WEAK TYPE (p, q)-INEQUALITIES FOR THE HAAR SYSTEM AND DIFFERENTIALLY SUBORDINATED MARTINGALES

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Abstract. For any $1 \leq p,\, q < \infty$, we determine the optimal constant $C_{p,q}$ such that the following holds. If $(h_k)_{k \geq 0}$ is the Haar system on $[0, 1]$, then for any vectors a_k from a separable Hilbert space $\mathcal{H}$ and $\varepsilon_k \in \{-1, 1\}$, $k = 0, 1, 2, \ldots$, we have

$$\left|\left|\sum_{k=0}^{n} \varepsilon_k a_k h_k\right|\right|_{q,\infty} \leq C_{p,q} \left|\left|\sum_{k=0}^{n} a_k h_k\right|\right|_p .$$

This is generalized to the sharp weak-type inequality

$$||Y||_{q,\infty} \leq C_{p,q} ||X||_p,$$

where X, Y stand for $\mathcal{H}$-valued martingales such that Y is differentially subordinate to X.

1. Introduction

The motivation of this paper comes from a basic question about the Haar system $(h_k)_{k \geq 0}$ on $[0, 1]$. Let us start with some related classical results from the literature. As shown by Marcinkiewicz [7] (see also Paley [12]), if $1 < p < \infty$, then there is a universal finite constant c_p such that

$$c_p^{-1} \left|\left|\sum_{k=0}^{n} a_k h_k\right|\right|_p \leq \left|\left|\sum_{k=0}^{n} \varepsilon_k a_k h_k\right|\right|_p \leq c_p \left|\left|\sum_{k=0}^{n} a_k h_k\right|\right|_p \tag{1.1}$$

for any n and any $a_k \in \mathbb{R}$, $\varepsilon_k \in \{-1, 1\}$, $k = 0, 1, 2, \ldots, n$. This result was extended by Burkholder [1] to the martingale setting. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, filtered by $(\mathcal{F}_k)_{k \geq 0}$, a nondecreasing family of sub-σ-fields of $\mathcal{F}$. Let $f = (f_k)_{k \geq 0}$ be a real-valued martingale with the difference sequence $(df_k)_{k \geq 0}$ given by $df_0 = f_0$ and $df_k = f_k - f_{k-1}$ for $k \geq 1$. Let g be a transform of f by a real predictable sequence $v = (v_k)_{k \geq 0}$ bounded in absolute value by 1: that is, $dg_k = v_k df_k$ for all $k \geq 0$ and by predictability we mean that each term v_k is measurable with respect to $\mathcal{F}_{(k-1) \vee 0}$. Then (cf. [1]) for $1 < p < \infty$ there is an absolute constant c_p' for which

$$||g||_p \leq c_p' ||f||_p. \tag{1.2}$$

Here we have used the notation $||f||_p = \sup_n ||f_n||_p$. Let $c_p(1.1)$, $c_p'(1.2)$ denote the optimal constants in (1.1) and (1.2), respectively. The Haar system is a martingale difference sequence with respect to its natural filtration (on the probability space being the Lebesgue's unit interval) and hence so is $(a_k h_k)_{k \geq 0}$, for given fixed real

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numbers $a_0,\ a_1,\ a_2,\ \ldots$. Therefore, $c_p(1.1) \le c'_p(1.2)$ for all $1 < p < \infty$. It follows from the results of Burkholder [2] and Maurey [8] that in fact the constants coincide: $c_p(1.1) = c'_p(1.2)$ for all $1 < p < \infty$. The question about the precise value of $c_p(1.1)$ was answered by Burkholder in [3]: $c_p(1.1) = p^* - 1$ (where $p^* = \max\{p, p/(p-1)\}$) for $1 < p < \infty$. Furthermore, the constant does not change if we allow the martingales and the coefficients a_k to take values in a separable Hilbert space $\mathcal{H}$. These results have been strengthened in [10], where the author determined the optimal universal constants $c_{p,q} \in [1,\infty]$, $1 \le p,\ q < \infty$, such that

$$||g||_q \le c_{p,q} ||f||_p, \tag{1.3}$$

for any $\mathcal{H}$-valued f, g as above. The description of these constants is quite complicated, so we do not present it here and refer the interested reader to [10]. Let us only mention here that $c_{p,q}$ are the best possible in

$$\left|\left|\sum_{k=0}^n a_k h_k\right|\right|_q \le c_{p,q} \left|\left|\sum_{k=0}^n \varepsilon_k a_k h_k\right|\right|_p,$$

even if we assume that the coefficients a_k are real. This follows, for example, from the reasoning presented in Section 10 of Burkholder [3].

For $p = 1$ the inequalities (1.1) and (1.2) do not hold with any finite constant; in other words, we have $c_{1,1} = \infty$ in (1.3). However, one can establish an appropriate weak type bound. Here is the result of Burkholder [3], valid for a wider range of parameters: if $1 \le p \le 2$, then for any real valued f, g as above we have the sharp inequality

$$||g||_{p,\infty} \le \left(\frac{2}{\Gamma(p+1)}\right)^{1/p} ||f||_p. \tag{1.4}$$

Here $||g||_{p,\infty} = \sup_{\lambda>0} \lambda(\mathbb{P}(\sup_n |g_n| \ge \lambda))^{1/p}$ denotes the weak p-th norm of g. For $p > 2$, Suh [13] showed that

$$||g||_{p,\infty} \le \left(p^{p-1}/2\right)^{1/p} ||f||_p \tag{1.5}$$

and that the constant $(p^{p-1}/2)^{1/p}$ is the best. Both (1.4), (1.5) extend to a Hilbert space setting and remain sharp even in the special case of the estimates for the Haar system with real coefficients.

In fact, all the inequalities above are valid under less restrictive assumption of differential subordination and can further be extended to the continuous-time setting. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is complete and equip it with a right-continuous filtration $(\mathcal{F}_t)_{t\ge0}$ such that $\mathcal{F}_0$ contains all the events of probability 0. Let X, Y be two adapted cadlag martingales taking values in $\mathcal{H}$ which, as we may and do assume from now on, is equal to ℓ^2. Following Wang [14], we say that Y is *differentially subordinate* to X, if the process $([X,X]_t - [Y,Y]_t)_{t\ge0}$ is nondecreasing and nonnegative as a function of t. Here $[X,Y] = \sum_{j=0}^\infty [X^j, Y^j]$, where X^j, Y^j stand for the j-th coordinates of X and Y, respectively, and $[X^j, Y^j]$ is the quadratic covariance process of X^j and Y^j (see e.g. Dellacherie and Meyer [6]). If we treat the discrete-time martingales $f = (f_k)_{k=0}^\infty$, $g = (g_k)_{k=0}^\infty$ as continuous-time processes (via $X_t = f_{\lfloor t\rfloor}$ and $Y_t = g_{\lfloor t\rfloor}$ for $t \ge 0$), then the above condition reads

$$|dg_k| \le |df_k| \qquad \text{for } k \ge 0,$$

which is the original definition of the differential subordination due to Burkholder [3]. Of course, this condition is satisfied by the martingale transforms studied above. Thus the following theorem (see [11], [13] and [14]) generalizes the previous inequalities (1.3), (1.4) and (1.5). We use the notation $||X||_p = \sup_t ||X_t||_p$ and $||X||_{p,\infty} = \sup_{\lambda>0} \lambda(\mathbb{P}(\sup_t |X_t| \geq \lambda))^{1/p}$, analogous to that of the discrete-time setting.

Theorem 1.1. *If Y is differentially subordinate to X, then*

$$||Y||_q \leq c_{p,q}||X||_p, \qquad 1 \leq p,\ q < \infty, \tag{1.6}$$

$$||Y||_{p,\infty} \leq \left(\frac{2}{\Gamma(p+1)}\right)^{1/p} ||X||_p, \qquad 1 \leq p \leq 2, \tag{1.7}$$

$$||Y||_{p,\infty} \leq \left(\frac{p^{p-1}}{2}\right)^{1/p} ||X||_p, \qquad 2 \leq p < \infty,$$

and the inequalities are sharp.

There is a natural and interesting question about the best constants in the corresponding weak type (p, q) estimates for the Haar system and the extension of these bounds to continuous-time differentially subordinated martingales. The purpose of this paper is to give a full answer to this question. Let

$$C_{p,q} = \begin{cases} \infty & \text{if } q > p, \\ 1 & \text{if } 1 \leq q \leq 2 \leq p < \infty, \\ \left(\frac{2}{\Gamma(p+1)}\right)^{1/p} & \text{if } 1 \leq q \leq p < 2, \\ 2^{1/p-2/q} q^{(p-1)/p} \left(\frac{p-q}{p-2}\right)^{(p-1)(p-q)/(pq)} & \text{if } 2 < q \leq p < \infty. \end{cases}$$

Our main result can be stated as follows.

Theorem 1.2. *Let X, Y be two Hilbert-space valued martingales such that Y is differentially subordinate to X. Then for any $1 \leq p,\ q < \infty$ we have*

$$||Y||_{q,\infty} \leq C_{p,q}||X||_p \tag{1.8}$$

and the constant $C_{p,q}$ is the best possible. It is already the best possible in the following estimate for the Haar system: for all n and all $a_k \in \mathbb{R}$, $\varepsilon_k \in \{-1, 1\}$, $k = 0, 1, 2, \ldots, n$,

$$\left|\left\{r \in [0,1] : \left|\sum_{k=0}^n \varepsilon_k a_k h_k(r)\right| \geq 1\right\}\right|^{1/q} \leq C_{p,q} \left|\left|\sum_{k=0}^n a_k h_k\right|\right|_p. \tag{1.9}$$

A few words about the proof and the organization of the paper. In the next section we establish the estimate (1.8). As we shall see, the heart of the matter lies in showing the inequality for $2 < q < p < \infty$; in the other cases the bound is either trivial or follows immediately from the results above. To deal with the non-trivial case, we shall exploit Burkholder's technique, which extracts the desired inequality from the existence of a certain special function on $\mathcal{H} \times \mathcal{H}$. Having completed the proof of (1.8), we turn to the optimality of the constants $C_{p,q}$ in the corresponding inequalities for the Haar system. This is done in Section 3. It turns out that this time there are two cases, $1 \leq q < p < 2$ and $2 < q < p < \infty$, which require some non-trivial reasoning. The final part of the paper contains the proofs of some technical facts needed in the earlier sections.

2. Proof of (1.8)

Of course, we may and do assume that $q \leq p$, since otherwise the estimate is trivial. Furthermore, the case $p = q$ follows from the works of Burkholder [3] and Suh [13] (see also Wang [14] and the author [11]). If $1 \leq q \leq 2 \leq p$, then

$$||Y||_{q,\infty} \leq ||Y||_q \leq ||Y||_2 \leq ||X||_2 \leq ||X||_p.$$

For $1 \leq q < p < 2$, the validity of (1.8) follows immediately from (1.7):

$$||Y||_{q,\infty} \leq ||Y||_{p,\infty} \leq \left(\frac{2}{\Gamma(p+1)}\right)^{1/p} ||X||_p.$$

Thus we are left with the case $2 < q < p < \infty$. As already mentioned in the Introduction, our approach rests on Burkholder's technique. To recall its main idea, fix a Borel function $V : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, a constant $c \in \mathbb{R}$ and suppose that we want to prove that

$$\mathbb{E}V(X_t, Y_t) \leq c, \qquad t \geq 0, \tag{2.1}$$

for all martingales X, Y such that Y is differentially subordinate to X (typically, the martingales are also assumed to satisfy certain integrability assumptions which guarantee the existence of the expectation in (2.1)). The method translates this problem into that of finding a special function U which majorizes V and satisfies $\mathbb{E}U(X_t, Y_t) \leq c$ for all $t \geq 0$. Usually, the latter condition is usually checked by proving that $(U(X_s, Y_s))_{s \geq 0}$ is a supermartingale such that $U(X_0, Y_0) \leq c$ almost surely: see [5] and [14]. However, in this paper we shall verify this condition directly.

At the first sight, this approach is not directly applicable here, since the inequality (1.8) is not of the form (2.1). To overcome this difficulty, we shall prove a larger family of related estimates, stated below in a separate theorem.

Theorem 2.1. *Let X, Y be two Hilbert-space valued martingales such that Y is differentially subordinate to X. Then for any $2 < p < \infty$ and any $\gamma \in [0, 1 - 2/p]$, we have*

$$\mathbb{P}(|Y_t| \geq 1) \leq \frac{p^{p-1}(1-\gamma)^p}{2}\mathbb{E}|X_t|^p + \frac{1}{4}\frac{\gamma^p p^{p-1}}{(p-2)^{p-1}}, \qquad t \geq 0. \tag{2.2}$$

The constant $\gamma^p p^{p-1}/(4(p-2)^{p-1})$ is the best possible, even in the following estimate for the Haar system: for all n and all $a_k \in \mathbb{R}$, $\varepsilon_k \in \{-1, 1\}$, $k = 0, 1, 2, \ldots, n$,

$$\left|\left\{r : \left|\sum_{k=0}^n \varepsilon_k a_k h_k(r)\right| \geq 1\right\}\right| \leq \frac{p^{p-1}(1-\gamma)^p}{2}\left|\left|\sum_{k=0}^n a_k h_k\right|\right|_p^p + \frac{1}{4}\frac{\gamma^p p^{p-1}}{(p-2)^{p-1}}. \tag{2.3}$$

Having proved (2.2), we shall deduce (1.8) simply by picking the optimal value of γ. Observe that the estimates of Theorem 2.1 are of the appropriate form and hence Burkholder's method can be used. Hence, introduce $V_{p,\gamma} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ by

$$V_{p,\gamma}(x, y) = 1_{\{|y| \geq 1\}} - \frac{p^{p-1}(1-\gamma)^p}{2}|x|^p.$$

To define the corresponding special function $U_{p,\gamma}$, let us first study an auxiliary object: a function $u_\infty : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ given by

$$u_\infty(x, y) = \begin{cases} 0 & \text{if } |x| + |y| < 1, \\ (|y| - 1)^2 - |x|^2 & \text{if } |x| + |y| \geq 1. \end{cases}$$

We shall need the following properties of this function.

Lemma 2.2. *(i) There is an absolute constant $A>0$ such that for all x, $y \in \mathcal{H}$,*

$$u_\infty(x,y) \leq A(|x|^2+|y|^2+1). \tag{2.4}$$

(ii) For all x, $y \in \mathcal{H}$ we have

$$u_\infty(x,y) \leq (|y|-1)^2-|x|^2. \tag{2.5}$$

(iii) If x, y, h, $k \in \mathcal{H}$ satisfy

$$|x|+|y| \leq 1, \quad |x+h|+|y+k| \geq 1 \quad \text{and} \quad |k| \leq |h|, \tag{2.6}$$

then $u_\infty(x+h,y+k) \leq 0$.

(iv) If x, $y \in \mathcal{H}$ satisfy $|y| \leq |x|$, then $u_\infty(x,y) \leq 0$.

Proof. (i), (ii) This follows immediately from the formula for u_∞.

(iii) The desired inequality is equivalent to $||y+k|-1| \leq |x+h|$. We have that $1-|y+k| \leq |x+h|$, which is the middle bound in (2.6). Furthermore, using this assumption and the triangle inequality, we see that

$$|y+k|-1 \leq |y|+|k|-1 \leq -|x|+|h| \leq |x+h|$$

and we are done.

(iv) The estimate is trivial if $|x|+|y| \leq 1$, while for remaining x, y,

$$u_\infty(x,y) = (|y|+|x|-1)(|y|-|x|-1) \leq 0. \quad \square$$

The key fact about u_∞ is described in the following lemma.

Lemma 2.3. *Suppose that martingales X, Y are bounded in L^2 and Y is differentially subordinate to X. Then for any $t \geq 0$,*

$$\mathbb{E}u_\infty(X_t,Y_t) \leq 0. \tag{2.7}$$

Proof. By (2.4), we see that the random variable $u_\infty(X_t,Y_t)$ is integrable. Introduce the stopping time $\tau = \inf\{s \geq 0 : |X_s|+|Y_s| > 1\}$. We will show the following three statements:

$$\mathbb{E}u_\infty(X_t,Y_t)1_{\{|X_0|+|Y_0|>1\}} \leq \mathbb{E}u_\infty(X_0,Y_0)1_{\{|X_0|+|Y_0|>1\}}, \tag{2.8}$$

$$u_\infty(X_t,Y_t) = u_\infty(X_0,Y_0) = 0 \quad \text{on } \{|X_0|+|Y_0| \leq 1,\, \tau > t\} \tag{2.9}$$

and

$$\mathbb{E}u_\infty(X_t,Y_t)1_{\{|X_0|+|Y_0|\leq 1,\, \tau \leq t\}} \leq \mathbb{E}u_\infty(X_0,Y_0)1_{\{|X_0|+|Y_0|\leq 1,\, \tau\leq t\}}. \tag{2.10}$$

These three facts yield the claim: indeed, they give $\mathbb{E}u_\infty(X_t,Y_t) \leq \mathbb{E}u_\infty(X_0,Y_0)$ and it suffices to note that $u_\infty(X_0,Y_0) \leq 0$, in view of the differential subordination and part (iv) of Lemma 2.2.

To prove (2.8), use (2.5) to get

$$\mathbb{E}\big[u_\infty(X_t,Y_t)|\mathcal{F}_0\big] \leq \mathbb{E}\big[|Y_t|^2-|X_t|^2|\mathcal{F}_0\big] - 2\mathbb{E}(|Y_t||\mathcal{F}_0)+1.$$

Of course, $\mathbb{E}(|Y_t||\mathcal{F}_0) \geq |Y_0|$. Moreover,

$$\mathbb{E}\big[(|Y_t|^2-|X_t|^2)-(|Y_0|^2-|X_0|^2)|\mathcal{F}_0\big] = \mathbb{E}\big[([Y,Y]_t-[X,X]_t)-([Y,Y]_0-[X,X]_0)|\mathcal{F}_0\big]$$

is nonpositive, due to the differential subordination. Consequently, on the set $\{|X_0|+|Y_0|>1\}$,

$$\mathbb{E}\big[u_\infty(X_t,Y_t)|\mathcal{F}_0\big] \leq |Y_0|^2-|X_0|^2-2|Y_0|+1 = u_\infty(X_0,Y_0),$$

which yields (2.8). The condition (2.9) is obvious, by the definition of u_∞ and τ. To get (2.10), we proceed as previously: on the set $\{|X_0|+|Y_0|\leq 1,\, \tau\leq t\}$ we have, by (2.5),

$$\begin{aligned}\mathbb{E}\big[u_\infty(X_t,Y_t)|\mathcal{F}_\tau\big] &= \mathbb{E}\big[|Y_t|^2-|X_t|^2|\mathcal{F}_\tau\big]-2\mathbb{E}(|Y_t||\mathcal{F}_\tau)+1\\ &\leq |Y_\tau|^2-|X_\tau|^2-2|Y_\tau|+1\\ &= u_\infty(X_\tau,Y_\tau).\end{aligned}$$

Now use part (iii) of Lemma 2.2 with $x=X_{\tau-}$, $y=Y_{\tau-}$, $h=\Delta X_\tau$ and $k=\Delta Y_\tau$: the first two conditions in (2.6) follow from the definition of τ, while the third one, $|\Delta Y_\tau|\leq|\Delta X_\tau|$, is due to the differential subordination. Thus, $u_\infty(X_\tau,Y_\tau)\leq 0 = u_\infty(X_0,Y_0)$ and the proof is complete. □

Introduce the auxiliary parameters

$$a=a_{p,\gamma}=\frac{\gamma}{(1-\gamma)(p-2)},\qquad b=b_{p,\gamma}=1-\frac{1}{p(1-\gamma)}. \tag{2.11}$$

It is easy to see that $a\leq b$. Next, define $k=k_{p,\gamma}:[0,\infty)\to[0,\infty)$ by

$$k(r)=\frac14 p^p(p-1)^{2-p}(p-2)(1-\gamma)^3(\gamma+(1-\gamma)r)^{p-3}r^2\,1_{[a,b]}(r)$$

and introduce $R_{p,\gamma}:\mathcal{H}\times\mathcal{H}\to\mathbb{R}$ by the formula

$$R_{p,\gamma}(x,y)=\frac14\frac{\gamma^{p-2}(1-\gamma)^2p^p}{(p-2)^{p-2}}(|y|^2-|x|^2)+\frac14\frac{\gamma^p p^{p-1}}{(p-2)^{p-1}}.$$

The special function $U_{p,\gamma}$ corresponding to (2.2) is given by

$$U_{p,\gamma}(x,y)=\int_0^\infty k(r)u_\infty(x/r,y/r)\mathrm{d}r+R_{p,\gamma}(x,y). \tag{2.12}$$

In the two lemmas below, we provide the explicit formula for $U_{p,\gamma}$ and prove the majorization $U_{p,\gamma}\geq V_{p,\gamma}$.

Lemma 2.4. *If $|x|+|y|<a$, then*

$$U_{p,\gamma}(x,y)=\frac14\frac{\gamma^{p-2}(1-\gamma)^2p^p}{(p-2)^{p-2}}(|y|^2-|x|^2)+\frac14\frac{\gamma^p p^{p-1}}{(p-2)^{p-1}}. \tag{2.13}$$

If $a\leq|x|+|y|\leq b$, then

$$\begin{aligned}&U_{p,\gamma}(x,y)\\ &=\frac12\left(\frac{p}{p-1}\right)^{p-1}\big(\gamma+(1-\gamma)(|y|-(p-1)|x|)\big)\big(\gamma+(1-\gamma)(|x|+|y|)\big)^{p-1}.\end{aligned} \tag{2.14}$$

Finally, if $|x|+|y|>b$, then

$$U_{p,\gamma}(x,y)=\frac{p^2}{4}\left[\left((1-\gamma)|y|-1-\gamma+\frac{2}{p}\right)^2-(1-\gamma)^2|x|^2+\frac{p-2}{p^3}\right]. \tag{2.15}$$

Lemma 2.5. *For all x, $y\in\mathcal{H}$ we have*

$$U_{p,\gamma}(x,y)\geq V_{p,\gamma}(x,y). \tag{2.16}$$

The proofs of these two statements are quite involved, so for the sake of clarity let us postpone them to Section 4 and proceed with the estimates of Theorem 2.1.

Proof of (2.2). We may assume that $\mathbb{E}|X_t|^p < \infty$, since otherwise the estimate is obvious. Thus the martingale $(X_s)_{s\leq t}$ is bounded in L^p and hence, by (1.6), so is $(Y_s)_{s\leq t}$. Furthermore, again by (1.6), we have $\mathbb{E}|Y_t|^2 \leq \mathbb{E}|X_t|^2$, which implies

$$\mathbb{E}R_{p,\gamma}(X_t, Y_t) \leq \frac{1}{4}\frac{\gamma^p p^{p-1}}{(p-2)^{p-1}}. \tag{2.17}$$

Next, by Lemma 2.3 and Fubini's theorem,

$$\mathbb{E}\int_0^\infty k(r)u_\infty(X_t/r, Y_t/r)\mathrm{d}r \leq \int_0^\infty k(r)\mathbb{E}u_\infty(X_t/r, Y_t/r)\mathrm{d}r \leq 0, \tag{2.18}$$

because Y/r is differentially subordinate to X/r for any $r > 0$. To see that Fubini's theorem is applicable, note that (2.4) gives the existence of a constant $\overline{A}$ depending only on p and γ such that

$$\left|\int_0^\infty k(r)u_\infty(x/r, y/r)\mathrm{d}r\right| \leq \overline{A}(|x|^2 + |y|^2 + 1)$$

for all x, y. Therefore, since X_t, $Y_t \in L^p$, we see that (2.18) holds true. Adding this bound to (2.17) gives

$$\mathbb{E}U_{p,\gamma}(X_t, Y_t) \leq \frac{1}{4}\frac{\gamma^p p^{p-1}}{(p-2)^{p-1}},$$

and the use of the majorization from Lemma 2.5 yields the claim. □

Proof of (1.8). Of course, we may assume that $||X||_p \neq 0$. First we use a well-known stopping time argument to strengthen (2.2) to a maximal weak-type bound

$$\mathbb{P}(\sup_s |Y_s| \geq 1) \leq \frac{p^{p-1}(1-\gamma)^p}{2}||X||_p^p + \frac{1}{4}\frac{\gamma^p p^{p-1}}{(p-2)^{p-1}}. \tag{2.19}$$

To do this, fix $\varepsilon > 0$ and let $\tau = \inf\{t : |Y_t| \geq 1 - \varepsilon\}$. We have

$$\{\sup_s |Y_s| \geq 1\} \subseteq \big\{|Y_t| \geq 1 - \varepsilon \text{ for some } t\big\} = \bigcup_{t\geq 0}\{|Y_{\tau\wedge t}| \geq 1 - \varepsilon\}.$$

The events $\{|Y_{\tau\wedge t}| \geq 1 - \varepsilon\}$ are non-decreasing. In consequence, applying (2.2) to a new pair $(X_{\tau\wedge t}/(1-\varepsilon))_{t\geq 0}$, $(Y_{\tau\wedge t}/(1-\varepsilon))_{t\geq 0}$ (for which the differential subordination is still valid), we obtain

$$\begin{aligned}
\mathbb{P}(\sup_s |Y_s| \geq 1) &\leq \lim_{t\to\infty} \mathbb{P}(|Y_{\tau\wedge t}| \geq 1 - \varepsilon) \\
&\leq \limsup_{t\to\infty} \frac{p^{p-1}(1-\gamma)^p}{2(1-\varepsilon)^p}\mathbb{E}|X_{\tau\wedge t}|^p + \frac{1}{4}\frac{\gamma^p p^{p-1}}{(p-2)^{p-1}} \\
&\leq \frac{p^{p-1}(1-\gamma)^p}{2(1-\varepsilon)^p}||X||_p^p + \frac{1}{4}\frac{\gamma^p p^{p-1}}{(p-2)^{p-1}}.
\end{aligned}$$

Since ε was arbitrary, (2.19) follows. Next, assume that $||X||_p \leq 1/2$ and put

$$\gamma = \Big(1 + (p-2)^{-1}(2||X||_p^p)^{-1/(p-1)}\Big)^{-1}.$$

Then $\gamma \leq 1 - 2/p$ and

$$1 - \gamma = \Big(1 + (p-2)(2||X||_p^p)^{1/(p-1)}\Big)^{-1}.$$

Plugging this into (2.19) yields

$$\mathbb{P}(\sup_s |Y_s| \geq 1) \leq \frac{p^{p-1}||X||_p^p}{2\left(1+(p-2)(2||X||_p^p)^{1/(p-1)}\right)^{p-1}} \tag{2.20}$$
$$= \frac{p^{p-1}||X||_p^{p-q}}{2\left(1+(p-2)(2||X||_p^p)^{1/(p-1)}\right)^{p-1}} \cdot ||X||_p^q.$$

To analyze the factor in front of $||X||_p^q$, we define the function $G:(0,\infty)\to\mathbb{R}$ by

$$G(t) = \frac{p^{p-1}t^{1-q/p}}{2\left(1+(p-2)(2t)^{1/(p-1)}\right)^{p-1}}.$$

A straightforward analysis shows that the maximum of G is equal to $C_{p,q}^q$. Thus,

$$\left(\mathbb{P}(\sup_s |Y_s| \geq 1)\right)^{1/q} \leq G(||X||_p^p)^{1/q}||X||_p \leq C_{p,q}||X||_p. \tag{2.21}$$

We have proved this under the assumption $||X||_p \leq 1/2$, but this is valid for all X. Indeed, suppose that $||X||_p > 1/2$ and use (1.7) to get

$$\mathbb{P}(\sup_s |Y_s| \geq 1) \leq ||X||_2^2 \leq ||X||_p^2 = \left[C_{p,q}^{-q}||X||_p^{2-q}\right] \cdot C_{p,q}^q ||X||_p^q.$$

However, the expression in the square brackets is smaller than

$$C_{p,q}^{-q}2^{q-2} = \left[\frac{2}{q}\left(\frac{p-2}{p-q}\right)^{(p-q)/q}\right]^{q(p-1)/p}$$

and since the function $x \mapsto (1+1/x)^x$ is increasing on $(0,\infty)$, we have

$$\frac{2}{q}\left(\frac{p-2}{p-q}\right)^{(p-q)/q} = \frac{2}{q}\left(1+\frac{q-2}{p-q}\right)^{(p-q)/q} < \frac{2}{q}e^{1-2/q} < 1.$$

Hence (2.21) holds, and it yields (1.8) by standard homogenization. □

3. Sharpness

In this section we shall prove that $C_{p,q}$ is optimal in (1.9) and that the constant $\gamma^p p^{p-1}/(4(p-2)^{p-1})$ is the best in (2.3). Let us first consider the trivial cases. It is clear that for $p < q$ the estimate (1.9) does not hold with any finite constant. Otherwise, by interpolation, we would have

$$\left|\left|\sum_{k=0}^n \varepsilon_k a_k h_k\right|\right|_{(p+q)/2} \leq c\left|\left|\sum_{k=0}^n a_k h_k\right|\right|_p$$

with some absolute $c < \infty$, which is impossible even for $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = \ldots = 1$. Next, it is obvious that the choice $C_{p,q} = 1$ is optimal for $q \leq 2 \leq p$, simply by taking $a_0 = 1$, $a_1 = a_2 = \ldots = 0$ and $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = \ldots = 1$. Thus we are left with the cases $1 \leq q \leq p < 2$ and $2 < q \leq p < \infty$, and these will be studied separately.

3.1. **Sharpness of** (1.9) **for** $1 \le q \le p < 2$**.** . We shall prove that for any $\varepsilon > 0$ there are sequences $(\varepsilon_k)_{k\ge 0}$, $(a_k)_{k\ge 0}$ and a nonnegative integer n such that

$$\left|\left\{r : \left|\sum_{k=0}^{n} \varepsilon_k a_k h_k(r)\right| \ge 1\right\}\right| = 1 \quad \text{and} \quad \left|\left|\sum_{k=0}^{n} a_k h_k\right|\right|_p^p \le \frac{\Gamma(p+1)}{2} + \varepsilon. \tag{3.1}$$

This will be accomplished by studying a corresponding boundary value problem given by (3.3) below; see Section 11 in [3] or Section 5 in [4] for related reasoning in the martingale setting.

For any $(x, y) \in \mathbb{R}^2$, let $\mathcal{M}(x, y)$ denote the class which consists of all functions of the form $\varphi = x + \sum_{k=1}^n a_k h_k$ for some n and real $a_1, a_2, \ldots, a_n$, which satisfy the following condition: there is a sequence $(\varepsilon_k)_{k=1}^n$ of signs such that

$$\left|y + \sum_{k=1}^{n} \varepsilon_k a_k h_k(r)\right| \ge 1 \quad \text{for all } r \in [0, 1]. \tag{3.2}$$

Of course, the class $\mathcal{M}(x, y)$ is nonempty for each $(x, y) \in \mathbb{R}^2$. Consider the function $W_p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$W_p(x, y) = \inf \left\{ ||\varphi||_p^p : \varphi \in \mathcal{M}(x, y) \right\}. \tag{3.3}$$

Lemma 3.1. *The function W_p has the following properties.*

(P1) If $|y| \ge 1$, then $W_p(x, y) = |x|^p$ for all x.

(P2) The function W_p is convex along the lines of slope ± 1.

Proof. To show (P1), apply Jensen's inequality to obtain $||\varphi||_p^p \ge |x|^p$ for any $\varphi \in \mathcal{M}(x, y)$. This gives $W_p(x, y) \ge |x|^p$ and the reverse bound follows from the fact that $\varphi \equiv x$ belongs to $\mathcal{M}(x, y)$. The property (P2) is a consequence of the so-called "splicing" argument (see e.g. page 77 in Burkholder [4]). To be more precise, fix a line L of slope 1, a point (x, y) lying on it and a positive number d. Pick two functions $\varphi^\pm \in \mathcal{M}(x \pm d, y \pm d)$ and splice them together using the formula

$$\varphi(r) = \begin{cases} \varphi^-(2r) & \text{if } r < 1/2, \\ \varphi^+(2r) & \text{if } r \ge 1/2. \end{cases} \tag{3.4}$$

It is evident from the structure of the Haar system that the splice φ is given by the finite Haar series $x - dh_1 + \sum_{k=2}^n a_k h_k$ and each number a_k coincides with an appropriate coefficient of φ^- or φ^+, depending on whether the support of h_k is contained in the left or the right half of the interval $[0, 1]$. In addition, it is clear that (3.2) is satisfied and hence $\varphi \in \mathcal{M}(x, y)$. Consequently, we have

$$W_p(x, y) \le ||\varphi||_p^p = \frac{1}{2}\left|\left|\varphi^-\right|\right|_p^p + \frac{1}{2}\left|\left|\varphi^+\right|\right|_p^p,$$

and taking infimum over all φ^-, φ^+ yields

$$W_p(x, y) \le (W_p(x - d, y - d) + W_p(x + d, y + d))/2.$$

Since x, y, and d were arbitrary, W_p is midpoint convex along L. Analogous arguments lead to the midpoint convexity of W_p along the lines of slope -1. However, it is not difficult to see that W_p is locally bounded from above (for example, use (P1) and the midpoint convexity just established). This proves (P2). □

Using the above lemma we shall show that $W_p(1/2,1/2)\le \Gamma(p+1)/2$, thus proving (3.1) (with $a_0=1/2$ and $\varepsilon_0=1$). To do this, observe that by (P2) and then by (P1),

$$W_p(1/2,1/2)\le \frac12 W_p(0,1)+\frac12 W_p(1,0)=\frac12 W_p(1,0). \tag{3.5}$$

Next, fix large integers K, N and put $\delta=K/(2N)$. For any $n=0,1,2,\ldots$ we have, by (P2),

$$\begin{aligned} W_p(1+2n\delta,0) &\le \frac{\delta}{1+\delta}W_p(2n\delta,1)+\frac{1}{1+\delta}W_p(1+2n\delta+\delta,-\delta)\\ &\le \frac{\delta}{1+\delta}W_p(2n\delta,1)+\frac{\delta}{1+\delta}W_p(2n\delta+2\delta,-1)\\ &\quad+\frac{1-\delta}{1+\delta}W_p(1+2n\delta+2\delta,0). \end{aligned} \tag{3.6}$$

However, $W_p(2n\delta,1)=(2n\delta)^p\le ((2n+2)\delta)^p=W_p((2n+2)\delta,-1)$ in view of (P1). Consequently, multiplying (3.6) throughout by $((1-\delta)/(1+\delta))^n$ gives

$$\begin{aligned} &\left(\frac{1-\delta}{1+\delta}\right)^n W_p(1+2n\delta,0)\\ &\qquad\le \frac{2\delta}{1+\delta}\left(\frac{1-\delta}{1+\delta}\right)^n((2n+2)\delta)^p+\left(\frac{1-\delta}{1+\delta}\right)^{n+1}W_p(1+2(n+1)\delta,0). \end{aligned}$$

Let us write these estimates for $n=0,1,2,\ldots,N-1$ and sum them. We obtain

$$W_p(1,0)\le \frac{2\delta}{1+\delta}\sum_{n=0}^{N-1}\left(\frac{1-\delta}{1+\delta}\right)^n((2n+2)\delta)^p+\left(\frac{1-\delta}{1+\delta}\right)^N W_p(1+2N\delta,0).$$

Finally, recall that $\delta=K/(2N)$ and note that by (P1) and (P2),

$$W_p(1+2N\delta,0)\le (W_p(2N\delta,-1)+W_p(2N\delta+2,1))/2=(K^p+(K+2)^p)/2.$$

Thus, if we keep K fixed and take N sufficiently large, then the upper bound for $W_p(1,0)$ we have just derived can be made arbitrarily close to

$$\int_0^K e^{-t}t^p\mathrm{d}t+e^{-K}(K^p+(K+2)^p).$$

Since we put no restrictions on K, we get $W_p(1,0)\le\Gamma(p+1)$ and the use of (3.5) completes the proof.

3.2. **The case** $2<q\le p<\infty$**.** Here the reasoning is more involved. Previously, the sequences $(a_k)_{k=0}^n$, $(\varepsilon_k)_{k=0}^n$ for which both sides of (1.9) were almost equal, satisfied the bound $|\sum_{k=0}^n\varepsilon_k a_k h_k|\ge 1$: see (3.1). Here this will be no longer true and we need to control both the size of $\sum_{k=0}^n a_kh_k$ and the measure of the set $\{r\in[0,1]:|\sum_{k=0}^n\varepsilon_ka_kh_k(r)|\ge 1\}$. This will slightly complicate the objects which were introduced in the previous subsection.

We start with an appropriate extension of the class $\mathcal{M}$ which this time will depend on three parameters. Namely, for any $(x,y)\in\mathbb{R}^2$ and $t\in[0,1]$, let $\mathcal{M}(x,y,t)$ consist of all functions of the form $\varphi=x+\sum_{k=1}^n a_kh_k$ for some n

and real $a_1, a_2, \ldots, a_n$, which satisfy the following condition: there is a sequence $(\varepsilon_k)_{k=1}^n$ of signs such that

$$\left|\left\{r \in [0,1] : \left|y + \sum_{k=1}^n \varepsilon_k a_k h_k(r)\right| \geq 1\right\}\right| \geq t. \tag{3.7}$$

Observe that the class $\mathcal{M}(x,y)$ studied previously coincides with $\mathcal{M}(x,y,1)$ in the new notation. Next, we introduce the function $W_p : \mathbb{R} \times \mathbb{R} \times [0,1] \to \mathbb{R}$ by

$$W_p(x,y,t) = \inf\left\{||\varphi||_p^p : \varphi \in \mathcal{M}(x,y,t)\right\}.$$

Let us state the analogue of Lemma 3.1.

Lemma 3.2. *The function W_p has the following properties.*

(P1') If $|y| \geq 1$ or $t = 0$, then $W_p(x,y,t) = |x|^p$.

(P2') For any x, $y \in \mathbb{R}$, $v \in \mathbb{R}$ and $\varepsilon \in \{-1,1\}$, the function $G = G_{x,y,v}$, given on $[0,1]$ by the formula $G(t) = W_p(x+tv, y+t\varepsilon v, t)$, is convex.

Proof. To establish (P1'), we use Jensen's inequality to get that $W_p(x,y,t) \geq |x|^p$ and obtain the reverse bound by noting that $\varphi \equiv x$ belongs to $\mathcal{M}(x,y,t)$ when $|y| \geq 1$ or $t = 0$. To show (P2'), we apply the splicing argument to prove that $G_{x,y,v}$ is midpoint convex and then deduce its true convexity using the local boundedness of W_p (which is obvious). The proof goes along the same lines, we only need to make the following simple observation: if we splice $\varphi^\pm \in \mathcal{M}(x \pm dv, y \pm \varepsilon dv, t \pm d)$ according to (3.4), then the coefficients $a_1, a_2, \ldots, a_n$ of the function φ we obtain satisfy (3.7) for an appropriate sequence $(\varepsilon_k)_{k=1}^n$ of signs. □

Recall the parameters a and b defined in (2.11). We shall need the following further properties of W_p.

Lemma 3.3. *(i) If $t \leq 1/2$, then*

$$W_p(a/2, a/2, t) \leq \frac{1}{2} W_p(0, a, 2t) + \frac{1}{2}\left(\frac{\gamma}{(1-\gamma)(p-2)}\right)^p. \tag{3.8}$$

(ii) We have

$$W_p(0, b, 1/2) \leq (p(1-\gamma))^{-p}. \tag{3.9}$$

(iii) Assume that $\gamma \in (0, 1-2/p)$, $\delta \in (0, (p-1)^{-1})$, $y > 0$ and put

$$\lambda_{p,\delta} = \frac{1-(p-1)\delta}{1+(p-1)\delta}.$$

Then for any $t \in [0, \lambda_{p,q}]$,

$$W_p(0,y,t) \leq \lambda_{p,\delta} W_p\left(0, y + 2\left(y + \frac{\gamma}{1-\gamma}\right)\delta, t\lambda_{p,\delta}^{-1}\right) + (1-\lambda_{p,\delta})\left[\frac{\gamma + (1-\gamma)y}{(p-1)(1-\gamma)}\right]^p.$$

We postpone the proof of this lemma to Section 4 and continue with the sharpness of (1.9) and (2.3).

Lemma 3.4. *For any $\gamma \in (0, 1-2/p)$ we have*

$$W_p\left(a/2, a/2, \frac{1}{4}\left(\frac{p\gamma}{p-2}\right)^{p-1}\right) \leq \frac{1}{2}\left(\frac{\gamma}{(1-\gamma)(p-2)}\right)^{p-1}. \tag{3.10}$$

Proof. If we set $F(s,t) = W_p(0, s - \gamma/(1-\gamma), t)$, then the inequality from (iii) can be rewritten in a more convenient form

$$F(s,t) \le \lambda_{p,\delta} F(s(1+2\delta), t\lambda_{p,\delta}^{-1}) + (1-\lambda_{p,\delta}) \left(\frac{s}{p-1}\right)^p,$$

where $s = y + \gamma/(1-\gamma)$. Hence, by induction, we have

(3.11)

$$F(s,t) \le \lambda_{p,\delta}^n F(s(1+2\delta)^n, t\lambda_{p,\delta}^{-n}) + (1-\lambda_{p,\delta}) \left(\frac{s}{p-1}\right)^p \frac{[\lambda_{p,\delta}(1+2\delta)^p]^n - 1}{\lambda_{p,\delta}(1+2\delta)^p - 1}$$

for all $s > \gamma/(1-\gamma)$, $\delta \in (0, (p-1)^{-1})$, any positive integer n and any $t \le \lambda_{p,\delta}^n$. Fix a large integer n and let s, δ, t be given by

$$s = a + \frac{\gamma}{1-\gamma} = \frac{(p-1)\gamma}{(p-2)(1-\gamma)}, \qquad (1+2\delta)^n = \frac{b + \gamma/(1-\gamma)}{a + \gamma/(1-\gamma)} = \frac{p-2}{p\gamma}$$

and $t = \lambda_{p,\delta}^n/2$. Note that if n is sufficiently large, then $\delta < (p-1)^{-1}$, so we may insert these parameters into (3.11) and let n go to ∞. We have

$$\lim_{n\to\infty} \lambda_{p,\delta}^n = \lim_{n\to\infty} \left(1 - \frac{2(p-1)\delta}{1+(p-1)\delta}\right)^n = \left(\frac{p\gamma}{p-2}\right)^{p-1},$$

$$\lim_{n\to\infty} \frac{1-\lambda_{p,\delta}}{\lambda_{p,\delta}(1+2\delta)^p - 1} = \lim_{\delta\to 0} \frac{2(p-1)\delta}{(1-(p-1)\delta)(1+2\delta)^p - 1 - (p-1)\delta} = p-1$$

and, by (P2'), the function $W_p(0,a,\cdot) : (0,1) \to \mathbb{R}$ is continuous. Therefore, in the limit, (3.11) becomes

$$\begin{aligned} W_p &\left(0, a, \frac{1}{2}\left(\frac{p\gamma}{p-2}\right)^{p-1}\right) \\ &\le \left(\frac{p\gamma}{p-2}\right)^{p-1} W_p\left(0, b, \frac{1}{2}\right) + (p-1)\left(\frac{\gamma}{(1-\gamma)(p-2)}\right)^p \left(\frac{p-2}{p\gamma} - 1\right) \\ &\le \left(\frac{\gamma}{(p-2)}\right)^{p-1} (1-\gamma)^{-p} \left(1 - \frac{p-1}{p-2}\gamma\right), \end{aligned}$$

where in the latter passage we have exploited (3.9). Plugging this estimate into (3.8) gives (3.10). □

Now we can easily show the sharpness of (1.9) and (2.3). Let us start with the second estimate. Fix $\varepsilon > 0$ and $\gamma \in (0, 1-2/p)$. Then, by (3.10) and the definition of W_p, there are finite sequences $(a_k)_{k=0}^n$ of real numbers and $(\varepsilon_k)_{k=0}^n$ of signs such that $a_0 = a/2$, $\varepsilon_0 = 1$,

$$\alpha := \left\|\sum_{k=0}^n a_k h_k\right\|_p^p \le \frac{1}{2}\left(\frac{\gamma}{(1-\gamma)(p-2)}\right)^{p-1} + \varepsilon$$

and

$$\beta := \left|\left\{r \in [0,1] : \left|\sum_{k=0}^n \varepsilon_k a_k h_k(r)\right| \ge 1\right\}\right| \ge \frac{1}{4}\left(\frac{p\gamma}{p-2}\right)^{p-1}.$$

Therefore,

$$\beta - \frac{p^{p-1}(1-\gamma)^p}{2}\alpha \ge \frac{1}{4}\frac{\gamma^p p^{p-1}}{(p-2)^{p-1}} - \frac{p^{p-1}(1-\gamma)^p}{2}\varepsilon$$

and hence (2.3) is sharp, since ε was arbitrary. The case $\gamma \in \{0, 1-2/p\}$ follows easily by passing to the limit. To deal with (1.9), pick $\gamma = 1-q/p$ and observe that

$$\frac{\beta}{\alpha^{q/p}} \geq \frac{1}{4}\left(\frac{p-q}{p-2}\right)^{p-1} \cdot \left[\frac{1}{2}\left(\frac{p-q}{q(p-2)}\right)^{p-1} + \varepsilon\right]^{-q/p}.$$

However, the right-hand side converges to $C_{p,q}^q$ when $\varepsilon \to 0$. This completes the proof.

4. Proofs of technical lemmas

4.1. **Proof of Lemma 2.4.** It is easy to show the formula for $U_{p,\gamma}(x,y)$ when $|x|+|y|<a$: then for any $r \in [a,b]$ we have $|x/r|+|y/r|<1$ and hence the integral in (2.12) vanishes. Now, suppose that $a \leq |x|+|y| \leq b$. Rewrite the integral from (2.12) in the form $I_1+I_2+I_3$, where

$$I_1 = (|y|^2-|x|^2)\int_a^{|x|+|y|}(\gamma+(1-\gamma)r)^{p-3}\mathrm{d}r = (|y|^2-|x|^2)(f_1(|x|+|y|)-f_1(a)),$$
$$I_2 = -2|y|\int_a^{|x|+|y|}(\gamma+(1-\gamma)r)^{p-3}r\mathrm{d}r = -2|y|(f_2(|x|+|y|)-f_2(a)),$$
$$I_3 = \int_a^{|x|+|y|}(\gamma+(1-\gamma)r)^{p-3}r^2\mathrm{d}r = f_3(|x|+|y|)-f_3(a).$$

Here, using integration by parts,

$$f_1(s) = \frac{(\gamma+(1-\gamma)s)^{p-2}}{(p-2)(1-\gamma)},$$
$$f_2(s) = \frac{(\gamma+(1-\gamma)s)^{p-2}s}{(p-2)(1-\gamma)} - \frac{(\gamma+(1-\gamma)s)^{p-1}}{(p-1)(p-2)(1-\gamma)^2},$$
$$f_3(s) = \frac{(\gamma+(1-\gamma)s)^{p-2}s^2}{(p-2)(1-\gamma)} - \frac{(\gamma+(1-\gamma)s)^{p-1}\cdot 2s}{(p-1)(p-2)(1-\gamma)^2} + \frac{2(\gamma+(1-\gamma)s)^p}{p(p-1)(p-2)(1-\gamma)^3}.$$

After some calculations, we get that

$$(|y|^2-|x|^2)f_1(|x|+|y|) - 2|y|f_2(|x|+|y|) + f_3(|x|+|y|)$$

is equal to the right hand side of (2.14) and

$$(|y|^2-|x|^2)f_1(a) - 2|y|f_2(a) + f_3(a) = R_{p,\gamma}(x,y).$$

This proves the validity of (2.14). To check the last formula, we use the above computation and see that

$$\begin{aligned} U_{p,\gamma}(x,y) &= (|y|^2-|x|^2)f_1(b) - 2|y|f_2(b) + f_3(b) \\ &\quad - ((|y|^2-|x|^2)f_1(a) - 2|y|f_2(a) + f_3(a)) + R_{p,\gamma}(x,y) \\ &= (|y|^2-|x|^2)f_1(b) - 2|y|f_2(b) + f_3(b). \end{aligned}$$

It can be verified readily that the latter expression is precisely the right-hand side of (2.15).

4.2. **Proof of Lemma 2.5.** Of course, we may assume that $\mathcal{H}=\mathbb{R}$ and it suffices to prove the estimate for x, $y\geq 0$. If $x+y\geq b$, the bound takes the form

$$\frac{p^2}{4}\left[\left((1-\gamma)y-1+\gamma+\frac{2}{p}\right)^2-(1-\gamma)^2x^2+\frac{p-2}{p^3}\right]\geq 1_{\{y\geq 1\}}-\frac{p^{p-1}(1-\gamma)^p}{2}x^p,$$

which is true for all nonnegative x, y. Indeed, the left hand side, as a function of y, is decreasing on $[0,1-2(p(1-\gamma))^{-1})$ and increasing on $(1-2(p(1-\gamma))^{-1},\infty)$, so it suffices to check the inequality for $y=2(p(1-\gamma))^{-1}$ and $y=1$. In both cases the estimate is equivalent to

$$(p^2(1-\gamma)^2x^2)^{p/2}-1\geq \frac{p}{2}\big(p^2(1-\gamma)^2x^2-1\big),$$

which follows from the mean value property. If $a\leq x+y<b$, then (2.16) can be rewritten in the form $F((1-\gamma)x,\gamma+(1-\gamma)y)\geq 0$, where

$$F(r,s)=(s-(p-1)r)(r+s)^{p-1}+(p-1)^{p-1}r^p.$$

However, $F(r,s)\geq 0$ for all r, $s\geq 0$: indeed, we have $F(r,(p-2)r)=0$ and, by a standard analysis, $F(r,\cdot)$ is decreasing on $[0,(p-2)r)$ and increasing on $((p-2)r,\infty)$ for any fixed r. Finally, if $x+y<a$, then the left hand side of (2.16) increases as y increases, so it suffices to prove the bound for $y=0$. This is equivalent to

$$\big((1-\gamma)^2x^2(p-2)^2\big)^{p/2}-(\gamma^2)^{p/2}\geq \frac{p}{2}(\gamma^2)^{p/2-1}\left((1-\gamma)^2x^2(p-2)^2-\gamma^2\right),$$

which, as previously, is a consequence of the mean value property.

4.3. **Proof of Lemma 3.3.** The claim follows from (P1') and (P2'). Indeed,

$$W_p(a/2,a/2,t)\leq\frac{1}{2}W_p(0,a,2t)+\frac{1}{2}W_p(a,0,0)=\frac{1}{2}W_p(0,a,2t)+\frac{1}{2}\left(\frac{\gamma}{(1-\gamma)(p-2)}\right)^p$$

and

$$W_p(0,b,1/2)\leq\frac{1}{2}W_p(b-1,2b-1,0)+\frac{1}{2}W_p(1-b,1,1)=(1-b)^p=(p(1-\gamma))^{-p}.$$

To check the third estimate, let $\kappa=y+\gamma/(1-\gamma)$ and note that

$$\begin{aligned}W_p(0,y,t)&\leq\frac{(p-1)\delta}{(p-1)\delta+1}W_p\left(\frac{\kappa}{p-1},y-\frac{\kappa}{p-1},0\right)\\&\quad+\frac{1}{(p-1)\delta+1}W_p\big(-\kappa\delta,y+\kappa\delta,((p-1)\delta+1)t\big)\\&=\frac{(p-1)\delta}{(p-1)\delta+1}\cdot\left(\frac{\kappa}{p-1}\right)^p\\&\quad+\frac{1}{(p-1)\delta+1}W_p\big(-\kappa\delta,y+\kappa\delta,((p-1)\delta+1)t\big).\end{aligned}$$

Similarly,

$$\begin{aligned}W_p\big(-\kappa\delta, y+\kappa\delta, ((p-1)\delta+1)t\big) &\leq (p-1)\delta\cdot W_p\left(-\frac{\kappa}{p-1}, y-\frac{\kappa}{p-1}+2\kappa\delta, 0\right)\\ &\quad + (1-(p-1)\delta)\cdot W_p\left(0, y+2\kappa\delta, t\lambda_{p,\delta}^{-1}\right)\\ &= (p-1)\delta\left(\frac{\kappa}{p-1}\right)^p\\ &\quad + (1-(p-1)\delta)W_p\left(0, y+2\kappa\delta, t\lambda_{p,\delta}^{-1}\right).\end{aligned}$$

Combining these two estimates gives the desired bound.

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