A WEAK-L^∞ INEQUALITY FOR WEAKLY DOMINATED MARTINGALES WITH APPLICATIONS TO HAAR SHIFT OPERATORS

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Abstract. Let $f = (f_n)_{n\geq 0}$ and $g = (g_n)_{n\geq 0}$ be two real-Hilbert-space-valued martingales such that $(g_n)_{n\geq 0}$ is weakly dominated by $(f_n)_{n\geq 0}$. The paper contains the proof of the inequality

$$\|g\|_{W(\Omega)} \leq 6\|f\|_{L^\infty},$$

where W is the weak-L^∞ space introduced by Bennett, DeVore and Sharpley. As an application, a related estimate for Haar shift operators is established.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a non-atomic probability space, filtered by $(\mathcal{F}_n)_{n\geq 0}$, a nondecreasing family of sub-σ-fields of $\mathcal{F}$. Suppose that $f = (f_n)_{n\geq 0}$ and $g = (g_n)_{n\geq 0}$ are two adapted martingales taking values in some separable Hilbert space $(H, |\cdot|)$. Let $(df_n)_{n\geq 0}$, $(dg_n)_{n\geq 0}$ denote the difference sequences of f and g respectively, given by

$$df_0 = f_0, \quad \text{and} \quad df_n = f_n - f_{n-1} \quad \text{for } n \geq 1,$$

and similarly for $(dg_n)_{n\geq 0}$. Following Burkholder [3, 4, 5], we say that g is differentially subordinate to f, if for any nonnegative integer n we have

$$|dg_n| \leq |df_n|,$$

with probability 1. This domination principle implies many interesting estimates between f and g, which can be further applied in numerous problems of harmonic analysis (see [1, 8]). Furthermore, there is a technique developed by Burkholder, which can be used in the search of the best constants in these estimates. This approach led to the celebrated sharp strong-type inequality

$$(\mathbb{E}|g_n|^p)^{1/p} \leq \max\{p-1, (p-1)^{-1}\}(\mathbb{E}|f_n|^p)^{1/p}, \qquad n = 0, 1, 2, \ldots \tag{1.1}$$

for $1 < p < \infty$ (see [3]). In the boundary case $p = 1$ the above moment inequality does not hold with any finite constant, but we have the corresponding weak-type bound

$$\mathbb{P}(|g_n| \geq 1) \leq 2\mathbb{E}|f_n|, \qquad n = 0, 1, 2, \ldots. \tag{1.2}$$

in which the constant 2 is also optimal (see [3]). A similar phenomenon occurs in the other boundary case, for $p = \infty$: the strong-type estimate fails to hold, but one can establish a suitable weak-type counterpart. To describe this estimate

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precisely, we need to define the weak-L^∞ space W, which was originally introduced by Bennett, DeVore and Sharpley in [2]. For a given random variable h, let $h^* : (0,1] \to [0,\infty)$ stand for its decreasing rearrangement, defined by

$$h^*(t) = \inf\{\lambda \geq 0 : \mathbb{P}(|h| > \lambda) \leq t\}.$$

Then h^{**}, the maximal function of h^*, is given by the formula

$$h^{**}(t) = \frac{1}{t}\int_0^t h^*(s)\,\mathrm{d}s\,, \qquad t \in (0,1].$$

There is an alternative definition of h^{**}, namely,

$$h^{**}(t) = \sup\left\{\frac{1}{\mathbb{P}(E)}\int_E |h|\,\mathrm{d}\mathbb{P} \,:\, E \in \mathcal{F},\ \mathbb{P}(E) = t\right\}.$$

Now we can define the weak-L^∞ space W by

$$W(\Omega) = \left\{h \,:\, \|h\|_{W(\Omega)} = \sup_{t\in(0,1]}(h^{**}(t) - h^*(t)) < \infty\right\}.$$

The above definitions extend easily to the case in which $(\Omega, \mathcal{F}, \mathbb{P})$ is replaced by an arbitrary non-atomic measure space.

There are several reasons for which the weak-L^∞ class is meaningful. One of the crucial features comes from interpolation theory. Observe that the classical definition of $L^{p,\infty}$ does not allow for a convenient extension for $p = \infty$ (it is customary to define $L^{\infty,\infty} = L^\infty$) and hence there is no Marcinkiewicz interpolation theorem between L^1 and L^∞ for operators which are unbounded on L^∞. The space W resolves this issue. It strictly contains L^∞ and enjoys the appropriate interpolation property: if an operator A is bounded from L^1 to $L^{1,\infty}$ and from L^∞ to W, then it has an extension which is bounded on L^p spaces, $1 < p < \infty$. Furthermore, there are close connections between W and the space BMO. For more detailed discussion on W and its interplay with the interpolation theory, see [2].

Equipped with the above definition, we return to the martingale setup and state the appropriate weak-L^∞ bound. It was proved in [10] that if g is differentially subordinate to f, then we have

$$\|g\|_{W(\Omega)} \leq 2\|f\|_{L^\infty}$$

and the constant 2 cannot be improved.

Similarly, general martingale inequalities (i.e., strong- or weak-type) can be studied when one imposes less restrictive types of subordination on the processes. In this paper, we will be particularly interested in the so-called weak domination. Following Kwapień and Woyczyński [7], we say that g is weakly dominated by f, if for any nonnegative integer n and any nonnegative number a we have the estimate

$$\mathbb{E}\big[(|dg_n| - a)_+|\mathcal{F}_{n-1}\big] \leq \mathbb{E}\big[(|df_n| - a)_+|\mathcal{F}_{n-1}\big]$$

almost surely. By standard approximation, this is equivalent to saying that for any n and any convex increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(|df_n|) \in L^1$, we have the inequality $\mathbb{E}\big[\phi(|dg_n|)|\mathcal{F}_{n-1}\big] \leq \mathbb{E}\big[\phi(|df_n|)|\mathcal{F}_{n-1}\big]$ almost surely.

We will give two simple and natural examples in which the weak domination holds. First, observe that if g is differentially subordinate to f, then we have the pointwise bound $(|dg_n|-a)_+ \leq (|df_n|-a)_+$ and hence the weak domination is satisfied. The second example concerns the so-called tangency relation. Namely, two martingales f and g are said to be tangent, if for any $n=0,1,2,\ldots$ the conditional distributions of df_n and dg_n with respect to $\mathcal{F}_{n-1}$ coincide. Obviously, if f and g are tangent, then g is weakly dominated by f and f is weakly dominated by g.

Inequalities for weakly dominated martingales were studied by a number of authors. It should be emphasized here that while Burkholder's technique is still available in this context, it does not seem to lead to sharp estimates; the best one can hope for are just 'tight' inequalities involving constants not far from optimal.

Let us formulate here the results from [9]: for any $n=0,1,2,\ldots$ we have

$$\mathbb{P}(|g_n|\geq 1)\leq 2\sqrt{2}\mathbb{E}|f_n|,$$

and, for $1<p<\infty$,

$$\left(\mathbb{E}|g_n|^p\right)^{1/p}\leq 3\max\{p-1,(p-1)^{-1}\}\left(\mathbb{E}|f_n|^p\right)^{1/p}. \tag{1.3}$$

As in the case of the differential subordination, we can ask about the weak-type counterpart of (1.3) for $p=\infty$. The following statement answers this question.

Theorem 1.1. *If f, g are two real-Hilbert-space-valued martingales such that g is weakly dominated by f, then we have*

$$\|g\|_{W(\Omega)}\leq 6\|f\|_{L^\infty}. \tag{1.4}$$

This result will be established in the next section. Section 3 contains some applications of (1.4) to Haar shift operators, important objects in harmonic analysis.

2. Proof of Theorem 1.1

For the sake of clarity, we split the contents of this section into two parts.

2.1. Special functions and their properties.

Introduce the strip

$$S=\{(x,y)\in H\times H: |x|\leq 1\}.$$

For any $\lambda\geq 0$, we consider the functions V_λ and U_λ given on S by the formulas

$$V_\lambda(x,y)=(|y|-\lambda-6)\chi_{\{|y|>\lambda\}}$$

and

$$U_\lambda(x,y) = \begin{cases} 0 & \text{if } 3|x|+|y|\leq\lambda+3,\\ \frac{1}{6}(|y|-\lambda-3)^2+\frac{4}{3}(|y|-\lambda-3)^2\chi_{\{|y|\geq\lambda+3\}}-\frac{3}{2}|x|^2 & \text{if } 3|x|+|y|>\lambda+3.\end{cases}$$

In the two lemmas below, we study certain crucial properties of U_λ and V_λ.

Lemma 2.1. *For any $\lambda\geq 0$ we have the majorization $U_\lambda\geq V_\lambda$.*

Proof. We consider three cases. If $|y| \leq \lambda$, then $3|x| + |y| \leq \lambda + 3$, so $U_\lambda(x, y) = V_\lambda(x, y) = 0$. If $|y| > \lambda$ and $3|x| + |y| \leq \lambda + 3$, then we have $U_\lambda(x, y) = 0$ and $V_\lambda(x, y) = |y| - \lambda - 6 \leq -3$, so the majorization is also satisfied. Finally, if $|y| > \lambda$ and $3|x| + |y| > \lambda + 3$, then

$$\begin{aligned} U_\lambda(x, y) - V_\lambda(x, y) &\geq \frac{1}{6}(|y| - \lambda - 3)^2 - \frac{3}{2}|x|^2 - (|y| - \lambda - 6) \\ &= \frac{1}{6}[(|y| - \lambda - 6)^2 + 9(1 - |x|^2)] \geq 0. \end{aligned}$$

The proof is complete. □

The second property is a certain concavity-type condition on U_λ. To formulate it, we need to define A, $B : S \to H$ by

$$A(x, y) = \begin{cases} 0 & \text{if } 3|x| + |y| \leq \lambda + 3, \\ -3x & \text{if } 3|x| + |y| > \lambda + 3 \end{cases}$$

and

$$B(x, y) = \begin{cases} 0 & \text{if } 3|x| + |y| \leq \lambda + 3, \\ \frac{1}{3}(|y| - \lambda - 3)(1 + 8\chi_{\{|y| > \lambda + 3\}})y' & \text{if } 3|x| + |y| > \lambda + 3, \end{cases}$$

where $y' = y/|y|$ stands for the normalization of the vector y. Note that outside the curve $3|x| + |y| = \lambda + 3$, A and B coincide with the partial derivatives U_x, U_y. Furthermore, for $(x, y) \in S$, let $\phi_{x,y} : [0, \infty) \to [0, \infty)$ be given as follows: if $3|x| + |y| \leq \lambda + 3$, set

$$\phi_{x,y}(s) = \begin{cases} 0 & \text{if} \quad s \leq \lambda + 3 - |y|, \\ \frac{3}{2}(s - \lambda - 3 + |y|)^2 & \text{if} \quad s > \lambda + 3 - |y|. \end{cases}$$

On the other hand, if $3|x| + |y| > \lambda + 3$, we put $\phi_{x,y}(s) = \frac{3}{2}s^2$. Note that for each $(x, y) \in S$, $\phi_{x,y}$ is a nondecreasing convex function.

Lemma 2.2. *For any $(x, y) \in S$ and any h, $k \in H$ such that $|x + h| \leq 1$ we have*

$$U_\lambda(x + h, y + k) \leq U_\lambda(x, y) + A(x, y) \cdot h + B(x, y) \cdot k + \phi_{x,y}(|k|) - \phi_{x,y}(|h|). \quad (2.1)$$

(Here the dot '·' denotes the scalar product in H.)

Proof. The verification of (2.1) is quite elaborate and splits naturally into several separate parts.

Case I: $3|x| + |y| \leq \lambda + 3$ and $3|x + h| + |y + k| \leq \lambda + 3$. Then the desired estimate is equivalent to $\phi_{x,y}(|h|) \leq \phi_{x,y}(|k|)$. If $|h| \leq \lambda + 3 - |y|$, then $\phi_{x,y}(|h|) = 0$ and the estimate holds; if $|h| > \lambda + 3 - |y|$, then

$$\begin{aligned} |k| = |-y + y + k| &\geq |y| - |y + k| \\ &\geq |y| + 3|x + h| - \lambda - 3 \\ &\geq |y| + 3|h| - 3|x| - \lambda - 3 \\ &\geq |h| + 0 + \lambda + 3 - |y| - 3|x| \geq |h|, \end{aligned}$$

so the estimate $\phi_{x,y}(|h|) \leq \phi_{x,y}(|k|)$ is also true.

Case II: $3|x|+|y|\leq\lambda+3$ *and* $|y+k|\geq\lambda+3$. Under these assumptions, the inequality (2.1) becomes

$$\frac{3}{2}(|y+k|-\lambda-3)^2-\phi_{x,y}(|k|)\leq\frac{3}{2}|x+h|^2-\phi_{x,y}(|h|).$$

But $|k|\geq|y+k|-|y|>\lambda+3-|y|$, which implies that the left-hand side above is nonpositive; thus it is enough to show that $\phi_{x,y}(|h|)\leq\frac{3}{2}|x+h|^2$. This is obvious if $|h|\leq\lambda+3-|y|$, since then $\phi_{x,y}(|h|)=0$. On the other hand, if $|h|>\lambda+3-|y|$, then we may write

$$\begin{aligned}\phi_{x,y}(|h|)&=\frac{3}{2}(|h|-\lambda-3+|y|)^2\\&\leq\frac{3}{2}(|x+h|+|x|-\lambda-3+|y|)^2\\&=\frac{3}{2}|x+h|^2+\frac{3}{2}(|x|-\lambda-3+|y|)\Big(|x|+2|x+h|-\lambda-3+|y|\Big)\\&\leq\frac{3}{2}|x+h|^2,\end{aligned}$$

where the last passage is due to the inequalities $|x|-\lambda-3+|y|\leq 0$ (see the assumptions of Case II) and $|x|+2|x+h|-\lambda-3+|y|\geq|h|-\lambda-3+|y|\geq 0$.

Case III: $3|x|+|y|\leq\lambda+3$ *and* $|y+k|<\lambda+3<3|x+h|+|y+k|$. For such parameters, the assertion takes the form

$$\frac{1}{6}(|y+k|-\lambda-3)^2-\frac{3}{2}|x+h|^2\leq\phi_{x,y}(|k|)-\phi_{x,y}(|h|).\tag{2.2}$$

If $|h|\leq\lambda+3-|y|$, then $\phi_{x,y}(|h|)=0$, so $\phi_{x,y}(|k|)-\phi_{x,y}(|h|)\geq 0$ and it is enough to prove that

$$\frac{1}{6}(|y+k|-\lambda-3)^2-\frac{3}{2}|x+h|^2\leq 0.\tag{2.3}$$

However, by the assumptions of the case we have $-3|x+h|<|y+k|-\lambda-3<0$, which implies $9|x+h|^2>(|y+k|-\lambda-3)^2$, an estimate equivalent to (2.3). On the other hand, $|h|>\lambda+3-|y|$, then $|h|>|x|$ (again, apply the first assumption of the case) and hence, adding one dimension to H if necessary, there exists a vector $\tilde{h}$ such that $|\tilde{h}|=|h|$ and $3|x+\tilde{h}|+|y+k|=\lambda+3$: indeed, we have $3|x+h|+|y+k|>\lambda+3$ and, simultaneously, $3(|x|-|h|)+|y+k|<\lambda+3$, which implies the existence of the intermediate $\tilde{h}$. But then $|x+h|>|x+\tilde{h}|$ and (2.2) follows from Case I.

Case IV: $3|x|+|y|>\lambda+3$. Consider the auxiliary function $\xi:H\to\mathbb{R}$, given by the formula

$$\xi(y)=\begin{cases}-\frac{1}{3}|y|(\lambda+3)+\frac{1}{6}(\lambda+3)^2-\frac{4}{3}|y|^2 & \text{if}\quad |y|<\lambda+3\\ -3|y|(\lambda+3)+\frac{3}{2}(\lambda+3)^2 & \text{if}\quad |y|\geq\lambda+3.\end{cases}$$

It is easy to check that ξ is concave and $U_\lambda(x,y)=\xi(y)+\frac{3}{2}|y|^2-\frac{3}{2}|x|^2$. Therefore, since $A(x,y)$, $B(x,y)$ are the partial derivatives of U_λ at the point (x,y), the right-hand side of (2.1) is not smaller than

$$\xi(y+k)+\frac{3}{2}|y+k|^2-\frac{3}{2}|x+h|^2.$$

It suffices to show that the latter expression majorizes $U_\lambda(x+h,y+k)$. If $3|x+h|+|y+k|\geq\lambda+3$, then we actually have the equality. If $3|x+h|+|y+k|<\lambda+3$, then in particular $|y+k|<\lambda+3$ and we must check that

$$0\leq-\frac{1}{3}|y+k|(\lambda+3)+\frac{1}{6}(\lambda+3)^2+\frac{1}{6}|y+k|^2-\frac{3}{2}|x+h|^2,$$

that is $(3|x+h|)^2\leq(\lambda+3-|y+k|)^2$. This follows directly from the assumption $3|x+h|+|y+k|<\lambda+3$ under which we work. ☐

2.2. **Proof of (1.4).** The functions U_λ and V_λ introduced above will allow us to prove the following auxiliary statement.

Lemma 2.3. *Suppose that f, g are martingales such that g is weakly dominated by f and f is bounded by 1. Then for any $\lambda\geq 0$ we have*

$$\mathbb{E}\big[(|g_n|-\lambda-6)\chi_{\{|g_n|>\lambda\}}\big]\leq 0,\qquad n=0,\,1,\,2,\,\ldots. \tag{2.4}$$

Proof. If f is bounded by 1, then in particular it is bounded in L^2 and hence, by the L^2 bound (1.3) formulated in the introductory section, g also has this property. Consequently, we see that for each n the random variables g_n, f_n, dg_n and df_n are square-integrable; this boundedness property will be needed in a moment to guarantee the integrability of a certain expression.

By Lemma 2.1, the estimate (2.4) will follow once we have proved the bound $\mathbb{E}U_\lambda(f_n,g_n)\leq 0$. To achieve this, fix a positive integer n and apply (2.1), with $x=f_{n-1}$, $y=g_{n-1}$, $h=df_n$ and $k=dg_n$, to obtain

$$\begin{aligned}U_\lambda(f_n,g_n)\leq{}&U_\lambda(f_{n-1},g_{n-1})+A(f_{n-1},g_{n-1})\cdot df_n+B(f_{n-1},g_{n-1})\cdot dg_n\\&-\phi_{f_{n-1},g_{n-1}}(|df_n|)+\phi_{f_{n-1},g_{n-1}}(|dg_n|).\end{aligned}$$

By the square-integrability of the variables g_n, f_n, dg_n and df_n mentioned above, both sides are integrable. Therefore, taking the conditional expectation with respect to $\mathcal{F}_{n-1}$ and using the weak domination condition, one gets

$$\mathbb{E}(U_\lambda(f_n,g_n)|\mathcal{F}_{n-1})\leq U_\lambda(f_{n-1},g_{n-1}).$$

This implies $\mathbb{E}U_\lambda(f_n,g_n)\leq\mathbb{E}U_\lambda(f_0,g_0)\leq 0$, where the latter estimate follows again from (2.1), this time applied to $x=y=0$ and $h=df_0=f_0$, $k=dg_0=g_0$. This gives the claim. ☐

We are ready to establish the assertion of Theorem 1.1.

Proof of (1.4). By homogeneity, we may assume that $\|f\|_{L^\infty}\leq 1$. Pick arbitrary nonnegative integer n, $t\in(0,1]$ and recall the alternative definition of g_n^{**}:

$$g_n^{**}(t)=\sup\left\{\frac{1}{\mathbb{P}(E)}\mathbb{E}\left(|g_n|\chi_E\right)\,:\,E\in\mathcal{F},\,\mathbb{P}(E)=t\right\}.$$

In particular, it implies that

$$g_n^{**}(t) - g_n^*(t) = \sup \left\{ \frac{1}{\mathbb{P}(E)} \mathbb{E}\big[\, (|g_n| - g_n^*(t)) \, \chi_E \big] \; : \; \mathbb{P}(E) = t \right\}.$$

By the definition of the decreasing rearrangement, we have $\mathbb{P}(|g_n| > \lambda) > t$ if $\lambda < g_n^*(t)$ and $\mathbb{P}(|g_n| > \lambda) \leq t$ if $\lambda > g_n^*(t)$, which in particular implies

$$\mathbb{P}(|g_n| \geq g_n^*(t)) \geq t \geq \mathbb{P}(|g_n| > g_n^*(t)).$$

Therefore, for any event E of probability t, we have

$$\frac{1}{\mathbb{P}(E)} \mathbb{E}\big[\, (|g_n| - g_n^*(t)) \, \chi_E \big] \leq \frac{1}{\mathbb{P}(|g_n| > g_n^*(t))} \mathbb{E}\big[(|g_n| - g_n^*(t)) \chi_{\{|g_n| > g_n^*(t)\}} \big],$$

which, by (2.4) applied to $\lambda = g_n^*(t)$, does not exceed 6. Taking the supremum over all E as above, we get the desired estimate. □

Remark 2.4. A closer look at the above proof shows that the weak domination can be significantly relaxed. Indeed, in the proof of (2.4) we exploit the requirement $\mathbb{E}(\phi(|dg_n|)|\mathcal{F}_{n-1}) \leq \mathbb{E}(\phi(|df_n|)|\mathcal{F}_{n-1})$ only for the functions $\phi_{x,y}$. If we want the proof to work for general bounded f (not necessarily bounded by 1), we need to take into account the scaled functions $\phi_{x,y}(\cdot/a)$, $a > 0$, as well. Summarizing, the inequality (1.4) remains valid if we assume that for any $n \geq 1$ and any $a \geq 0$,

$$\mathbb{E}\big[(|dg_n| - a)_+^2 | \mathcal{F}_{n-1} \big] \leq \mathbb{E}\big[(|df_n| - a)_+^2 | \mathcal{F}_{n-1} \big].$$

3. Estimates for Haar shift operators

The contribution of this section is to show how the inequalities for weakly dominated martingales imply the corresponding results for Haar shift operators; actually, as we shall see, the approach is very flexible and it allows a successful treatment of much more general class of dyadic shift operators. Let us introduce the necessary background and notation, to place these results into an appropriate context. Assume that $K : (-\infty, 0) \cup (0, \infty) \to \mathbb{R}$ is an odd, twice differentiable function (in the sense that K' is absolutely continuous) which satisfies the conditions

$$\lim_{x \to \infty} K(x) = \lim_{x \to \infty} K'(x) = 0 \tag{3.1}$$

and

$$x^3 K''(x) \in L^\infty(\mathbb{R}). \tag{3.2}$$

Let T_K be the associated one-dimensional singular integral operator, defined by

$$T_K f(x) = \mathrm{p.\,v.} \int_{\mathbb{R}} f(x-y) K(y) \mathrm{d}y.$$

For instance, for the choice $K(x) = (\pi x)^{-1}$ we obtain the Hilbert transform, the fundamental object in harmonic analysis.

As shown by Vagharshakyan [11], if K satisfies (3.1) and (3.2), then the operator T_K can be expressed as an average of appropriate one-dimensional dyadic shifts, whose definition we recall now. Let F^-, F^+, G and $H : \mathbb{R} \to \mathbb{R}$ be functions supported on the unit interval $[0,1]$ and given by

$$F^-(x) = \begin{cases} -\sqrt{2} & \text{if } 0 \le x < 1/4, \\ 0 & \text{if } 1/4 \le x < 3/4, \\ \sqrt{2} & \text{if } 3/4 \le x \le 1, \end{cases} \qquad F^+(x) = \begin{cases} 0 & \text{if } 0 \le x < 1/4, \\ -\sqrt{2} & \text{if } 1/4 \le x < 1/2, \\ \sqrt{2} & \text{if } 1/2 \le x < 3/4, \\ 0 & \text{if } 3/4 \le x \le 1, \end{cases}$$

$$G(x) = \begin{cases} -1 & \text{if } 0 \le x < 1/4, \\ 1 & \text{if } 1/4 \le x < 3/4, \\ -1 & \text{if } 3/4 \le x \le 1 \end{cases} \qquad \text{and} \qquad H(x) = \begin{cases} 7 & \text{if } 0 < x < 1/4, \\ -1 & \text{if } 1/4 \le x < 1/2, \\ 1 & \text{if } 1/2 \le x < 3/4, \\ -7 & \text{if } 3/4 \le x \le 1. \end{cases}$$

For any function $f : \mathbb{R} \to \mathbb{R}$ and any interval $I = [a,b]$, the rescaled version $f_I : \mathbb{R} \to \mathbb{R}$ is given by the formula

$$f_I(x) = \frac{1}{\sqrt{b-a}} f\left(\frac{x-a}{b-a}\right).$$

Next, for any $\beta = \{\beta_l\} \in \{0,1\}^{\mathbb{Z}}$ and any $r \in [1,2)$, consider the dyadic grid

$$\mathbb{D}_{r,\beta} = \left\{ r2^n \left([0,1) + k + \sum_{i<n} 2^{i-n}\beta_i \right) \right\}_{n\in\mathbb{Z}, k\in\mathbb{Z}}.$$

We equip $\{0,1\}^{\mathbb{Z}}$ with the uniform probability measure μ uniquely determined by the requirement $\mu(\{\beta : (\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_n}) = a\}) = 2^{-n}$ for any n, any sequence $i_1 < i_2 < \ldots < i_n$ of integers and any $a \in \{0,1\}^n$.

The aforementioned result of Vagharshakyan asserts the following.

Theorem 3.1. *Suppose that the kernel K satisfies (3.1) and (3.2). Then there exists a coefficient function $\gamma : (0,\infty) \to \mathbb{R}$ satisfying*

$$\|\gamma\|_{L^\infty} \le C\|x^2 K''(x)\|_{L^\infty}$$

such that

$$K(x-y) = \int_{\{0,1\}^{\mathbb{Z}}} \int_1^2 \sum_{I \in \mathbb{D}_{r,\beta}} \gamma(|I|) H_I(x) G_I(y) \frac{dr}{r}\, d\mu(\beta) \tag{3.3}$$

for all $x \ne y$. Here C is some absolute constant and the series on the right is absolutely convergent almost everywhere.

In other words, we see that the singular integral T_K can be expressed as an average of the Haar shift operators

$$T_{r,\beta} f = \sum_{I \in \mathbb{D}_{r,\beta}} \gamma(|I|) \langle f, G_I \rangle H_I.$$

In particular, this justifies the interest in various estimates for these objects. As we will see now, such inequalities can be handled with the use of weakly dominated martingales. Consider the probability space $([0,1], \mathcal{B}([0,1]), |\cdot|)$ equipped with the filtration $(\mathcal{F}_n)_{n\ge 0}$, where $\mathcal{F}_n$ is generated by $\mathcal{C}_n$, the class of all dyadic subintervals of $[0,1]$ having measure 4^{-n}. It is easy to show that the collection $\{F_I^{\pm}, G_I : I \in \mathcal{C}_n,\ n = 0, 1, 2, \ldots\}$ forms an orthonormal basis of the Hilbert space $L_0^2(\Omega)$ of mean-zero square integrable variables. Indeed, the orthonormality is evident, and the fact that the collection spans the space follows from the well-known fact that the classical Haar basis $(h_j)_{j\geq 1}$ enjoys this property (and the fact that each Haar function h_j can be expressed as a combination of G_I, F_I^- and F_I^+ for some interval I). Consequently, any $f \in L^2$ can be uniquely represented as the series

$$f = \int_I f + \sum_{n\geq 1} \sum_{I\in\mathcal{C}_{n-1}} \Big[\langle f, F_I^-\rangle F_I^- + \langle f, F_I^+\rangle F_I^+ + \langle f, G_I\rangle G_I\Big] = \sum_{n\geq 0} df_n.$$

The key observation is that the martingale $(7\|\gamma\|_{L^\infty} f_n)_{n\geq 0}$ is weakly dominating the martingale g given by

$$g_n = \sum_{k=0}^{n} \sum_{I\in\mathcal{C}_{n-1}} \gamma(I)\langle f, G_I\rangle H_I, \qquad n = 0, 1, 2, \ldots.$$

Indeed, fix an arbitrary convex and increasing function $\phi : [0,\infty) \to [0,\infty)$. For any $n \geq 1$ and any interval $I \in \mathcal{C}_{n-1}$ we have, by Jensen's inequality, the orthogonality of $F_I^{\pm}$ and G_I and the estimate $7|G| \geq |H|$,

$$\begin{aligned}\mathbb{E}\big[\phi\big(\big|7\|\gamma\|_\infty df_n\big|\big)|I\big] &\geq \mathbb{E}\big[\phi\big(\big|7\|\gamma\|_\infty \langle f, G_I\rangle G_I\big|\big)|I\big]\\ &\geq \mathbb{E}\big[\phi\big(\big|\gamma(I)\langle f, G_I\rangle H_I\big|\big)|I\big].\end{aligned}$$

This shows that $\mathbb{E}(\phi(7\|\gamma\|_\infty|df_n|)|\mathcal{F}_{n-1}) \geq \mathbb{E}(\phi(|dg_n|)|\mathcal{F}_{n-1})$, which is the aforementioned weak domination (formally, let us record here that $dg_0 = 0$, so the estimate holds also for the case $n = 0$). We can rescale the initial interval $[0,1]$ to an arbitrary $I \in \mathbb{D}_{r,\beta}$ and hence any estimate for weakly dominated martingales immediately yields the corresponding statement for the Haar shift operators. We will illustrate this transference on the example of the estimate (1.4). One can obtain other (e.g., moment or weak-type) estimates using the same argumentation; we leave the details to the reader.

Theorem 3.2. *For any Hilbert-space-valued function f we have the estimate*

$$\|T_{r,\beta} f\|_{W(\mathbb{R})} \leq 42\|\gamma\|_{L^\infty}\|f\|_{L^\infty}.$$

Proof. We may assume that $\|f\|_{L^\infty} \leq 1$. Let I be an arbitrary element of $\mathbb{D}_{r,\beta}$ and let $T_{r,\beta,N}$ be the truncated Haar shift operator given by

$$T_{r,\beta,N} f = \sum_{\substack{J\in\mathbb{D}_{r,\beta}, J\subseteq I,\\ |J|/|I|\geq 4^{-N}}} \gamma(|J|)\langle f, G_J\rangle H_J.$$

By the above discussion, the inequality (2.4) yields

$$\int_I \left(\frac{|T_{r,\beta,N} f|}{7\|\gamma\|_{L^\infty}} - \lambda - 6\right) \chi_{\{|T_{r,\beta,N} f| > 7\|\gamma\|_{L^\infty}\lambda\}} \mathrm{d}s \leq 0$$

for any finite N. Now letting $N \to \infty$ and then $|I| \to \infty$ gives

$$\int_I \left(\frac{|T_{r,\beta} f|}{7\|\gamma\|_{L^\infty}} - \lambda - 6\right) \chi_{\{|T_{r,\beta} f| > 7\|\gamma\|_{L^\infty}\lambda\}} \mathrm{d}s \leq 0,$$

by Fatou's lemma. It remains to repeat, word-by-word, the argumentation from the end of the previous section to obtain the desired estimate. □

Remark 3.3. The above approach enables the study of much wider classes of Haar shifts; we will briefly discuss an example of such an extension. Let n be an arbitrary integer and let G and $H : [0,1] \to \mathbb{R}$ be functions measurable with respect to the σ-algebra $\mathcal{A}$ generated by the dyadic subintervals of $[0,1]$ having measure 2^{-n} (in the above considerations, we had $n = 2$). Assume in addition that there is a constant a such that for any convex increasing function ϕ we have $\int_0^1 \phi(|aG|)\mathrm{d}s \geq \int_0^1 \phi(|H|)\mathrm{d}s$. Replacing $F^{\pm}$ by an arbitrary collection of functions which together with G form an orthonormal basis of $L^2([0,1],\mathcal{A})$ and repeating the above arguments, we obtain

$$\|T_{r,\beta}f\|_{W(\mathbb{R})} \leq 6|a|\|\gamma\|_{L^\infty}\|f\|_{L^\infty}.$$

We omit further details in this direction.

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