begin{vmatrix} H(F) & \nabla F^{\mathrm{T}} \\ \nabla F & 0 \end{vmatrix}}{|\nabla F|^{n+2}}.$$

Mean curvature

$$k = \frac{\nabla F * H(F) * \nabla F^{\mathrm{T}} - |\nabla F|^2\,\mathrm{Trace}(H(F))}{|\nabla F|^3},$$

$$K_M = \frac{\nabla F * H(F) * \nabla F^{\mathrm{T}} - |\nabla F|^2\,\mathrm{Trace}(H(F))}{2|\nabla F|^3},$$

$$(K_M)_n(F) = \frac{\nabla F * H(F) * \nabla F^{\mathrm{T}} - |\nabla F|^2 \operatorname{Trace}(H(F))}{n|\nabla F|^3},$$

$$(K_M)_n = \frac{(-1)^{n-1} \operatorname{coeff}(\lambda^{n-1}) \operatorname{in} \begin{vmatrix} H(F)-\lambda I & \nabla F^{\mathrm{T}} \\ \nabla F & 0 \end{vmatrix}}{n|\nabla F|^3},$$

$$(K_M)_n = -\nabla \bullet \left(\frac{\nabla F}{|\nabla F|}\right).$$

6.3. Curvature and torsion formulas for implicit space curves

Notation

Implicit space curve: $\{F(x, y, z) = 0\} \cap \{G(x, y, z) = 0\}$.
Curve tangent: $T^* = \nabla F \times \nabla G$.
Additional notation:

- $T^{**} = T^* * \nabla T^* = (\nabla F \times \nabla G) * \nabla(\nabla F \times \nabla G)$
- $$\begin{aligned} T^{***} &= T^* * \nabla(\nabla T^*) * T^{*\mathrm{T}} + T^* * \nabla T^* * \nabla T^* \\ &= (\nabla F \times \nabla G) * \nabla(\nabla(\nabla F \times \nabla G)) * (\nabla F \times \nabla G)^{\mathrm{T}} \\ &\quad + (\nabla F \times \nabla G) * \nabla(\nabla F \times \nabla G) * \nabla(\nabla F \times \nabla G). \end{aligned}$$

Curvature and torsion formulas

Parametric curves:

$$k = \frac{|P' \times P''|}{|P'|^3}, \qquad \tau = \frac{\operatorname{Det}(P'\ P''\ P''')}{|P' \times P''|^2}.$$

Implicit curves:

$$k = \frac{|T^* \times T^{**}|}{|T^*|^3}, \qquad \tau = \frac{\operatorname{Det}(T^*\ T^{**}\ T^{***})}{|T^* \times T^{**}|^2}.$$

6.4. Curvature formulas for implicit curves in $(n + 1)$-dimensions

Notation

Implicit curve: $\{F_1(x_1, \ldots, x_{n+1}) = 0\} \cap \cdots \cap \{F_n(x_1, \ldots, x_{n+1}) = 0\}$.
Gradients of scalars: $\nabla F_1 = (F_{1x_1}, \ldots, F_{1x_{n+1}}), \ldots, \nabla F_n = (F_{nx_1}, \ldots, F_{nx_{n+1}})$.
Gradients of vectors: $\nabla(F_1, \ldots, F_n) = ((\nabla F_1)^{\mathrm{T}}, \ldots, (\nabla F_n)^{\mathrm{T}})$.

Curvature formulas

Curves in 2-dimensions:

$$k = \frac{\operatorname{Det}((\operatorname{Tan}(F) * \nabla(\operatorname{Tan}(F)))^{\mathrm{T}} (\operatorname{Tan}(F))^{\mathrm{T}})}{|\operatorname{Tan}(F)|^3},$$

$$\operatorname{Tan}(F) = (-F_y, F_x).$$

Curves in 3-dimensions:

$$k = \frac{|(\mathrm{Tan}(F, G) * \nabla(\mathrm{Tan}(F, G))) \times \mathrm{Tan}(F, G)|}{|\mathrm{Tan}(F, G)|^3},$$
$$\mathrm{Tan}(F, G) = \nabla F \times \nabla G.$$

Curves in $(n+1)$-dimensions:

$$k = \frac{|(\mathrm{Tan}(F_1, \ldots, F_n) * \nabla(\mathrm{Tan}(F_1, \ldots, F_n))) \wedge \mathrm{Tan}(F_1, \ldots, F_n)|}{|\mathrm{Tan}(F_1, \ldots, F_n)|^3},$$
$$\mathrm{Tan}(F_1, \ldots, F_n) = \mathrm{Det}\big(e^{\mathrm{T}}\ (\nabla F_1)^{\mathrm{T}}\ \ldots\ (\nabla F_n)^{\mathrm{T}}\big),$$
$$|u \wedge v|^2 = |u|^2|v|^2 - (u \bullet v)^2.$$

7. Open questions

Although we have presented many elegant curvature formulas for implicit curves and surfaces, there still remains some work to be done. We close with a few open problems for future research.

Problem 1. *Derive closed formulas for higher order curvatures for implicit curves in* $(n+1)$*-dimensions*. Here we have derived closed formulas only for the curvature in $(n+1)$-dimensions and for the torsion in 3-dimensions. Can elegant closed formulas be derived for the higher order analogues of the torsion for implicit curves in $(n+1)$-dimensions?

Problem 2. *Develop curvature formulas for curves lying on implicit surfaces*. Here we have developed curvature formulas only for curves lying in n-dimensional space. What can be said about the curvature for curves lying on implicit surfaces?

Problem 3. *Compute curvature formulas for implicit surfaces in* $(n+1)$*-dimensions with co-dimension* > 1. Apparently at least some of these formulas are known in the German literature (Dombrowski, 1968, p. 164), but I have been unable to derive these expressions or to find these formulas in an easily accessible form.

Problem 4. *Investigate numerical efficiency and robustness.* We have provided many different equivalent formulations for the curvature of an implicit planar curve. In theory, all of these expression are mathematically equivalent, but, in practice, some of these expressions may be more computationally efficient or numerically robust for performing actual calculations. We have not considered these issues here, but these questions should be looked at in the future.

Acknowledgement

I would like to thank the participants of the Erbach Workshop on Differential Geometry and Geometric Modeling, who provided me with several pertinent references both in English and in German and who encouraged me to complete work on this paper.

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