ON THE BEST CONSTANT IN THE WEAK TYPE INEQUALITY FOR THE SQUARE FUNCTION OF A CONDITIONALLY SYMMETRIC MARTINGALE

ADAM OSĘKOWSKI

Abstract. Let f be a real conditionally symmetric martingale and $S(f)$ denote its square function. The purpose of this note is to show that the inequality

$$\sup_{\lambda>0} (\lambda \mathbb{P}(S(f) \geq \lambda)) \leq K||f||_1, \quad K = \exp\Big(-\frac{1}{2}\Big) + \int_0^1 \exp\Big(-\frac{t^2}{2}\Big)dt \approx 1,4622,$$

due to Bollobás, is sharp.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, filtered by a nondecreasing family $(\mathcal{F}_n)$ of sub-σ-fields of $\mathcal{F}$. Assume $f = (f_n)$ is a martingale, that is, an adapted sequence of integrable variables satisfying $\mathbb{E}(f_n|\mathcal{F}_{n-1}) = f_{n-1}$ almost surely for $n = 1, 2, \ldots$. We define the square function $S(f)$ of f by

$$S(f) = \left(\sum_{k=0}^{\infty} |df_k|^2\right)^{1/2},$$

where (df_k) is a difference sequence of f, given by $df_0 = f_0$ and $df_n = f_n - f_{n-1}$ for $n \geq 1$. We will also use the notation $S_n(f) = (\sum_{k=0}^n |df_k|^2)^{1/2}$, $n = 0, 1, 2, \ldots$.

We will be interested in special classes of martingales. A martingale is *conditionally symmetric* if for any n, the conditional distributions of df_n and $-df_n$ with respect to $\mathcal{F}_{n-1}$ coincide (we set $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$). In particular, all dyadic martingales are conditionally symmetric. A martingale on the Lebesgue unit interval is called dyadic, if it has dyadic differences: for all n, its n-th difference and the norm of $n+1$-st difference are both constant on the interval $[(k-1)/2^n, k/2^n)$ for all $k = 1, 2, \ldots, 2^n$.

In [1], Bollobás established the weak type $(1,1)$ inequality for the square function of a dyadic martingale with a constant

$$K = \exp\Big(-\frac{1}{2}\Big) + \int_0^1 \exp\Big(-\frac{t^2}{2}\Big)dt \approx 1,4622,$$

and proved that the best constant is not smaller than $1,44$. As explained in the paper [2] by Burkholder, the optimal constant does not change if we allow the martingale to be conditionally symmetric. In this note we will show that the constant K is the best possible. Here is the precise statement.

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Theorem 1.1. *Let f be conditionally symmetric martingale. Then for any $\lambda > 0$,*

$$\lambda \mathbb{P}(S(f) \geq \lambda) \leq K||f||_1 \tag{1.1}$$

and the constant K can not be replaced by a smaller one.

Clearly, by homogeneity, it suffices to deal with the case $\lambda = 1$ only.

2. The sharpness

Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion starting from 0 and ε be a Rademacher random variable independent of B. Introduce a stopping time $\tau = \inf\{t : B_t^2 + t \geq 1\}$, satisfying $\tau \leq 1$ almost surely, and let the process $X = (X_t)_{t\geq 0}$ be given by

$$X_t = B_{\tau\wedge t} + \varepsilon B_\tau I_{\{t\geq 1\}}.$$

The process X is a Brownian motion, which stops at the moment τ, and then at time 1 jumps to one of the points 0, $2B_\tau$ with probability 1/2 and stays there forever. Clearly, it is a martingale with respect to its natural filtration. Its square bracket process $[X]$ (which is a continuous-time extension of a square function, see e.g. Dellacherie and Meyer [3]) satisfies

$$[X]_1 = [B]_\tau + |B_\tau|^2 = \tau + B_\tau^2 = 1 \quad \text{almost surely,}$$

and, as we shall prove now, $||X||_1 = 1/K$. Observe that $||X||_1 = ||X_1||_1 = ||X_\tau||_1 = ||B_\tau||_1$. Let $U : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be given by

$$U(t,x) = \sqrt{1-t}\exp(-\frac{x^2}{2(1-t)}) + |x| \int_0^{|x|/\sqrt{1-t}} \exp(-s^2/2)ds,$$

if $t + x^2 < 1$, and $U(t,x) = K|x|$ otherwise. It can be verified readily that U is continuous and satisfies the heat equation $U_t + \frac{1}{2}U_{xx} = 0$ on the set $\{(t,x) : t + x^2 < 1\}$. This implies that $(U(\tau \wedge t, B_{\tau\wedge t}))_{t\geq 0}$ is a martingale adapted to $\mathcal{F}^B$ and therefore

$$K||B_\tau||_1 = \mathbb{E}U(\tau, B_\tau) = U(0,0) = 1.$$

This shows the sharpness of (1.1) in the continuous-time setting. Now the passage to the discrete-time case can be carried out using standard approximation techniques. However, our proof will be different. Suppose that the best constant in the inequality (1.1) for dyadic martingales equals K_0. Exploiting the ideas of Burkholder (cf. [2]), we see that this implies the existence of a function $W : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfying the following three conditions:

(i) $W(0,0) \leq 0$,
(ii) $W(t,x) \geq I_{\{t\geq 1\}} - K_0|x|$ for all $t \geq 0$, $x \in \mathbb{R}$,
(iii) $W(t+d^2, x-d) + W(t+d^2, x+d) - 2W(t,x) \leq 0$ for all $t \geq 0$, x, $d \in \mathbb{R}$.

Indeed, one takes

$$W(t,x) = \sup\{\mathbb{P}(t + x^2 - S^2(f) \geq 1) - K_0||f||_1\},$$

where the supremum is taken over all the simple martingales starting from x and dyadic differences df_n, $n = 1, 2, \ldots$. It is not difficult to see that W is continuous. To see this, let f, $f_0 \equiv x$, be as in the definition of W. Fix x' and let $f' = f + x' - x$. Then we have $x^2 - S^2(f) = (x')^2 - S^2(f')$ and, for any $t \geq 0$,

$$\mathbb{P}(t+x^2-S^2(f) \geq 1) - K_0||f||_1 \leq \mathbb{P}(t+(x')^2-S^2(f') \geq 1) - K_0||f'||_1 + K_0|x-x'|,$$

which implies $W(t,x) \leq W(t,x') + K_0|x-x'|$ and hence, for fixed t, $W(t,\cdot)$ is K_0-Lipschitz. Hence, applying (iii), for any $s<t$ and any x,

$$W(s,x) \geq \frac{1}{2}[W(t,x-\sqrt{t-s}) + W(t,x+\sqrt{t-s}] \geq W(t,x) - K_0\sqrt{t-s}.$$

On the other hand, $W(s,x) \leq W(t,x)$ by the definition of W. Therefore, for any x, $W(x,\cdot)$ is continuous. This yields the continuity of W.

Now extend W to the whole $\mathbb{R}^2$ by setting $W(t,x) = W(0,x)$ for $t<0$. Let $\delta>0$ and convolve W with a nonnegative smooth function g^δ satisfying $||g^\delta||_1 = 1$ and supported on the ball centered at $(0,0)$ and radius δ. As the result, we obtain a smooth function W^δ, for which (iii) is still valid. Dividing this inequality by d^2 and letting $d \to 0$ gives $W^\delta_t + W^\delta_{xx} \leq 0$ and hence, by Itô's formula, $\mathbb{E}W^\delta(\tau, B_\tau) \leq W^\delta(0,0)$. Now let $\delta \to 0$ and use the continuity of W and Lebesgue's dominated convergence theorem to conclude that $\mathbb{E}W(\tau,B_\tau) \leq W(0,0)$. The final step is that, by (iii),

$$W(\tau + B_\tau^2, B_\tau - B_\tau) + W(\tau + B_\tau^2, B_\tau + B_\tau) \leq 2W(\tau, B_\tau) \quad \text{almost surely},$$

which yields $\mathbb{E}W([X]_1, X_1) \leq W(0,0)$ and, by (i) and (ii), $K_0 \geq K$.

References

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Department of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland

E-mail address: ados@mimuw.edu.pl