WEAK-TYPE INEQUALITIES FOR FOURIER MULTIPLIERS WITH APPLICATIONS TO THE BEURLING-AHLFORS TRANSFORM

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Abstract. The paper contains the study of weak-type constants of Fourier multipliers resulting from modulation of the jumps of Lévy processes. We exhibit a large class of functions $m : \mathbb{R}^d \to \mathbb{C}$, for which the corresponding multipliers T_m satisfy the estimates

$$||T_m f||_{L^{p,\infty}(\mathbb{R}^d)} \leq \left[\frac{1}{2}\Gamma\left(\frac{2p-1}{p-1}\right)\right]^{(p-1)/p} ||f||_{L^p(\mathbb{R}^d)}$$

for $1 < p < 2$, and

$$||T_m f||_{L^{p,\infty}(\mathbb{R}^d)} \leq \left[\frac{p^{p-1}}{2}\right]^{1/p} ||f||_{L^p(\mathbb{R}^d)}$$

for $2 \leq p < \infty$. The proof rests on a novel duality method and a new sharp inequality for differentially subordinated martingales. We also provide lower bounds for the weak-type constants by constructing appropriate examples for the Beurling-Ahlfors operator on $\mathbb{C}$.

1. Introduction

The martingale theory plays a fundamental role in obtaining the L^p bounds for many important singular integrals and Fourier multipliers, and the purpose of this paper is to explore further this connection. We shall introduce a new method which will allow us to deduce sharp weak type (p, p) inequalities for a large class of Fourier multipliers from an appropriate bound for differentially subordinated martingales.

A celebrated theorem of Burkholder [7] states that if X, Y are Hilbert-space-valued martingales such that Y is differentially subordinate to X (see the next section for the necessary definitions), then we have the sharp estimate

$$||Y||_p \leq (p^* - 1)||X||_p, \qquad 1 < p < \infty, \tag{1.1}$$

where $p^* = \max\{p, p/(p-1)\}$. The inequality breaks down for $p = 1$, but we have the corresponding weak-type bound [7]:

$$\mathbb{P}(\sup_{t\geq 0} |Y_t| \geq 1) \leq \frac{2}{\Gamma(p+1)}||X||_p^p, \qquad 1 \leq p \leq 2, \tag{1.2}$$

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and the constant is the best possible. The optimal constant in the case $2 < p < \infty$ was determined by Suh in [16]: we have

$$\mathbb{P}(\sup_{t\geq 0}|Y_t| \geq 1) \leq \frac{p^{p-1}}{2}||X||_p^p. \tag{1.3}$$

The moment inequality (1.1) can be used to obtain tight L^p bounds for a wide class of Fourier multipliers. Recall that for any bounded function $m : \mathbb{R}^d \to \mathbb{C}$, there is a unique bounded linear operator T_m on $L^2(\mathbb{R}^d)$, called *the Fourier multiplier with the symbol* m, which is given by the following relation between Fourier transforms:

$$\widehat{T_m f} = m\hat{f}.$$

The norm of T_m on $L^2(\mathbb{R}^d)$ is equal to $||m||_{L^\infty(\mathbb{R}^d)}$ and it has been long of interest to study those m, for which the corresponding Fourier multiplier extends to a bounded linear operator on $L^p(\mathbb{R}^d)$, $1 < p < \infty$. One of the fundamental examples of such multipliers is the collection of Riesz transforms R_1, R_2, ..., R_d in $\mathbb{R}^d$, which correspond to the symbols $i\xi_1/|\xi|$, $i\xi_2/|\xi|$, ..., $i\xi_d/|\xi|$, respectively. Using (1.1), Bañuelos and Wang [6] showed the following bound for the vector $R = (R_1, R_2, \ldots, R_d)$:

$$||Rf||_{L^p(\mathbb{R}^d;\mathbb{R}^d)} \leq 2(p^* - 1)||f||_{L^p(\mathbb{R}^d)}, \qquad 1 < p < \infty.$$

See also Iwaniec and Martin [12] for related results, obtained by a purely analytic approach.

In the present paper we shall consider the following class of symbols, studied by Bañuelos and Bogdan [2] and Bañuelos, Bielaszewski and Bogdan [3]. Let ν be a Lévy measure on $\mathbb{R}^d$: that is, a nonnegative Borel measure on $\mathbb{R}^d$ such that $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d} \min\{|x|^2, 1\}\nu(\mathrm{d}x) < \infty.$$

Assume further that μ is a finite Borel measure on the unit sphere $\mathbb{S}$ of $\mathbb{R}^d$ and fix two Borel functions ϕ on $\mathbb{R}^d$ and ψ on $\mathbb{S}$ which take values in the unit ball of $\mathbb{C}$. We define the associated multiplier $m = m_{\phi,\psi,\mu,\nu}$ on $\mathbb{R}^d$ by

$$m(\xi) = \frac{\frac{1}{2}\int_{\mathbb{S}}\langle \xi, \theta\rangle^2\psi(\theta)\mu(\mathrm{d}\theta) + \int_{\mathbb{R}^d}[1 - \cos\langle \xi, x\rangle]\phi(x)\nu(\mathrm{d}x)}{\frac{1}{2}\int_{\mathbb{S}}\langle \xi, \theta\rangle^2\mu(\mathrm{d}\theta) + \int_{\mathbb{R}^d}[1 - \cos\langle \xi, x\rangle]\nu(\mathrm{d}x)} \tag{1.4}$$

if the denominator is not 0, and $m(\xi) = 0$ otherwise. Here $\langle \cdot, \cdot\rangle$ stands for the scalar product in $\mathbb{R}^d$. As proved in [2] and [3] (see also Section 3 below), the Fourier multipliers corresponding to these symbols can be given a martingale representation by the use of appropriate transformations of jumps of Lévy processes. Combining this representation with Burkholder's inequality (1.1), Bañuelos, Bielaszewski and Bogdan established the following L^p estimate.

Theorem 1.1. *Let $1 < p < \infty$ and let $m = m_{\phi,\psi,\mu,\nu}$ be given by (1.4). Then for any $f \in L^p(\mathbb{R}^d)$ we have*

$$||T_m f||_{L^p(\mathbb{R}^d)} \leq (p^* - 1)||f||_{L^p(\mathbb{R}^d)}. \tag{1.5}$$

See also [5] for related lower bounds. In particular, this theorem yields interesting estimates for the Beurling-Ahlfors transform $\mathcal{BA}$ on $\mathbb{C}$. Recall that this operator is given by the singular integral

$$\mathcal{BA}\, f(z) = -\text{p.v.}\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(w)}{(z-w)^2}\,\mathrm{d}w, \qquad z \in \mathbb{C},$$

where the integration is with respect to the Lebesgue's measure on the complex plane. Alternatively, $\mathcal{BA}$ can be defined as the Fourier multiplier with the symbol $m(\xi) = (\xi/|\xi|)^2$, $\xi \in \mathbb{C} \setminus \{0\}$ (with the standard identification $\mathbb{C} \simeq \mathbb{R}^2$). This operator plays a fundamental role in the study of quasiconformal mappings, partial differential equations and complex analysis; its importance lies in the fact that it changes the complex derivative $\bar{\partial}$ into ∂. Formally,

$$\mathcal{BA}(\bar{\partial} f) = \partial f \tag{1.6}$$

for every f in the Sobolev space $W^{1,2}(\mathbb{C},\mathbb{C})$. There is an important question about the precise value of the norm of $\mathcal{BA}$ acting on $L^p(\mathbb{C})$, $1 < p < \infty$; a celebrated and long-standing hypothesis of Iwaniec [10] states that $||\mathcal{BA}||_{L^p(\mathbb{C})\to L^p(\mathbb{C})} = p^* - 1$. While the lower bound $p^* - 1$ was shown by Lehto [13], the question about the upper bound remains open. The estimate (1.5) yields $||\mathcal{BA}||_{L^p(\mathbb{C})\to L^p(\mathbb{C})} \leq 2(p^* - 1)$ (see Section 4 below). This can be further improved; the best result so far is the inequality $||\mathcal{BA}||_{L^p(\mathbb{C})\to L^p(\mathbb{C})} \leq 1.575(p^* - 1)$, obtained by Bañuelos and Janakiraman [4] by the use of a refined version of (1.1).

There is a natural question whether the interplay between the martingale theory and Fourier multipliers, which has been so fruitful in the case of L^p bounds, carries over to the weak-type (p, p) estimates. The objective of this paper is to propose an approach which will yield the affirmative answer to this question. It should be stressed here that the repetition of the arguments leading to the L^p-estimates and replacing (1.1) by (1.2) or (1.3) in the middle does not produce the weak-type bounds. Roughly speaking, the problem lies in the fact that the representation of a given Fourier multiplier in terms of Lévy processes involves the use of an appropriate conditional expectation; this operation is a contraction on L^p, but no longer on $L^{p,\infty}$. Thus a refinement of the method is needed, and we have invented a duality argument to handle this problem. Of course, since $L^p \subset L^{p,\infty}$, we immediately obtain the rough bound for $1 < p < \infty$:

$$||T_m||_{L^p(\mathbb{R}^d)\to L^{p,\infty}(\mathbb{R}^d)} \leq ||T_m||_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} \leq p^* - 1. \tag{1.7}$$

We shall establish the following significant improvement of this estimate. Introduce the constants

$$K_p = \begin{cases} \left[\frac{1}{2}\Gamma\left(\frac{2p-1}{p-1}\right)\right]^{(p-1)/p} & \text{if } 1 < p < 2, \\ p^{p-1}/2 & \text{if } p \geq 2. \end{cases}$$

Theorem 1.2. *Assume that* $1 < p < \infty$ *and* m *is given by* (1.4), *with* ν, μ, ϕ *and* ψ *satisfying the above assumptions. Then for any* $f \in L^p(\mathbb{R}^d)$ *we have*

$$|\{x \in \mathbb{R}^d : |T_m f(x)| \geq 1\}| \leq K_p ||f||^p_{L^p(\mathbb{R}^d)}, \tag{1.8}$$

that is, $||T_m||_{L^p(\mathbb{R}^d)\to L^{p,\infty}(\mathbb{R}^d)} \leq K_p^{1/p}$.

It is not difficult to prove that $K_p < (p-1)^{-1}$ for $1 < p < 2$, so (1.8) is better than (1.7); however, it is very likely that this bound can be further improved. On the other hand, we strongly believe that for $p \geq 2$ the constant K_p is the best possible. To justify this conjecture, note that K_p coincides with the optimal constant from (1.3) and hence the bound (1.8) seems to be the farthest point where the martingale methods can take us. Unfortunately, we have only managed to find examples showing that the weak-type constant K_p is not smaller than $p^{p-1}/2^{p+1}$. Nevertheless, these examples are very interesting on their own, for they exhibit further close connections between martingale transforms and the Beurling-Ahlfors operator. See Section 4 below for details.

We have organized the remainder of this paper as follows. In the next section we study an inequality for differentially subordinated martingales, which constitutes the foundation for our further considerations. In Section 3 we combine this estimate with the representation of Fourier multipliers (1.4) in terms of Lévy processes, and provide the proof of Theorem 1.2. This section contains also a version of Theorem 1.2 for vector-valued multipliers. Finally, in Section 4 we apply our results to the study of the weak type constants of the Beurling-Ahlfors transform and provide examples which yield the corresponding lower bounds.

2. A martingale inequality

The key ingredient of the proof of the announced estimate (1.8) is an appropriate inequality for differentially subordinated martingales. We begin with introducing the necessary probabilistic background and notation. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, equipped with $(\mathcal{F}_t)_{t\geq 0}$, a nondecreasing family of sub-σ-fields of $\mathcal{F}$, such that $\mathcal{F}_0$ contains all the events of probability 0. Let X, Y be two adapted martingales taking values in a certain separable Hilbert space $(\mathcal{H}, |\cdot|)$; with no loss of generality, we may put $\mathcal{H} = \ell^2$. As usual, we assume that the processes have right-continuous trajectories with the limits from the left. The symbol $[X, Y]$ will stand for the quadratic covariance process of X and Y. See e.g. Dellacherie and Meyer [9] for details in the case when the processes are real-valued, and extend the definition to the vector setting by $[X, Y] = \sum_{k=0}^{\infty}[X^k, Y^k]$, where X^k, Y^k are the k-th coordinates of X, Y, respectively. Following Bañuelos and Wang [6] and Wang [17], we say that Y is differentially subordinate to X, if the process $([X, X]_t - [Y, Y]_t)_{t\geq 0}$ is nonnegative and nondecreasing as a function of t.

Now we are ready to formulate our main probabilistic result, a dual estimate to (1.2) and (1.3). For $1 < q < \infty$, let

$$C_q = \begin{cases} 2^{1-q}q/(q-1) & \text{if } 1 < q \leq 2, \\ \Gamma(q+1)/2 & \text{if } q > 2. \end{cases}$$

We use the notation $||X||_p = \sup_{t\geq 0} ||X_t||_p$, $1 \leq p \leq \infty$.

Theorem 2.1. *Assume that X, Y are $\mathcal{H}$-valued martingales such that Y is differentially subordinate to X. Then for any $1 < q < \infty$,*

$$||Y||_q^q \leq C_q ||X||_1 ||X||_\infty^{q-1}. \tag{2.1}$$

For each q, the constant C_q is the best possible.

The proof rests on Burkholder's method: we shall deduce the inequality (2.1) from the existence of a family $\{V_q\}_{q\in(1,\infty)}$ of certain special functions defined on the set $S = \{(x, y) \in \mathcal{H} \times \mathcal{H} : |x| \leq 1\}$. In order to simplify the technicalities, we shall combine the technique with an "integration argument", invented in [14] (see also [15]): first we introduce two simple functions v_1, $v_\infty : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, for which the calculations are relatively easy; then define V_q by integrating these two objects against appropriate nonnegative kernels. Let

$$v_1(x, y) = \begin{cases} |y|^2 - |x|^2 & \text{if } |x| + |y| \leq 1, \\ 1 - 2|x| & \text{if } |x| + |y| > 1 \end{cases}$$

and

$$v_\infty(x,y) = \begin{cases} 0 & \text{if } |x|+|y|\leq 1,\\ (|y|-1)^2-|x|^2 & \text{if } |x|+|y|>1. \end{cases}$$

We have the following fact (see Lemma 2.2 in [15] for a slightly stronger statement in which the differential subordination is replaced by a less restrictive assumption).

Lemma 2.2. *For all $\mathcal{H}$-valued martingales X, Y such that Y is differentially subordinate to X we have*

$$\mathbb{E}v_1(X_t,Y_t)\leq 0 \qquad \text{for all } t\geq 0.$$

If in addition X satisfies $||X||_2<\infty$, then

$$\mathbb{E}v_\infty(X_t,Y_t)\leq 0 \qquad \text{for all } t\geq 0.$$

Recall that $S=\{(x,y)\in\mathcal{H}\times\mathcal{H}:|x|\leq 1\}$. For $1<q<2$, define $V_q:S\to\mathbb{R}$ by

$$V_q(x,y)=\frac{q(2-q)}{2}\int_0^{1/2} r^{q-1}v_1(x/r,y/r)\mathrm{d}r+\frac{q}{2^{q-1}}(|y|^2-|x|^2).$$

A little calculation shows that if $|x|+|y|\leq 1/2$, then

$$V_q(x,y)=\frac{1}{q-1}(|x|+|y|)^{q-1}(-|x|+(q-1)|y|),$$

while for $|x|+|y|>1/2$,

$$V_q(x,y)=\frac{q(2-q)}{2}\left[\frac{1}{q2^q}-\frac{|x|}{(q-1)2^{q-2}}\right]+\frac{q}{2^{q-1}}(|y|^2-|x|^2).$$

If $q=2$, then we set $V_q(x,y)=|y|^2-|x|^2$. Finally, when $2<q<\infty$, define $V_q:S\to\mathbb{R}$ by

$$V_q(x,y)=\int_1^\infty k_q(r)v_\infty(x/r,y/r)\mathrm{d}r+\frac{\Gamma(q+1)}{2}(|y|^2-|x|^2),$$

where, for $r>1$,

$$k_q(r)=\frac{q(q-1)r^2}{2}\left[e^r\int_r^\infty e^{-s}(s-1)^{q-2}\mathrm{d}s-(r-1)^{q-2}\right].$$

After some lengthy, but straightforward computations, we check that $V_q(x,y)$ equals

$$\begin{cases} \Gamma(q+1)(|y|^2-|x|^2)/2 & \text{if } |x|+|y|\leq 1,\\ (|x|+|y|-1)^q+q(1-|x|)\int_{|x|+|y|}^\infty e^{|x|+|y|-s}(s-1)^{q-1}\mathrm{d}s-C_q & \text{if } |x|+|y|>1. \end{cases}$$

We shall need the following majorization property.

Lemma 2.3. *For any $1<q<\infty$ we have*

$$V_q(x,y)\geq |y|^q-C_q|x| \qquad \text{for all } (x,y)\in S. \tag{2.2}$$

Proof. We may assume that $q\neq 2$, since for $q=2$ the bound reduces to a trivial estimate $|y|^2-|x|^2\geq|y|^2-1$. Obviously, it suffices to prove the majorization for $\mathcal{H}=\mathbb{R}$. Furthermore, since V_q satisfies the symmetry condition $V_q(x,y)=V_q(-x,y)=V_q(x,-y)$ for all $(x,y)\in S$, we may restrict ourselves to x, $y\geq 0$. The next observation is that V_q is linear along the line segments of slope -1 contained in $S_+=[0,1]\times[0,\infty)$, while the right-hand side of (2.2) is convex along these segments. Consequently, it suffices to verify the majorization at the boundary of the strip S_+. The final reduction is that V_q is concave along the segment $[0,1]\times\{0\}$;

thus we will be done if we show (2.2) for $x \in \{0,1\}$ and $y \geq 0$. Let us consider the cases $1 < q < 2$ and $2 < q < \infty$ separately.

The case $1 < q < 2$. If $x = 0$ and $y \leq 1/2$, then both sides of (2.2) are equal. If $x = 0$ and $y > 1/2$, or $x = 1$, then (2.2) can be transformed into the equivalent estimate

$$\frac{2-q}{2^{q+1}} + \frac{q}{2^{q-1}}y^2 \geq y^q,$$

or

$$(y^2)^{q/2} - (2^{-2})^{q/2} \leq \frac{q}{2}(2^{-2})^{q/2-1}(y^2 - 2^{-2}),$$

which follows immediately from the mean value property.

The case $2 < q < \infty$. Both sides of (2.2) are equal when $x = 1$. If $x = 0$ and $|y| \leq 1$, we have

$$V_q(x,y) - |y|^q + C_q|x| = |y|^2\left(\frac{\Gamma(q+1)}{2} - |y|^{q-2}\right) \geq 0. \tag{2.3}$$

Finally, if $x = 0$ and $y > 1$, the majorization can be rewritten in the form

$$\kappa(y) := (y-1)^q e^{-y} + q\int_y^\infty e^{-s}(s-1)^{q-1}\mathrm{d}s - y^q e^{-y} - \frac{\Gamma(q+1)}{2}e^{-y} \geq 0.$$

We see that

$$\kappa'(y)e^y = y^q - (y-1)^q - qy^{q-1} + \frac{\Gamma(q+1)}{2} \to -\infty \qquad \text{as } y \to \infty,$$

and

$$(\kappa'(y)e^y)' = q\big(y^{q-1} - (y-1)^{q-1} - (q-1)y^{q-2}\big) \leq 0,$$

by the mean value property. Thus there is $y_0 \geq 1$ such that κ is increasing on $[1, y_0]$ and decreasing on $[y_0, \infty)$. Since $\kappa \to 0$ as $y \to \infty$ and $\kappa(1) \geq 0$, as we have already checked in (2.3), the majorization follows. □

Now we are ready to establish Theorem 2.1.

Proof of (2.1). It suffices to show that $||Y||_q^q \leq C_q||X||_1$ for any X, Y as in the statement satisfying the additional condition $||X||_\infty \leq 1$. Suppose first that $1 < q < 2$ and fix $t \geq 0$. We have $\mathbb{E}|Y_t|^2 \leq \mathbb{E}|X_t|^2 \leq 1$, by Burkholder's inequality (1.1) for $p = 2$ and the boundedness of X. Therefore, Lemma 2.2 and Fubini's theorem imply

$$\mathbb{E}V_q(X_t, Y_t) \leq \frac{q(2-q)}{2}\int_0^{1/2} r^{q-1}\mathbb{E}v_1(X_t/r, Y_t/r)\mathrm{d}r \leq 0. \tag{2.4}$$

To see that Fubini's theorem is applicable, note that $|v_1(x,y)| \leq c(|x| + |y| + 1)$ for all x, $y \in \mathcal{H}$ and some absolute constant c; thus

$$\mathbb{E}\int_0^{1/2} r^{q-1}|v_1(X_t/r, Y_t/r)|\mathrm{d}r \leq \tilde{c}\mathbb{E}(|X_t| + |Y_t| + 1) < \infty,$$

where $\tilde{c}$ is another universal constant. Combining (2.4) with (2.2) yields $\mathbb{E}|Y_t|^q \leq C_q\mathbb{E}|X_t|$ and it suffices to let $t \to \infty$ to get the claim. The case $2 \leq q < \infty$ is dealt with in a similar manner; the only thing which must be checked is that the kernel k_q is nonnegative. But this is evident: for $r > 1$,

$$e^r\int_r^\infty e^{-s}(s-1)^{q-2}\mathrm{d}s \geq e^r\int_r^\infty e^{-s}(r-1)^{q-2}\mathrm{d}s = (r-1)^{q-2}.$$

This completes the proof. □

Remark 2.4. It is well known that in general Burkholder's function (that is, the special function leading to a given martingale inequality) is not unique, see e.g. [8]. Sometimes it is of interest to determine the optimal (that is, the least) of the possible ones, at least for $\mathcal{H} = \mathbb{R}$. Though we shall not need this, we would like to mention here that V_q is optimal in the real case when $2 \leq q < \infty$. When $1 < q < 2$, the optimal function is given by the following formula. First define $v_q : [0,1] \times [0,\infty) \to \mathbb{R}$ by

$$v_q(x,y) = \begin{cases} (-x+(q-1)y)(x+y)^{q-1}/(q-1) + qx/(q-1) & \text{if } x+y \leq 1, \\ (x+y)^q - qxe^{-x-y}\int_1^{x+y} e^s s^{q-1} \mathrm{d}s & \text{if } x+y > 1. \end{cases}$$

Then the optimal $\overline{V}_q : [-1,1] \times \mathbb{R} \to \mathbb{R}$ is given by

$$\overline{V}_q(x,y) = v_q\big(1 - \big|2|x|-1\big|, 2|y|\big)/2^q - C_q|x|.$$

We omit the further details.

Sharpness of (2.1), $1 < q \leq 2$. If $q = 2$, the sharpness is trivial: simply take $X = Y \equiv 1$. Suppose then, that $q < 2$. Let $N \geq 1$ be a fixed integer and put $\delta = (4N)^{-1}$. Consider a sequence $\xi_0, \xi_1, \ldots, \xi_{2N}$ of independent random variables with the following distributions: $\xi_0 \equiv \delta$,

$$\mathbb{P}(\xi_n = \delta) = 1 - \mathbb{P}(\xi_n = -n\delta) = \frac{n}{n+1}, \qquad n = 1, 2, \ldots, 2N-1,$$

and $\mathbb{P}(\xi_{2N} = -1/2) = \mathbb{P}(\xi_{2N} = 1/2) = 1/2$. Introduce the stopping time

$$\tau = \inf\{n \leq 2N : \xi_0 + \xi_1 + \xi_2 + \ldots + \xi_n \in \{0,1\}\}.$$

Define the processes X, Y by

$$X_t = \xi_0 + \xi_1 + \ldots + \xi_{\tau\wedge\lfloor t\rfloor} \quad \text{and} \quad Y_t = \xi_0 - \xi_1 + \xi_2 - \xi_3 + \ldots + (-1)^{\lfloor t\rfloor}\xi_{\tau\wedge\lfloor t\rfloor},$$

for $t \geq 0$. Since the variables ξ_k are centered (for $k > 0$), both X and Y are martingales. We have that $||X||_\infty = 1$, since X takes values in $[0,1]$ and $\mathbb{P}(X_{2N} = 1) > 0$. Moreover, Y is differentially subordinate to X: we have $[X,X]_t = [Y,Y]_t$ for all $t \geq 0$. Next, $||Y||_q^q = \mathbb{E}|Y_{2N}|^q$ and the distribution of $|Y_{2N}|$ is given as follows. We have $|Y_{2N}| \in \{2\delta, 4\delta, \ldots, 1/2\}$ and, for $k = 1, 2, \ldots, N-1$,

$$\begin{aligned} \mathbb{P}(|Y_{2N}| = 2k\delta) &= \mathbb{P}(\tau = 2k-1 \text{ or } \tau = 2k) \\ &= \mathbb{P}(\xi_1 > 0, \xi_2 > 0, \ldots, \xi_{2k-2} > 0, \xi_{2k-1} < 0) \\ &\quad + \mathbb{P}(\xi_1 > 0, \xi_2 > 0, \ldots, \xi_{2k-1} > 0, \xi_{2k} < 0) \\ &= \frac{1}{2k(2k-1)} + \frac{1}{2k(2k+1)} = \frac{2}{(2k+1)(2k-1)}. \end{aligned}$$

Finally, we have

$$\mathbb{P}(|Y_{2N}| = 1/2) = 1 - \mathbb{P}(|Y_{2N}| < 1/2) = \frac{1}{2N-1}.$$

Recalling that $\delta = (4N)^{-1}$, we see that

$$\frac{||Y||_q^q}{||X||_1 ||X||_\infty^{q-1}} = \frac{\mathbb{E}|Y_{2N}|^q}{\mathbb{E}X_0} = 2\delta \sum_{k=1}^{N-1} \frac{(2k\delta)^q}{(2k+1)\delta \cdot (2k-1)\delta} + \frac{2^{1-q}}{(4N-2)\delta}.$$

If we tend with N to ∞, the first term on the right converges to $\int_0^{1/2} s^{q-2}\mathrm{d}s = 2^{1-q}/(q-1)$ and the second to 2^{1-q}; thus, if N is taken sufficiently large, then the ratio $||Y||_q^q/(||X||_1||X||_\infty^{q-1})$ can be made arbitrarily close to C_q. This proves the optimality of this constant in (2.1). □

Sharpness of (2.1), $q > 2$. As previously, fix a large positive integer N and put $\delta = (4N)^{-1}$. Consider independent random variables ξ_0, ξ_1, ξ_2, … such that $\xi_0 \equiv 1/2$, $\mathbb{P}(\xi_1 = -1/2) = \mathbb{P}(\xi_1 = 1/2) = 1/2$ and, for $k = 1, 2, \ldots$,

$$\mathbb{P}(\xi_{2k} = \delta) = 1 - \mathbb{P}(\xi_{2k} = -1) = \frac{1}{1+\delta},$$
$$\mathbb{P}(\xi_{2k+1} = -\delta) = 1 - \mathbb{P}(\xi_{2k+1} = 1-\delta) = 1 - \delta.$$

Next, let $\tau = \inf\{n : \xi_0 + \xi_1 + \ldots + \xi_n \in \{-1, 1\}\}$ and define martingales X, Y by

$$X_t = \xi_0 + \xi_1 + \ldots + \xi_{\tau\wedge\lfloor t\rfloor} \quad \text{and} \quad Y_t = \xi_0 - \xi_1 + \xi_2 - \xi_3 + \ldots + (-1)^{\lfloor t\rfloor}\xi_{\tau\wedge\lfloor t\rfloor},$$

for $t \geq 0$. We easily verify that $||X||_1 = ||X||_\infty = 1$ and that Y is differentially subordinate to X. It is also easy to see that the martingale Y converges almost surely to a random variable Y_∞, which takes values in the set $\{0, 2\delta, 4\delta, \ldots\}$. We compute that

$$\mathbb{P}(Y_\infty = 0) = \mathbb{P}(\tau = 1 \text{ or } \tau = 2) = \mathbb{P}(\xi_1 > 0) + \mathbb{P}(\xi_1 < 0,\, \xi_2 < 0) = \frac{1+2\delta}{2(1+\delta)}$$

and, for $k = 1, 2, \ldots$,

$$\mathbb{P}(Y_\infty = 2k\delta) = \mathbb{P}(\tau = 2k+1 \text{ or } \tau = 2k+2) = \frac{\delta(1-\delta)^{k-1}}{(1+\delta)^{k+1}}.$$

Consequently, we have

$$\frac{||Y||_q^q}{||X||_1||X||_\infty^{q-1}} = \mathbb{E}Y_\infty^q = \frac{\delta}{(1-\delta)(1+\delta)}\sum_{k=1}^\infty (2k\delta)^q\left(\frac{1-\delta}{1+\delta}\right)^k.$$

If we let $N \to \infty$, then $\delta \to 0$ and the right-hand side converges to $\frac{1}{2}\int_0^\infty s^q e^{-s}\mathrm{d}s = C_q$. This proves that the constant C_q cannot be replaced in (2.1) by a smaller number. □

3. Proof of Theorem 1.2

Let $m = m_{\phi,\psi,\mu,\nu}$ be a multiplier as in (1.4). By the results in [3], we may assume that the Lévy measure ν satisfies the symmetry condition $\nu(B) = \nu(-B)$ for all Borel subsets B of $\mathbb{R}^d$. More precisely, there are $\bar{\mu}$, $\bar{\nu}$, $\bar{\phi}$, $\bar{\psi}$ such that $\bar{\nu}$ is symmetric and $m_{\phi,\psi,\mu,\nu} = m_{\bar{\phi},\bar{\psi},\bar{\mu},\bar{\nu}}$. Assume in addition that $|\nu| = \nu(\mathbb{R}^d)$ is finite and nonzero, and define $\tilde{\nu} = \nu/|\nu|$. Consider the independent random variables T_{-1}, T_{-2}, …, Z_{-1}, Z_{-2}, … such that for each $n = -1, -2, \ldots$, T_n has exponential distribution with parameter $|\nu|$ and Z_n takes values in $\mathbb{R}^d$ and has $\tilde{\nu}$ as the distribution. Next, put $S_n = -(T_{-1} + T_{-2} + \ldots + T_n)$ for $n = -1, -2, \ldots$ and let

$$X_{s,t} = \sum_{s<S_j\leq t} Z_j, \qquad X_{s,t-} = \sum_{s<S_j<t} Z_j, \qquad \Delta X_{s,t} = X_{s,t} - X_{s,t-},$$

for $-\infty < s \leq t \leq 0$. For a given $f \in L^\infty(\mathbb{R}^d)$, define its parabolic extension $\mathcal{U}_f$ to $(-\infty, 0] \times \mathbb{R}^d$ by

$$\mathcal{U}_f(s,x) = \mathbb{E}f(x + X_{s,0}).$$

Next, fix $x \in \mathbb{R}^d$, $s < 0$ and $f \in L^\infty(\mathbb{R}^d)$. We introduce the processes $F = (F_t^{x,s,f})_{t\in[s,0]}$ and $G = (G_t^{x,s,f,\phi})_{t\in[s,0]}$ by

$$\begin{aligned} F_t &= \mathcal{U}_f(t, x + X_{s,t}), \\ G_t &= \sum_{s<u\le t} \big[\Delta F_u \cdot \phi(\Delta X_{s,u})\big] \\ &\quad - \int_s^t \int_{\mathbb{R}^d} \big[\mathcal{U}_f(v, x + X_{s,v-} + z) - \mathcal{U}_f(v, x + X_{s,v-})\big]\phi(z)\nu(\mathrm{d}z)\mathrm{d}v. \end{aligned} \tag{3.1}$$

Note that the sum in the definition of G can be seen as the result of modulating of the jumps of F by ϕ, and the subsequent double integral can be regarded as an appropriate compensator. We have the following statement, proved in [2].

Lemma 3.1. *For any fixed x, s, f as above, the processes $F^{x,s,f}$, $G^{x,s,f,\phi}$ are martingales with respect to $(\mathcal{F}_t)_{t\in[s,0]}$. Furthermore, if $||\phi||_\infty \le 1$, then $G^{x,s,f,\phi}$ is differentially subordinate to $F^{x,s,f}$.*

Now, fix $s < 0$ and define the operator $\mathcal{S} = \mathcal{S}^{s,\phi,\nu}$ by the bilinear form

$$\int_{\mathbb{R}^d} \mathcal{S}f(x)g(x)\mathrm{d}x = \int_{\mathbb{R}^d} \mathbb{E}\big[G_0^{x,s,f,\phi} g(x + X_{s,0})\big]\mathrm{d}x, \tag{3.2}$$

where f, $g \in C_0^\infty(\mathbb{R}^d)$. We have the following fact, proved in [2]. It constitutes the crucial part of the aforementioned representation of Fourier multipliers in terms of Lévy processes.

Lemma 3.2. *Let $1 < p < \infty$ and $d \ge 2$. The operator $\mathcal{S}^{s,\phi,\nu}$ is well defined and extends to a bounded operator on $L^p(\mathbb{R}^d)$, which can be expressed as a Fourier multiplier with the symbol*

$$\begin{aligned} M(\xi) &= M_{s,\phi,\nu}(\xi) \\ &= \left[1 - \exp\left(2s \int_{\mathbb{R}^d} (1 - \cos\langle \xi, z\rangle)\nu(dz)\right)\right] \frac{\int_{\mathbb{R}^d}(1 - \cos\langle\xi, z\rangle)\phi(z)\nu(dz)}{\int_{\mathbb{R}^d}(1 - \cos\langle\xi, z\rangle)\nu(dz)} \end{aligned}$$

if $\int_{\mathbb{R}^d}(1 - \cos\langle\xi, z\rangle)\nu(dz) \ne 0$, and $M(\xi) = 0$ otherwise.

We are ready to establish the following dual version of (1.8).

Theorem 3.3. *Assume that $1 < q < \infty$ and let $m : \mathbb{R}^d \to \mathbb{C}$ be a multiplier as in Theorem 1.2. Then for any function $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ we have*

$$||T_m f||^q_{L^q(\mathbb{R}^d)} \le C_q ||f||_{L^1(\mathbb{R}^d)} ||f||^{q-1}_{L^\infty(\mathbb{R}^d)}. \tag{3.3}$$

Proof. By homogeneity, it suffices to establish the bound for f bounded by 1. Furthermore, we may and do assume that at least one of the measures μ, ν is nonzero. It is convenient to split the reasoning into two parts.

Step 1. First we show the estimate for the multipliers of the form

$$M_{\phi,\nu}(\xi) = \frac{\int_{\mathbb{R}^d}(1 - \cos\langle\xi, z\rangle)\phi(z)\nu(\mathrm{d}z)}{\int_{\mathbb{R}^d}(1 - \cos\langle\xi, z\rangle)\nu(\mathrm{d}z)}. \tag{3.4}$$

Assume that $0 < \nu(\mathbb{R}^d) < \infty$, so that the above machinery using Lévy processes is applicable. Fix $s < 0$ and functions f, $g \in C_0^\infty(\mathbb{R}^d)$ such that f is bounded by

1; of course, then the martingale $F^{x,s,f}$ also takes values in the unit ball of $\mathbb{C}$. By Hölder's inequality, Fubini's theorem and (2.1), we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d}\mathbb{E}\big[G_0^{x,s,f,\phi}g(x+X_{s,0})\big]\mathrm{d}x\right| \\
&\qquad\leq\left(\int_{\mathbb{R}^d}\mathbb{E}|G_0^{x,s,f,\phi}|^q\mathrm{d}x\right)^{1/q}\left(\int_{\mathbb{R}^d}\mathbb{E}|g(x+X_{s,0})|^p\mathrm{d}x\right)^{1/p} \\
&\qquad=\left(\int_{\mathbb{R}^d}\mathbb{E}|G_0^{x,s,f,\phi}|^q\mathrm{d}x\right)^{1/q}||g||_{L^p(\mathbb{R}^d)} \qquad (3.5)\\
&\qquad\leq\left(C_q\int_{\mathbb{R}^d}\mathbb{E}|F_0^{x,s,f}|\mathrm{d}x\right)^{1/q}||g||_{L^p(\mathbb{R}^d)} \\
&\qquad=(C_q||f||_1)^{1/q}\,||g||_{L^p(\mathbb{R}^d)}.
\end{aligned}$$

Plugging this into the definition of $\mathcal{S}$, we obtain

$$||\mathcal{S}^{s,\phi,\nu}f||^q_{L^q(\mathbb{R}^d)}\leq C_q||f||_{L^1(\mathbb{R}^d)}.$$

Now if we let $s\to-\infty$, then $M_{s,\phi,\nu}$ converges pointwise to the multiplier $M_{\phi,\nu}$ given by (3.4). By Plancherel's theorem, $\mathcal{S}^{s,\phi,\nu}f\to T_{M_{\phi,\nu}}f$ in $L^2(\mathbb{R}^d)$ and hence there is a sequence $(s_n)_{n=1}^\infty$ converging to $-\infty$ such that $\lim_{n\to\infty}\mathcal{S}^{s_n,\phi,\nu}f\to T_{M_{\phi,\nu}}f$ almost everywhere. Thus Fatou's lemma yields the desired bound for the multiplier $T_{M_{\phi,\nu}}$.

Step 2. Now we deduce the result for the general multipliers as in (1.4) and drop the assumption $0<\nu(\mathbb{R}^d)<\infty$. For a given $\varepsilon>0$, define a Lévy measure ν_ε in polar coordinates $(r,\theta)\in(0,\infty)\times\mathbb{S}$ by

$$\nu_\varepsilon(\mathrm{d}r\mathrm{d}\theta)=\varepsilon^{-2}\delta_\varepsilon(\mathrm{d}r)\mu(d\theta).$$

Here δ_ε denotes Dirac measure on $\{\varepsilon\}$. Next, consider a multiplier $M_{\varepsilon,\phi,\psi,\mu,\nu}$ as in (3.4), in which the Lévy measure is $1_{\{|x|>\varepsilon\}}\nu+\nu_\varepsilon$ and the jump modulator is given by $1_{\{|x|>\varepsilon\}}\phi(x)+1_{\{|x|=\varepsilon\}}\psi(x/|x|)$. Note that this Lévy measure is finite and nonzero, at least for sufficiently small ε. If we let $\varepsilon\to 0$, we see that

$$\begin{aligned}
\int_{\mathbb{R}^d}[1-\cos\langle\xi,x\rangle]\psi(x/|x|)\nu_\varepsilon(\mathrm{d}x)&=\int_{\mathbb{S}}\langle\xi,\theta\rangle^2\phi(\theta)\frac{1-\cos\langle\xi,\varepsilon\theta\rangle}{\langle\xi,\varepsilon\theta\rangle^2}\mu(d\theta)\\
&\to\frac{1}{2}\int_{\mathbb{S}}\langle\xi,\theta\rangle^2\phi(\theta)\mu(\mathrm{d}\theta)
\end{aligned}$$

and, consequently, $M_{\varepsilon,\phi,\psi,\mu,\nu}\to m_{\phi,\psi,\mu,\nu}$ pointwise. This yields the claim by the similar argument as above, using of Plancherel's theorem and the passage to the subsequence which converges almost everywhere. □

Now we shall apply duality to deduce (1.8).

Proof of Theorem 1.2. Observe that the class (1.4) is closed under the complex conjugation: we have $\bar{m}=m_{\bar{\phi},\bar{\psi},\mu,\nu}$. Fix $f\in L^p(\mathbb{R}^d)$ and put

$$g=\frac{T_mf}{|T_mf|}1_{\{x\in\mathbb{R}^d:|T_mf(x)|\geq 1\}}.$$

By Hölder's inequality and Parseval's identity,

$$
\begin{aligned}
|\{x \in \mathbb{R}^d : |T_m f(x)| \geq 1\}| &\leq \int_{\mathbb{R}^d} T_m f(x)\overline{g(x)} \mathrm{d}x \\
&= \int_{\mathbb{R}^d} \widehat{T_m f}(x)\overline{\widehat{g}(x)} \mathrm{d}x \\
&= \int_{\mathbb{R}^d} \widehat{f}(x)\overline{\widehat{T_{\bar{m}} g}(x)} \mathrm{d}x \qquad (3.6)\\
&= \int_{\mathbb{R}^d} f(x)\overline{T_{\bar{m}} g(x)} \mathrm{d}x \\
&\leq ||f||_{L^p(\mathbb{R}^d)} ||T_{\bar{m}} g||_{L^q(\mathbb{R}^d)} \\
&\leq ||f||_{L^p(\mathbb{R}^d)} \left(C_q ||g||_{L^1(\mathbb{R}^d)}\right)^{1/q}.
\end{aligned}
$$

Here in the latter passage we have used (3.3) and the fact that g takes values in the unit ball of $\mathbb{C}$. However, $||g||_{L^1(\mathbb{R}^d)} = |\{x \in \mathbb{R}^d : |T_m f(x)| \geq 1\}|$ and $C_q^{p/q} = K_q$. This completes the proof of the weak type estimate. □

In the remainder of this section we discuss the possibility of extending the assertion of Theorem 1.2 to the vector-valued multipliers. For any bounded function $m = (m_1, m_2, \ldots, m_n) : \mathbb{R}^d \to \mathbb{C}^n$, we may define the associated Fourier multiplier acting on complex valued functions on $\mathbb{R}^d$ by the formula $T_m f = (T_{m_1} f, T_{m_2} f, \ldots, T_{m_n} f)$. As we shall see, the reasoning presented above can be easily modified to yield the following statement.

Theorem 3.4. *Let ν, μ be two measures on $\mathbb{R}^d$ and $\mathbb{S}$, respectively, satisfying the assumptions of Theorem 1.2. Assume further that ϕ, ψ are two Borel functions on $\mathbb{R}^d$ taking values in the unit ball of $\mathbb{C}^n$ and let $m : \mathbb{R}^d \to \mathbb{C}^n$ be the associated symbol given by* (1.4)*. Then for any Borel function $f : \mathbb{R}^d \to \mathbb{C}$ we have*

$$
||T_m f||^q_{L^q(\mathbb{R}^d;\mathbb{C}^n)} \leq C_q ||f||_{L^1(\mathbb{R}^d)} ||f||^{q-1}_{L^\infty(\mathbb{R}^d)}, \qquad 1 < q < \infty,
$$

and

$$
||T_m f||_{L^{p,\infty}(\mathbb{R}^d;\mathbb{C}^n)} \leq K_p^{1/p} ||f||_{L^p(\mathbb{R}^d)}, \qquad 1 < p < \infty.
$$

Proof. Suppose first that ν is finite. For a given function $f \in C_0^\infty(\mathbb{R}^d)$ bounded by 1, we introduce the martingales F and $G = (G^1, G^2, \ldots, G^n)$ by (3.1). It is not difficult to check that Lemma 3.1 is also valid in the vector-valued setting (repeat the reasoning from [2]). Applying the representation (3.2) to each coordinate of G separately, we obtain the associated multiplier $\mathcal{S} = (\mathcal{S}^1, \mathcal{S}^2, \ldots, \mathcal{S}^n)$, where S^j has symbol $M_{\phi_j,\nu}$ defined in (3.4). Now we repeat the reasoning from (3.5), with a vector-valued function $g : \mathbb{R}^d \to \mathbb{C}^n$ (the expression $G_0^{x,s,f,\phi} g(x + X_{s,0})$ under the first integral is replaced with the corresponding scalar product). An application of (2.1) gives

$$
||S^{s,\phi,\nu} f||^q_{L^q(\mathbb{R}^d;\mathbb{C}^n)} \leq C_q ||f||_{L^1(\mathbb{R}^d)},
$$

which extends to general f by standard density arguments. The passage to general m as in (1.4) is carried over in the same manner as in the scalar case; this yields the vector version of Theorem 3.3. The duality argument explained in (3.6) extends to the vector-valued setting with no difficulty (one only has to replace appropriate multiplications by scalar products) and thus Theorem 1.2 holds true for the multipliers on $\mathbb{C}^n$. □

4. Weak-type bounds for the Beurling-Ahlfors transform

For the sake of clarity, we have decided to split this section into three parts.

4.1. **Upper bounds.** Let us rewrite the symbol corresponding to the Beurling-Ahlfors transform in the form

$$m(\xi)=\frac{\xi^2}{|\xi|^2}=\frac{\xi_1^2-\xi_2^2}{\xi_1^2+\xi_2^2}+i\frac{2\xi_1\xi_2}{\xi_1^2+\xi_2^2}.$$

The real and imaginary parts of m belong to the class (1.4). For instance, the choice $d=2$, $\mu=\delta_{(1,0)}+\delta_{(0,1)}$, $\psi(1,0)=-1=-\psi(0,1)$ and $\nu=0$ gives $T_m=\mathrm{Re}\,(\mathcal{BA})$; likewise, $d=2$, $\mu=\delta_{(1/\sqrt{2},1/\sqrt{2})}+\delta_{(1/\sqrt{2},-1/\sqrt{2})}$, $\psi(1/\sqrt{2},1/\sqrt{2})=1=-\psi(1/\sqrt{2},-1/\sqrt{2})$ and $\nu=0$ leads to $T_m=\mathrm{Im}\,(\mathcal{BA})$. Analogously, it can be shown that $\frac{1}{2}\mathcal{BA}$ also has the symbol as in (1.4). Thus Theorem 1.2 yields the following.

Theorem 4.1. *For any $1<p<\infty$ and $f\in L^p(\mathbb{C})$ we have*

$$|\{z\in\mathbb{C}:|Re\,\mathcal{BA}f(z)|\geq 1\}|\leq K_p||f||^p_{L^p(\mathbb{C})},$$
$$|\{z\in\mathbb{C}:|Im\,\mathcal{BA}f(z)|\geq 1\}|\leq K_p||f||^p_{L^p(\mathbb{C})}$$

and

$$|\{z\in\mathbb{C}:|\mathcal{BA}f(z)|\geq 1\}|\leq 2K_p||f||^p_{L^p(\mathbb{C})}.$$

4.2. **Lower bounds.** We consider the cases $1\leq p\leq 2$ and $p\geq 2$ separately.

Theorem 4.2. *For any $1\leq p\leq 2$ there is a real-valued function $f\in L^p(\mathbb{C})$ which satisfies*

$$|\{z\in\mathbb{C}:|\mathcal{BA}f(z)|\geq 1\}|=\left(\int_0^\infty|1-t|^pe^{-t}\,dt\right)^{-1}\int_{\mathbb{C}}|f(z)|^p\,dz.$$

Proof. Consider the function $w:\mathbb{C}\to\mathbb{C}$ given by $w(z)=\bar{z}\log|z|^2 1_{\{|z|<1\}}$. We easily derive that the complex partial derivatives of w are

$$\bar{\partial}w(z)=(1+\log|z|^2)1_{\{|z|<1\}}\qquad\text{and}\qquad\partial w(z)=\left(\frac{\bar{z}}{|z|}\right)^2 1_{\{|z|<1\}}.$$

Put $f=\overline{\partial}w$. Then, using the polar coordinates,

$$\int_{\mathbb{C}}|f(z)|^p\mathrm{d}z=2\pi\int_0^1|1+\log(r^2)|^pr\mathrm{d}r=\pi\int_0^\infty|1-t|^pe^{-t}\mathrm{d}t$$

and, since $\mathcal{BA}\,f=\partial w$ (see (1.6)),

$$|\{z\in\mathbb{C}:|\mathcal{BA}f(z)|\geq 1\}|=\pi.$$

This completes the proof. □

The corresponding lower bound in the case $p\geq 2$ is much more interesting. We obtain the same constant as in the martingale inequality (1.3) of Suh.

Theorem 4.3. *For any $p\geq 2$ and any $c<p^{p-1}/2$ there is a function f on $\mathbb{C}$ such that*

$$|\{z\in\mathbb{C}:|\mathcal{BA}f(z)|\geq 1\}|>c\int_{\mathbb{C}}|f(z)|^p\,dz-\varepsilon.$$

The further interesting fact is that the examples we are going to present are based on appropriate extremal martingales X, Y in (1.3) (i.e., those which yield the sharpness of this estimate).

Proof of Theorem 4.3. Fix a positive number δ, let $\alpha = (p-1)^{-1}$ and consider the sequences $(r_k)_{k\geq 0}$, $(a_k)_{k\geq 1}$ given as follows. The term $r_0 < 1$ will be specified later, while for $k \geq 0$,

$$r_{k+1} = (1+\delta)^{-k/(2\alpha)},$$

$$a_{2k+1} = \frac{p-1}{p}(1+\delta)^{1-2k} \quad \text{and} \quad a_{2k+2} = \frac{p-1}{p}(1+\delta)^{2k+1}.$$

Define $w : \mathbb{C} \to \mathbb{C}$ by

$$w(z) = \begin{cases} a_1 r_0^{2-2\alpha} \bar{z}^{-1} & \text{if } |z| \geq r_0, \\ a_{2k+1} z|z|^{-2\alpha} & \text{if } r_{2k+1} \leq |z| < r_{2k}, \\ a_{2k+2} z|z|^{2\alpha} & \text{if } r_{2k+2} \leq |z| < r_{2k+1} \end{cases} \tag{4.1}$$

for $k = 0, 1, 2, \ldots$. The detailed explanation of how we have discovered this function is given in Subsection 4.3 below. We easily compute the complex derivatives

$$\bar{\partial} w(z) = \begin{cases} -a_1 r_0^{2-2\alpha} \bar{z}^{-2} & \text{if } |z| \geq r_0, \\ -\alpha a_{2k+1} z^2 |z|^{-2\alpha-2} & \text{if } r_{2k+1} \leq |z| < r_{2k}, \\ \alpha a_{2k+2} z^2 |z|^{2\alpha-2} & \text{if } r_{2k+2} \leq |z| < r_{2k+1} \end{cases}$$

and

$$\partial w(z) = \begin{cases} 0 & \text{if } |z| \geq r_0, \\ (1-\alpha) a_{2k+1} |z|^{-2\alpha} & \text{if } r_{2k+1} \leq |z| < r_{2k}, \\ (1+\alpha) a_{2k+2} |z|^{2\alpha} & \text{if } r_{2k+2} \leq |z| < r_{2k+1}, \end{cases}$$

$k = 0, 1, 2, \ldots$. Note that for each k and each $z \in \mathbb{C}$ satisfying $r_{2k+2} \leq |z| < r_{2k+1}$ we have

$$\partial w(z) \geq (1+\alpha) a_{2k+2} r_{2k+2}^{2\alpha} = 1.$$

Consequently, we may write

$$\begin{aligned} |\{z \in \mathbb{C} : |\partial w| \geq 1\}| &\geq \pi \sum_{k=0}^{\infty} \left(r_{2k+1}^2 - r_{2k+2}^2\right) \\ &= \pi \sum_{k=0}^{\infty} \left[(1+\delta)^{-2k/\alpha} - (1+\delta)^{-(2k+1)/\alpha}\right] \\ &= \pi \left[1 + (1+\delta)^{-1/\alpha}\right]^{-1} \to \pi/2 \end{aligned}$$

as $\delta \to 0$. We turn to the integral $\int_{\mathbb{C}} |\bar{\partial} w|^p$. We have

$$\int_{\{|z|\geq r_0\}} |\bar{\partial} w(z)|^p \mathrm{d}z = 2\pi \int_{r_0}^{\infty} |a_1 r_0^{2-2\alpha}|^p r^{1-2p} \mathrm{d}r = \frac{\pi(p-1)^{p-1}(1+\delta)^p}{p^p} r_0^{2-2\alpha p}$$

and (recall that $r_1 = 1$)

$$\begin{aligned} \int_{\{r_1 \leq |z| < r_0\}} |\bar{\partial} w(z)|^p \mathrm{d}z &= 2\pi \int_1^{r_0} |\alpha a_1|^p r^{1-2\alpha p} \mathrm{d}r \\ &= \frac{\pi(p-1)(1+\delta)^p}{p^p}(1 - r_0^{2-2\alpha p}). \end{aligned}$$

Next, we easily check that $1/p \leq |\bar{\partial} w| \leq (1+\delta)/p$ on $\{|z| \leq r_1\}$ and hence

$$\frac{\pi}{p^p} \leq \int_{\{|z|\leq r_1\}} |\bar{\partial} w(z)|^p \mathrm{d}z \leq \frac{\pi(1+\delta)^p}{p^p}.$$

Combining the above three facts, we see that if we take r_0 sufficiently large and δ sufficiently small, then the integral $\int_{\mathbb{C}} |\bar{\partial} w|^p$ can be made arbitrarily close to $\pi(p-1)p^{-p} + \pi p^{-p} = \pi p^{-p+1}$. Thus, for any $\varepsilon > 0$ we have

$$\frac{|\{z \in \mathbb{C} : |\partial w| \geq 1\}|}{\int_{\mathbb{C}} |\bar{\partial} w(z)|^p \mathrm{d}z} \geq \frac{p^{p-1}}{2} - \varepsilon$$

with the appropriate choice of the parameters r_0 and δ. This completes the proof. $\square$

Since $\frac{1}{2}\mathcal{BA}$ has the symbol belonging to (1.4), we get the following.

Corollary 4.4. *For* $1 < p < \infty$, *let* κ_p *be the best constant in* (1.8). *Then*

$$\kappa_p \geq \begin{cases} \left(\int_0^\infty |1-t|^p e^{-t} dt\right)^{-1}/2^p & \text{if } 1 < p < 2, \\ p^{p-1}/2^{p+1} & \text{if } p \geq 2. \end{cases}$$

4.3. **On the search of the function w in the case $p \geq 2$.** Let us now sketch some steps which led us to the discovery of the function w above. First we present a pair (X, Y) of martingales which implies the sharpness of (1.3). Fix $\varepsilon \in (0, 1 - p^{-1})$, a positive integer N and put $\delta = (1 - p^{-1} - \varepsilon)/(2N)$. We assume that N is large enough so that $\varepsilon > (p-3)\delta$ and $\delta < (2p)^{-1}$. Consider the sequence $(\xi_n)_{n \geq 0}$ of independent mean-zero random variables with the distributions uniquely determined by the following assumptions:

(i) $\xi_0 \equiv \varepsilon/2$, $\xi_1 \in \{-\varepsilon/2, \varepsilon/2\}$,

(ii) for $n = 0, 1, 2, \ldots, N$,

$$\xi_{2n+2} \in \left\{\delta, -\frac{\varepsilon + 2n\delta}{p-1}\right\} \quad \text{and} \quad \xi_{2n+3} \in \left\{-\delta, \frac{\varepsilon + 2(n+1)\delta}{p-1} - \delta\right\},$$

(iii) we have

$$\xi_{2N+4} \in \{-\delta, p^{-1}\} \quad \text{and} \quad \xi_{2N+5} \in \{\delta, -p^{-1} + \delta\},$$

(iv) for $n \geq 2N + 6$, the random variable ξ_n has the same distribution as ξ_{n-4}.

Next, introduce the stopping time τ by

$$\tau = \begin{cases} 1 & \text{if } \xi_1 = \varepsilon/2, \\ \inf\{n \geq 2 : |\xi_n| \neq \delta\} & \text{if } \xi_1 = -\varepsilon/2 \end{cases}$$

and for any $t \geq 0$, define

$$X_t = \xi_0 + \xi_1 + \ldots + \xi_{\tau \wedge \lfloor t \rfloor} \quad \text{and} \quad Y_t = \xi_0 - \xi_1 + \xi_2 - \ldots + (-1)^{\lfloor t \rfloor} \xi_{\tau \wedge \lfloor t \rfloor}.$$

Clearly, Y is differentially subordinate to X. Moreover, it can be checked that the ratio $\mathbb{P}(\sup_{t \geq 0} |Y_t| \geq 1)/||X||_p^p = \mathbb{P}(|Y_\tau| \geq 1)/||X_\tau||_p^p$ can be made arbitrarily close to $p^{p-1}/2$, by choosing sufficiently small ε, δ and sufficiently large N. In fact, a careful analysis of this example reveals the following further properties of the terminal variable (X_τ, Y_τ):

1° If $\tau = 1$, then $(X_\tau, Y_\tau) = (\varepsilon, 0)$.

2° If $\tau \leq 2N + 3$, then $Y_\tau = (p-2)|X_\tau|$.

3° If $\tau \in \{2N + 4k, 2N + 4k + 1\}$, $k = 1, 2, \ldots$, then $Y_\tau \geq 1$ and $|X_\tau| = p^{-1}$.

4° If $\tau \in \{2N + 4k + 2, 2N + 4k + 3\}$, $k = 1, 2, \ldots$, then $Y_\tau \geq 1 - 2p^{-1}$ and $Y_\tau = (p-2)|X_\tau|$.

These four conditions are the key in the construction of the extremal functions announced in Theorem 4.3. Let us explain the connection now. First of all, it is clear that the complex plane $\mathbb{C}$ should correspond to Ω and the pair $(\overline{\partial}w, \partial w)$ should play the role of the terminal value (X_τ, Y_τ). Motivated by the examples of Baernstein and Montgomery-Smith [1], Iwaniec [11], Lehto [13] and others, it is natural to work with the functions of the form

$$w(z) = \begin{cases} b_0 z|z|^{2\beta_0} & \text{if } |z| \geq R_0, \\ b_n z|z|^{2\beta_n} & \text{if } |z| \in [R_n, R_{n-1}),\ n = 1,\ 2,\ \ldots, \end{cases}$$

for some parameters $(b_n)_{n\geq 0}$, $(\beta_n)_{n\geq 0}$ and $(R_n)_{n\geq 0}$ to be found. We derive that

$$\overline{\partial}w(z) = \begin{cases} b_0\beta_0 z^2|z|^{2\beta_0-2} & \text{if } |z| > R_0, \\ b_n\beta_n z^2|z|^{2\beta_n-2} & \text{if } |z| \in (R_n, R_{n-1}),\ n = 1,\ 2,\ \ldots, \end{cases}$$

and

$$\partial w(z) = \begin{cases} b_0(\beta_0+1)|z|^{2\beta_0} & \text{if } |z| > R_0, \\ b_n(\beta_n+1)|z|^{2\beta_n} & \text{if } |z| \in (R_n, R_{n-1}),\ n = 1,\ 2,\ \ldots. \end{cases}$$

A little thought and experimentation suggests that the set $\{z \in \mathbb{C} : |z| \geq R_0\}$ should correspond to the event $\{\tau = 1\}$; the annulus $\{z \in \mathbb{C} : R_0 \leq |z| < R_1\}$ should be the analogue of $\{\tau \leq 2N+3\}$; finally, that $\{z \in \mathbb{C} : R_n \leq |z| < R_{n-1}\}$, $n \geq 2$, should play the role of the set $\{\tau \in \{2N+4k, 2N+4k+1\}\}$ or $\{\tau \in \{2N+4k+2, 2N+4k+3\}\}$, depending on the parity of n. Now we exploit the algebraic relations between X_τ and Y_τ described in 1°–4°. The first condition suggests the equality $\partial w(z) = 0$ for $|z| > R_0$, since Y_τ vanishes on $\{\tau = 1\}$. This yields $\beta_0 = -1$. Next, the relation $Y_\tau = (p-2)|X_\tau|$, valid on $\{\tau \leq 2N+3\}$ and $\{\tau \in \{2N+4k+2, 2N+4k+3\}\}$, $k = 1,\ 2,\ \ldots$, implies $\beta_n = -(p-1)^{-1}$ for odd n. On the remaining annuli, motivated by 3°, we impose the condition $\partial w = p|\overline{\partial}w|$, which yields $\beta_n = (p-1)^{-1}$. The parameters b_n and R_n are determined by the condition $w \in W^{1,2}(\mathbb{C}, \mathbb{C})$ and the further requirements

$$\inf_{R_n < |z| < R_{n-1}} \partial w(z) = 1 \qquad \text{for even } n$$

and

$$\inf_{R_n < |z| < R_{n-1}} \partial w(z) = 1 - 2p^{-1} \qquad \text{for odd } n,$$

which are suggested by 3° and 4°. This yields the function w given by (4.1).

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