WEAK-TYPE ESTIMATES FOR MAXIMAL OPERATORS

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Abstract. The paper contains the study of sharp extensions of weak-type estimates for a martingale maximal function. Given $1 < p < \infty$ and a pair (x, y) of nonnegative numbers satisfying $x^p \leq y$, we identify the optimal upper bounds for $\|| \sup_n f_n |\|_{p,\infty}$, for nonnegative martingales $f = (f_n)_{n\geq 0}$ satisfying $\|f\|_1 = x$ and $\|f\|_p^p = y$.

1. Introduction

As evidenced in numerous works, maximal inequalities play a distinguished role in harmonic analysis and probability theory. The purpose of this paper is to present a refined study of certain weak-type estimates arising in the context of martingales.

We start with the description of the background and notation used throughout the text. In what follows, $(\Omega, \mathcal{F}, \mathbb{P})$ will denote the probability space, equipped with the discrete-time filtration $(\mathcal{F}_n)_{n\geq 0}$. For an adapted martingale $f = (f_n)_{n\geq 0}$, the symbol $f^* = \sup_{n\geq 0} |f_n|$ will stand for the associated maximal function, we will also use the notation $f_n^* = \max_{0\leq k\leq n} |f_k|$ for its truncated version ($n = 0, 1, 2, \ldots$).

Estimates between f and f^* (equivalent to the boundedness of the maximal operator on various function spaces) are of fundamental importance to the martingale theory and form the base for stochastic integration. For example, we have the classical weak- and strong-type estimates (cf. [2], see also [1, 5] for a different perspective)

$$\|f^*\|_{p,\infty} \leq \|f\|_p, \qquad 1 \leq p < \infty,$$
$$\|f^*\|_p \leq \frac{p}{p-1}\|f\|_p, \qquad 1 < p \leq \infty,$$

where $\|f^*\|_{p,\infty} = \sup_{\lambda>0} \left(\lambda \mathbb{P}(f^* \geq \lambda)\right)^{1/p}$ stands for the weak-L^p norm of f^* and $\|f\|_p = \sup_{n\geq 0} \left(\mathbb{E}|f_n|^p\right)^{1/p}$ is the L^p norm of a martingale f. Both inequalities are sharp: the constants 1 and $p/(p-1)$ cannot be decreased without additional conditions on f.

The purpose of this paper is to study a certain modification and extension of the weak-type bound. Consider the following alternative norming of the Lorentz space $L^{p,\infty}$: for $1 < p < \infty$ and an arbitrary random variable ξ, put

$$|||\xi|||_{p,\infty} = \sup\left\{\mathbb{P}(A)^{1/p-1}\int_A |\xi| \mathrm{d}\mathbb{P}\right\},$$

where the supremum is taken over all events A of positive probability. It is well-known that the quasinorms $\|\cdot\|_{p,\infty}$ and $|||\cdot|||_{p,\infty}$ are equivalent for $1 < p < \infty$ (cf. [3]): we have $\|\xi\|_{p,\infty} \leq |||\xi|||_{p,\infty} \leq c_p\|\xi\|_{p,\infty}$ for some constant c_p depending only on p. We will identify the optimal constant in the weak-type estimate under this new norming.

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Theorem 1.1. *For any $1 < p < \infty$ and any martingale f we have the sharp estimate*

$$|||f^*|||_{p,\infty} \leq \Gamma\left(\frac{2p-1}{p-1}\right)^{1-1/p} \|f\|_p. \tag{1.1}$$

Note that it is enough to study the above estimate for nonnegative martingales f only. Indeed, given an arbitrary, real-valued L^p-bounded martingale f, let us denote its pointwise limit by f_∞. Then the passage from f to the nonnegative martingale $(\mathbb{E}(|f_\infty|\,|\mathcal{F}_n))_{n\geq 0}$ does not change the right-hand side of (1.1), while the left-hand side can only increase. Thus, from now on, we will restrict ourselves to nonnegative martingales. We will be able to study the following much more precise version of (1.1). Namely, fix $1 < p < \infty$ and suppose that $f = (f_n)_{n\geq 0}$ is a nonnegative, L^p bounded martingale satisfying $\|f\|_1 = x$ and $\|f\|_p^p = y$. Here x, y are arbitrary positive numbers with $x^p \leq y$ (it is easy to see that for such x and y, there is at least one nonnegative martingale satisfying the norm requirements). What is optimal upper bound for $|||f^*|||_{p,\infty}$? Of course, (1.1) will give $|||f^*|||_{p,\infty} \leq \Gamma\left(\frac{2p-1}{p-1}\right)^{1-1/p} y^{1/p}$, but this does not have to be sharp: for example, if $x^p = y$, then f must be a constant martingale: $f_0 = f_1 = f_2 = \ldots \equiv x$ and hence $|||f^*|||_{p,\infty} = x$.

Our main result can be formulated as follows. Suppose that x, y are arbitrary positive numbers with $x^p \leq y$. Introduce the function

$$\mathbb{B}_p(x,y) = \sup\Big\{|||f^*|||_{p,\infty}\,:\, f\geq 0,\, \|f\|_1 = x,\, \|f\|_p^p = y\Big\}.$$

Theorem 1.2. *The function $\mathbb{B}_p$ is given by*

$$\mathbb{B}_p(x,y) = \begin{cases} \Gamma\left(\frac{2p-1}{p-1}\right)^{1-1/p} y^{1/p} & \text{if } \frac{p-1}{p}\Gamma\left(\frac{p}{p-1}\right)^{p-1} y < x^p, \\ c_*^{1-p}y + x + (p-1)c_*\gamma\left(\frac{x}{c_*}\right)^p - p\gamma\left(\frac{x}{c_*}\right)^{p-1} x & \text{otherwise}, \end{cases}$$

where $c_* = c_*(x,y)$ is defined in (2.5) below.

By a standard approximation, the above result extends to the continuous-time context. That is, if $(X_t)_{t\geq 0}$ is a nonnegative, continuous-time cádlág martingale satisfying $\|X\|_1 = x$ and $\|X\|_p^p = y$, then its maximal function $X^* = \sup_{t\geq 0}|X_t|$ satisfies

$$|||X^*|||_{p,\infty} \leq \mathbb{B}_p(x,y).$$

Furthermore, the constant on the right cannot be decreased: for each x, y there is a martingale X with prescribed first and p-th norms, for which both sides above are equal.

Let us say a few words about our approach. A natural idea is to apply Burkholder's method (sometimes referred to in the literature as the Bellman function technique). This approach relates a given martingale inequality to the existence of a certain special function, enjoying appropriate size and concavity requirements: convenient references on this subject are [1] and [4]. However, a direct application of the method requires the invention of a complicated special function of *four* variables (which control the first norm of f, the p-th norm of f, the size of the maximal function and the size of the event A which appears in the definition of the weak norm, respectively). To overcome this technical difficulty, we propose an alternative novel approach which is of independent interest. Namely, appropriate optimization and homogenization arguments allow to reduce the problem to the investigation of a much simpler martingale inequality. To prove this inequality, we will still use Burkholder's method, but this time the special functions will involve two variables only.

The paper is organized as follows. In the next section we introduce the technical background needed for our investigation and establish the simple martingale estimate discussed above. The final part of the paper contains the proofs of Theorem 1.1 and 1.2.

2. An auxiliary estimate

Throughout this section, we assume that $1 < p < \infty$ is a fixed parameter.

2.1. **Some technical facts.** We start our analysis with the introduction of a certain special function of one variable. Let $g : [0, \infty) \to [0, \infty)$ be given by

$$g(s) = p(p-1)\exp(ps^{p-1}) \int_s^\infty u^{p-1} \exp(-pu^{p-1}) \mathrm{d}u.$$

We will need the following properties of this object.

Lemma 2.1. *The function g is increasing and satisfies $g(s) > s$, $\lim_{s\to\infty} g(s)/s = 1$.*

Proof. The asymptotics $\lim_{s\to\infty} g(s)/s = 1$ follows easily by de l'Hospital rule. The estimate $g(s) > s$ is equivalent to

$$p(p-1) \int_s^\infty u^{p-1} \exp(-pu^{p-1}) \mathrm{d}u - s \exp(-ps^{p-1}) > 0.$$

It is enough to note that the left-hand side vanishes at infinity and its derivative at s equals $-\exp(-ps^{p-1}) < 0$. Finally, the monotonicity of g is a direct consequence of the equation

$$g'(s) = p(p-1)s^{p-2}(g(s) - s) \tag{2.1}$$

and the estimate $g(s) > s$ we have just established. □

Put $\lambda_0 = g(0) = p^{-1/(p-1)}\Gamma\left(\frac{p}{p-1}\right)$ and let $\gamma : [\lambda_0, \infty) \to [0, \infty)$ be the inverse to g. Then the estimate $g(s) > s$ implies that $\gamma(t) < t$ for $t \geq \lambda_0$. Furthermore, plugging $s = \gamma(t)$ into (2.1) and noting that $g'(\gamma(t))\gamma'(t) = 1$, we see that γ satisfies the differential equation

$$\gamma'(t) = \big(p(p-1)\gamma(t)^{p-2}(t - \gamma(t))\big)^{-1}, \qquad t > 0. \tag{2.2}$$

We extend γ to the function on the whole half-line $[0, \infty)$, setting $\gamma(t) = 0$ for $t < \lambda_0$. Later on, we will need the following property of γ.

Lemma 2.2. *The function $\xi : (0, \infty) \to \mathbb{R}$ given by*

$$\xi(x) = \left(\gamma(x)^p + \frac{x}{p-1}\right) x^{-p}$$

is nonincreasing and satisfies $\lim_{x\to 0} \xi(x) = \infty$, $\lim_{x\to\infty} \xi(x) = 1$.

Proof. The equality $\lim_{x\to 0} \xi(x) = \infty$ is evident, the identity $\lim_{x\to\infty} \xi(x) = 1$ follows directly from the asymptotics $\lim_{s\to\infty} g(s)/s = 1$ established in the previous lemma. We turn our attention to the monotonicity of ξ. This property holds on the interval

$x \in (0, \lambda_0]$, because $\gamma(x) = 0$ there. For $x \in (\lambda_0, \infty)$, we make the substitution $x = g(y)$; since g is increasing, we see that we must prove that the function

$$y \mapsto \left(y^p + \frac{g(y)}{p-1} \right) g(y)^{-p}$$

is nonincreasing on $(0, \infty)$. By the direct differentiation of this function and (2.1), it suffices to show that

$$g(y)^2 < p(g(y) - y)((p-1)y^p + g(y)). \tag{2.3}$$

However, integrating by parts, we obtain

$$g(y) = p(p-1)\exp(py^{p-1}) \int_y^\infty s^{p-1} \exp(-ps^{p-1}) \mathrm{d}s,$$

$$g(y) - y = \exp(py^{p-1}) \int_y^\infty \exp(-ps^{p-1}) \mathrm{d}s$$

and

$$(p-1)y^p + g(y) = p(p-1)^2 \exp(py^{p-1}) \int_y^\infty s^{2p-2} \exp(-ps^{p-1}) \mathrm{d}s. \tag{2.4}$$

Plugging these three identities into (2.3) we obtain the equivalent bound

$$\left(\int_y^\infty s^{p-1} \exp(-ps^{p-1}) \mathrm{d}s \right)^2 \leq \int_y^\infty \exp(-ps^{p-1}) \mathrm{d}s \cdot \int_y^\infty s^{2p-2} \exp(-ps^{p-1}) \mathrm{d}s,$$

which follows by Schwarz' inequality. □

Finally, we will need the following statement.

Lemma 2.3. *Assume that positive numbers x, y satisfy the condition $(p-1)\lambda_0^{p-1} y \leq x^p < y$. Then there is a unique root $c_* = c_*(x, y) \leq x/\lambda_0$ of the equation*

$$(1-p)c^{-p}y + (p-1)\gamma \left(\frac{x}{c} \right)^p + \frac{x}{c} = 0. \tag{2.5}$$

Furthermore, the function $c \mapsto c^{1-p}y + c(p-1)\gamma(x/c)^p - p\gamma(x/c)^{p-1}x$, considered on $(0, x/\lambda_0]$, attains its minimum for $c = c_*(x, y)$.

Proof. The equation (2.5) is equivalent to $\xi(x/c) = y/x^p$, so the existence and the uniqueness of the root follows at once from the previous lemma. To show that $c_*(x, y) \leq x/\lambda_0$ (that is, $x/c_*(x, y) \geq \lambda_0$), we use the monotonicity of ξ together with the estimate

$$\xi(\lambda_0) = \frac{\lambda_0^{1-p}}{p-1} \geq \frac{y}{x^p},$$

which is assumed in the statement of the lemma. The second part of the assertion follows from differentiation. Indeed, the derivative of the function in question is precisely the left-hand side of (2.5); obviously, this derivative is a continuous function and, as we have just proved, it has a unique zero. Thus it suffices to note that its value at $c = x/\lambda_0$ is nonnegative and its limit as $c \to 0$ is negative. The first inequality has already been analyzed above, the negativity of the limit follows at once from observing that

$$(1-p)c^{-p}y + (p-1)\gamma \left(\frac{x}{c} \right)^p + \frac{x}{c} = (1-p)c^{-p}(y - x^p) + (p-1) \left(\gamma \left(\frac{x}{c} \right)^p - \left(\frac{x}{c} \right)^p \right) + \frac{x}{c},$$

and recalling that $y > x^p$ and $\gamma(t) < t$ for all t. □

2.2. **Two martingale inequalities.** We are ready to introduce a family $(U_\lambda)_{\lambda\geq 0}$ of special functions, defined on the angular domain $D = \{(x, y) : 0 \leq x \leq y\}$, which will play a central role in this paper. First we need to consider appropriate subdomains D_i^λ of D. We consider two cases. If $\lambda \geq \lambda_0$, we introduce three domains D_0^λ, D_1^λ and D_2^λ, given by

$$\begin{aligned} D_1^\lambda &= \{(x, y) \in D : \lambda \leq y \leq g(x)\},\\ D_2^\lambda &= \{(x, y) \in D : x \geq \gamma(\lambda),\ y < \lambda\},\\ D_0^\lambda &= D \setminus (D_1^\lambda \cup D_2^\lambda). \end{aligned}$$

For $0 \leq \lambda < \lambda_0$, there are four domains, defined by

$$\begin{aligned} D_1^\lambda &= \{(x, y) \in D : \lambda_0 \leq y \leq g(x)\},\\ D_2^\lambda &= \{(x, y) \in D : y < \lambda\},\\ D_3^\lambda &= \{(x, y) \in D : \lambda \leq y < \lambda_0\},\\ D_0^\lambda &= D \setminus (D_1^\lambda \cup D_2^\lambda \cup D_3^\lambda). \end{aligned}$$

See Figure 1 below.

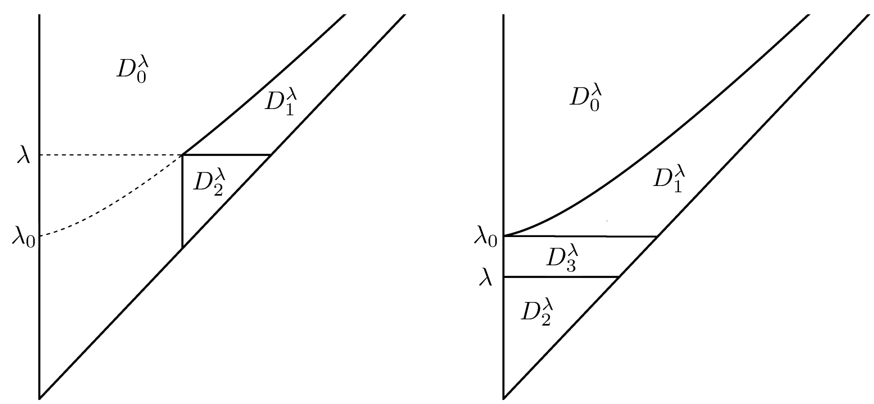

FIGURE 1. The subdomains D_i^λ in the case $\lambda \geq \lambda_0$ (left) and $\lambda < \lambda_0$ (right). On the right picture, the common boundary between D_0^λ and D_1^λ is the graph of the function g.

To define $U_\lambda : D \to \mathbb{R}$, we also consider two cases. For $\lambda \geq \lambda_0$, we let

$$U_\lambda(x, y) = \begin{cases} (y - \lambda)_+ - x^p & \text{if } (x, y) \in D_0^\lambda,\\ y - \lambda + (p - 1)\gamma(y)^p - p\gamma(y)^{p-1}x & \text{if } (x, y) \in D_1^\lambda,\\ (p - 1)\gamma(\lambda)^p - p\gamma(\lambda)^{p-1}x & \text{if } (x, y) \in D_2^\lambda, \end{cases}$$

while for $0 \leq \lambda < \lambda_0$, we put

$$U_\lambda(x, y) = \begin{cases} (y - \lambda)_+ - x^p & \text{if } (x, y) \in D_0^\lambda,\\ y - \lambda + (p - 1)\gamma(y)^p - p\gamma(y)^{p-1}x & \text{if } (x, y) \in D_1^\lambda,\\ x \ln(\lambda_0/\lambda) & \text{if } (x, y) \in D_2^\lambda,\\ y - \lambda + x \ln(\lambda_0/y) & \text{if } (x, y) \in D_3^\lambda. \end{cases}$$

It is easy to check that for any fixed $y>0$, the function $U_\lambda(\cdot,y)$ is of class C^1 on $[0,y]$, in particular, the partial derivative $\partial_x U_\lambda(x,y)$ exists for any $x\in[0,y]$. In the lemma below we study two further important properties of the above functions.

Lemma 2.4. *(i) For any $(x,y)\in D$ we have the majorization*

$$U_\lambda(x,y)\geq (y-\lambda)_+-x^p. \tag{2.6}$$

(ii) For any $(x,y)\in D$ and any $h\geq x$ we have the estimate

$$U_\lambda(x+h,(x+h)\vee y)\leq U_\lambda(x,y)+\partial_x U_\lambda(x,y)h \tag{2.7}$$

(we set $\partial_x U_\lambda(0,0)=0$).

Proof. We will only check the case $\lambda\geq\lambda_0$, for $0\leq\lambda<\lambda_0$ the reasoning is similar.

(i) The claim is trivial for $(x,y)\in D_0^\lambda$ (both sides are equal). If $(x,y)\in D_1^\lambda$ or $(x,y)\in D_2^\lambda$, the estimate (2.6) is equivalent to $(p-1)\gamma(y)^p+x^p\geq p\gamma(y)^{p-1}x$ or $(p-1)\gamma(\lambda)^p+x^p\geq p\gamma(\lambda)^{p-1}x$, respectively, which follows at once from Young's inequality.

(ii) It is obvious from the formulas on D_0^λ, D_1^λ and D_2^λ that for each y, the function $U_\lambda(\cdot,y):[0,y]\to\mathbb{R}$ is concave. Therefore, the estimate (2.7) holds true for $x+h\leq y$ and we may restrict ourselves to $x+h>y$. Exploiting the concavity of $U_\lambda(\cdot,y)$ again, we may write

$$\begin{aligned}U_\lambda(x,y)+\partial_x U_\lambda(x,y)h&=U_\lambda(x,y)+\partial_x U_\lambda(x,y)(y-x)+\partial_x U_\lambda(x,y)(x+h-y)\\&\geq U_\lambda(y,y)+\partial_x U_\lambda(y,y)(x+h-y).\end{aligned}$$

Thus we will be done if we show that $U_\lambda(x+h,x+h)\leq U_\lambda(y,y)+\partial_x U_\lambda(y,y)(x+h-y)$. To this end, we make three observations. First, the function $y\mapsto U_\lambda(y,y)$ is of class C^1 (straightforward); second, the function $y\mapsto\partial_x U_\lambda(y,y)$ is nonincreasing (this is also very simple); finally, we have $\partial_y U_\lambda(y,y)=0$ for any $y>0$: this is clear if $(y,y)\in D_0^\lambda\cup D_2^\lambda$, and follows from the differential equation (2.2) for $(y,y)\in D_1^\lambda$. Putting these observations together, we obtain that the function $y\mapsto U_\lambda(y,y)$ is concave and hence

$$\begin{aligned}U_\lambda(x+h,x+h)&\leq U_\lambda(y,y)+\big(\partial_x U_\lambda(y,y)+\partial_y U_\lambda(y,y)\big)(x+h-y)\\&=U_\lambda(y,y)+\partial_x U_\lambda(y,y)(x+h-y).\end{aligned}$$

The proof is complete. □

We are ready to prove the main results of this section.

Theorem 2.5. *Suppose that $(f_n)_{n\geq0}$ is an arbitrary nonnegative martingale bounded in L^p. Then for any $\lambda\geq0$ we have the estimate*

$$\mathbb{E}(f^*-\lambda)_+\leq\|f\|_p^p+U_\lambda(\mathbb{E}f_0,\mathbb{E}f_0). \tag{2.8}$$

Proof. Let us extend the filtration $(\mathcal{F}_n)_{n\geq0}$ by setting $\mathcal{F}_{-1}=\{\emptyset,\Omega\}$; this adds the variable $f_{-1}\equiv\mathbb{E}f_0$ to the martingale $(f_n)_{n\geq0}$ (and possibly increases its maximal function, but this will not affect the proof). The key observation is that the composition $(U_\lambda(f_n,f_n^*))_{n\geq-1}$ is a supermartingale. Indeed, the integrability of $U_\lambda(f_n,f_n^*)$ follows from the estimate $U_\lambda(x,y)\leq c_p(1+x^p+y^p)$, valid for some constant c_p depending only on p, and the assumed L^p-boundedness of f. Furthermore, for any $n\geq-1$ we have

$$\begin{aligned}\mathbb{E}\big[U_\lambda(f_{n+1},f_{n+1}^*)|\mathcal{F}_n\big]&=\mathbb{E}\Big[U_\lambda(f_n+df_{n+1},(f_n+df_{n+1})\vee f_n^*)|\mathcal{F}_n\Big]\\&\leq U_\lambda(f_n,f_n^*)+\partial_x U_\lambda(f_n,f_n^*)\mathbb{E}(df_{n+1}|\mathcal{F}_n)=U_\lambda(f_n,f_n^*),\end{aligned}$$

where the inequality follows directly from (2.7), applied with $x = f_n$, $y = f_n^*$ and $h = df_{n+1}$. Consequently, for any n we have $\mathbb{E}U_\lambda(f_n, f_n^*) \leq \mathbb{E}U_\lambda(f_{-1}, f_{-1}^*) = U_\lambda(\mathbb{E}f_0, \mathbb{E}f_0)$, which combined with (2.6) yields

$$\mathbb{E}(f_n^* - \lambda)_+ - \|f\|_p^p \leq \mathbb{E}(f_n^* - \lambda)_+ - \mathbb{E}f_n^p \leq \mathbb{E}U_\lambda(f_n, f_n^*) \leq U_\lambda(\mathbb{E}f_0, \mathbb{E}f_0).$$

It remains to let $n \to \infty$ and apply Lebesgue's monotone convergence theorem. □

We will also need the following refined version of the above estimate.

Theorem 2.6. *Suppose that $(f_n)_{n\geq 0}$ is an arbitrary nonnegative martingale with $\|f\|_1 = x$ and $\|f\|_p^p = y$. If $(p-1)\lambda_0^{p-1} y \leq x^p < y$, then for any $\lambda > 0$ we have*

$$\mathbb{E}(f^* - \lambda)_+ \leq \frac{K(x,y)^p}{p-1}\lambda^{1-p}, \tag{2.9}$$

where

$$K(x,y) = \frac{p-1}{p}\left[c_*^{1-p}y + x + (p-1)c_*\gamma\left(\frac{x}{c_*}\right)^p - p\gamma\left(\frac{x}{c_*}\right)^{p-1}x\right]$$

and $c_* = c_*(x,y)$ is given by (2.5).

Proof. We will consider two major cases $x \geq \lambda$ and $x < \lambda$ separately.

Case $x \geq \lambda$. Apply (2.8) to f/c_* and λ/c_* to obtain an estimate equivalent to

$$\mathbb{E}(f^* - \lambda)_+ \leq c_*^{1-p}\|f\|_p^p + c_* U_{\lambda/c_*}\left(\frac{x}{c_*}, \frac{x}{c_*}\right).$$

But $x/c_* \geq \lambda_0$, so we get

$$\mathbb{E}(f^* - \lambda)_+ \leq c_*^{1-p}y + x - \lambda + c_*(p-1)\gamma\left(\frac{x}{c_*}\right)^p - p\gamma\left(\frac{x}{c_*}\right)^{p-1}x$$

(no matter whether $\lambda/c_* \geq \lambda_0$ or not; in both cases, the formula for $U_{\lambda/c_*}(x/c_*, x/c_*)$ is the same, since $(x/c_*, x/c_*) \in D_1^{\lambda/c_*}$). That is, we have shown that $\mathbb{E}(f^* - \lambda)_+ \leq pK(x,y)/(p-1) - \lambda$, and the latter expression does not exceed $K(x,y)^p\lambda^{1-p}/(p-1)$, by virtue of Young's inequality.

Case $x < \lambda$. Pick the parameter $c = \lambda c_*/x$ and proceed as in the previous case to get

$$\mathbb{E}(f^* - \lambda)_+ \leq c^{1-p}\|f\|_p^p + cU_{\lambda/c}\left(\frac{x}{c}, \frac{x}{c}\right). \tag{2.10}$$

Now there are two sub-cases. If $x/c \geq \gamma(\lambda/c)$, this is equivalent to

$$\begin{aligned}\mathbb{E}(f^* - \lambda)_+ &\leq c^{1-p}y + (p-1)c\gamma\left(\frac{\lambda}{c}\right)^p - p\gamma\left(\frac{\lambda}{c}\right)^{p-1}x \\ &= \left(\frac{\lambda}{x}\right)^{1-p}\left[c_*^{1-p}y + (p-1)c_*\gamma\left(\frac{x}{c_*}\right)^p \cdot \left(\frac{\lambda}{x}\right)^p - p\gamma\left(\frac{x}{c_*}\right)^{p-1}x \cdot \left(\frac{\lambda}{x}\right)^{p-1}\right].\end{aligned}$$

Let us optimize the expression in the square brackets, considered as a function of λ/x. We know that $1 \leq \lambda/x \leq (x/c_*)/\gamma(x/c_*)$ (the second inequality is equivalent to $x/c \geq$

$\gamma(\lambda/c))$. Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[c_*^{1-p}y+(p-1)c_*\gamma\left(\frac{x}{c_*}\right)^p\cdot s^p-p\gamma\left(\frac{x}{c_*}\right)^{p-1}x\cdot s^{p-1}\right]$$
$$=p(p-1)\gamma\left(\frac{x}{c_*}\right)^p s^{p-2}c_*\left(s-\frac{x/c_*}{\gamma(x/c_*)}\right)<0$$

for $s=\lambda/x$. That is, the expression in the square brackets is the largest for $s=1$, and hence we obtain

$$\mathbb{E}(f^*-\lambda)_+\leq\left(\frac{\lambda}{x}\right)^{1-p}\left[c_*^{1-p}y+(p-1)c_*\gamma\left(\frac{x}{c_*}\right)^p-p\gamma\left(\frac{x}{c_*}\right)^{p-1}x\right], \tag{2.11}$$

or $\mathbb{E}(f^*-\lambda)_+\leq(\lambda/x)^{1-p}\left(pK(x,y)/(p-1)-x\right)$. It remains to note that the latter expression is not bigger than $K(x,y)^p\lambda^{1-p}/(p-1)$, by Young's inequality.

Finally, we need to consider the second sub-case $x/c<\gamma(\lambda/c)$. Under this assumption, the estimate (2.10) becomes

$$\mathbb{E}(f^*-\lambda)_+\leq c^{1-p}(y-x^p)=\left(\frac{\lambda}{x}\right)^{1-p}c_*^{1-p}(y-x^p),$$

which implies (2.11) and the assertion, by the Young inequality again. □

3. Proofs of Theorems 1.1 and 1.2

We start with Theorem 1.2. If $y=x^p$, then the claim is trivial: the only martingale which satisfies the conditions $\|f\|_1=x$ and $\|f\|_p^p=x^p$ is the constant one: $f\equiv x$, for which $|||f|||_{p,\infty}=x$. Hence, from now on, we will assume that $y>x^p$.

3.1. **Proof of the upper bound for $\mathbb{B}_p$.** We consider two major cases: $(p-1)\lambda_0^{p-1}y\geq x^p$ and $(p-1)\lambda_0^{p-1}y<x^p$. In the first case, we apply the estimate (2.8) with $\lambda=\lambda_0$ and the martingale f/c, where c is a positive parameter which will be specified in a moment. Since $U_{\lambda_0}(s,s)\leq 0$ for all $s\geq 0$, we obtain $\mathbb{E}(f^*/c-\lambda_0)_+\leq\|f/c\|_p^p$, or

$$\mathbb{E}(f^*-c\lambda_0)_+\leq c^{1-p}\|f\|_p^p=c^{1-p}y.$$

Now pick an arbitrary event A of positive probability. We may write

$$\int_A f^*\mathrm{d}\mathbb{P}=c\lambda_0\mathbb{P}(A)+\int_A(f^*-c\lambda_0)\mathrm{d}\mathbb{P}\leq c\lambda_0\mathbb{P}(A)+\int_A(f^*-c\lambda_0)_+\mathrm{d}\mathbb{P}$$
$$\leq c\lambda_0\mathbb{P}(A)+\mathbb{E}(f^*-c\lambda_0)_+\leq c\lambda_0\mathbb{P}(A)+c^{1-p}y.$$

The latter expression, considered as a function of c, attains its minimal value for $c=\left((p-1)y\lambda_0^{-1}\mathbb{P}(A)^{-1}\right)^{1/p}$. Plugging this choice above, we get

$$\int_A f^*\mathrm{d}\mathbb{P}\leq\frac{p}{p-1}\lambda_0^{1-1/p}((p-1)y)^{1/p}\mathbb{P}(A)^{1-1/p}=\Gamma\left(\frac{2p-1}{p-1}\right)^{1-1/p}y^{1/p}\cdot\mathbb{P}(A)^{1-1/p}.$$

Since A was arbitrary, the estimate follows. In the case $(p-1)\lambda_0^{p-1}y<x^p$ the reasoning is similar, but we apply Theorem 2.6 instead. Namely, we take an arbitrary event A with $\mathbb{P}(A)>0$ and argue as above, obtaining

$$\int_A f^*\mathrm{d}\mathbb{P}\leq\lambda\mathbb{P}(A)+\frac{K(x,y)^p}{p-1}\lambda^{1-p}.$$

The expression on the right, considered as a function of λ, attains its minimum for $\lambda = K(x,y)\mathbb{P}(A)^{-1/p}$. Plugging this special λ above, we obtain the claim.

3.2. **Proof of the lower bound for $\mathbb{B}_p$.** We will proceed directly and construct appropriate examples. It is enough to consider continuous-time martingales: as we have mentioned in the introductory section. As previously, we consider two cases.

Case $(p-1)\lambda_0^{p-1}y \leq x^p$. Let $c_* = c_*(x,y)$ be the number defined in (2.5). Consider the probability space equal to the interval $[0,1]$, equipped with its Borel subsets and the Lebesgue measure. We introduce the continuous-time filtration $(\mathcal{F}_t)_{t\in[0,1]}$, where the σ-algebra $\mathcal{F}_t$ is generated by the interval $[0,1-t]$ and all Borel subsets of $(1-t,1]$. Consider the random variable

$$\xi(\omega) = c_*\left(\gamma\left(\frac{x}{c_*}\right)^{p-1} - \frac{\ln\omega}{p}\right)^{1/(p-1)}$$

and the associated martingale $(\xi_t)_{t\in[0,1]} = (\mathbb{E}(\xi|\mathcal{F}_t))_{t\in[0,1]}$. We compute that

$$\begin{aligned}
\mathbb{E}\xi &= c_*\int_0^1 \left(\gamma\left(\frac{x}{c_*}\right)^{p-1} - \frac{\ln\omega}{p}\right)^{1/(p-1)} \mathrm{d}\omega\\
&= c_*\int_0^\infty \left(\gamma\left(\frac{x}{c_*}\right)^{p-1} + \frac{u}{p}\right)^{1/(p-1)} e^{-u}\mathrm{d}u\\
&= c_*p^{-1/(p-1)}\int_{p\gamma(x/c_*)^{p-1}}^\infty u^{1/(p-1)}\exp\left(-u+p\gamma\left(\frac{x}{c_*}\right)^{p-1}\right)\mathrm{d}u\\
&= c_*p(p-1)\exp\left(p\gamma\left(\frac{x}{c_*}\right)^{p-1}\right)\int_{\gamma(x/c_*)}^\infty t^{p-1}\exp(-pt^{p-1})\mathrm{d}t\\
&= c_*g\left(\gamma\left(\frac{x}{c_*}\right)\right) = c_*\cdot\frac{x}{c_*} = x
\end{aligned}$$

and, similarly,

$$\begin{aligned}
\mathbb{E}\xi^p &= c_*^p\int_0^1 \left(\gamma\left(\frac{x}{c_*}\right)^{p-1} - \frac{\ln\omega}{p}\right)^{p/(p-1)} \mathrm{d}\omega\\
&= c_*^p p(p-1)\exp\left(p\gamma\left(\frac{x}{c_*}\right)^{p-1}\right)\int_{\gamma(x/c_*)}^\infty t^{2p-2}\exp(-pt^{p-1})\mathrm{d}t\\
&= c_*^p\left[\gamma\left(\frac{x}{c_*}\right)^p + \frac{g\left(\gamma\left(x/c_*\right)\right)}{p-1}\right] = c_*^p\left[\gamma\left(\frac{x}{c_*}\right)^p + \frac{x/c_*}{p-1}\right],
\end{aligned}$$

where in the third line we have applied (2.4). However, by (2.5), the last expression above is equal to y: thus, $\mathbb{E}\xi^p = y$. Finally, the maximal function of $(\xi_t)_{t\in[0,1]}$ satisfies $\xi^*(\omega) \geq \xi_\omega(\omega) = (1-\omega)^{-1}\int_0^{1-\omega}\xi(s)\mathrm{d}s$. Carrying out similar calculations to those above, we obtain

$$|||\xi^*|||_{p,\infty} \geq \mathbb{E}\xi^* \geq c_*\left(\frac{px/c_*}{p-1} + p\gamma\left(\frac{x}{c_*}\right)^p - p\gamma\left(\frac{x}{c_*}\right)^{p-1}\frac{x}{c_*}\right),$$

which is the desired lower bound, by (2.5).

Case $(p-1)\lambda_0^{p-1}y > x^p$. Consider the auxiliary parameters

$$\bar{x} = (p-1)^{1/(p-1)}\left(\frac{y}{x}\right)^{1/(p-1)}\lambda_0 \qquad \text{and} \qquad \bar{y} = (p-1)^{1/(p-1)}\left(\frac{y}{x}\right)^{p/(p-1)}\lambda_0.$$

Note that $\bar{x} > x$, by the assumption of our case. Furthermore, we check easily that $(p-1)\lambda_0^{p-1}\bar{y} = \bar{x}^p$, so by the construction from the previous case, there is a martingale $\bar{\xi}$ satisfying $\|\bar{\xi}\|_1 = \bar{x}$, $\|\bar{\xi}\|_p^p = \bar{y}$ and

$$\mathbb{E}\bar{\xi}^* = \mathbb{B}_p(\bar{x},\bar{y}) = \bar{x} + \lambda_0^{p-1}\bar{x}^{1-p}\bar{y} = p(p-1)^{1/(p-1)-1}\lambda_0\left(\frac{y}{x}\right)^{1/(p-1)}.$$

We introduce the martingale $(\xi_t)_{t\in[0,1]}$ as follows. First we pick an event A of probability $x/\bar{x}$. We assume that on A^c, the compliment of A, the martingale $(\xi_t)_{t\in[0,1]}$ is constant and equal to zero; on the other hand, on A its conditional distribution is the same as that of $\bar{\xi}$. Then we have $\|\xi\|_1 = \|\bar{\xi}\|_1 \cdot x/\bar{x} = x$, $\|\xi\|_p^p = \|\bar{\xi}\|_p^p \cdot x/\bar{x} = x\bar{y}/\bar{x} = y$ and

$$|||\xi|||_{p,\infty} \geq \mathbb{P}(A)^{1/p-1}\int_A \xi^* \mathrm{d}\mathbb{P} = \mathbb{P}(A)^{1/p}\cdot\mathbb{E}\bar{\xi}^* = (x/\bar{x})^{1/p}\cdot\mathbb{B}_p(\bar{x},\bar{y}).$$

3.3. **Proof of Theorem 1.1.** To show (1.1), it is enough to prove the estimate

$$\mathbb{B}_p(x,y) \leq \Gamma\left(\frac{2p-1}{p-1}\right)^{1-1/p} y^{1/p}.$$

If $(p-1)\lambda_0^{p-1}y \geq x^p$, then both sides are equal; otherwise, the inequality is equivalent to

$$c_*^{1-p}y + x + (p-1)c_*\gamma\left(\frac{x}{c_*}\right)^p - p\gamma\left(\frac{x}{c_*}\right)^{p-1}x \leq \frac{p\lambda_0^{1-1/p}}{(p-1)^{1-1/p}}y^{1/p}.$$

Lemma 2.3 implies that the left-hand side of above inequality is not greater than

$$h(x) = a_y^{1-p}y + x + (p-1)a_y\gamma\left(\frac{x}{a_y}\right)^p - p\gamma\left(\frac{x}{a_y}\right)^{p-1}x,$$

where $a_y = [(p-1)y]^{1/p}\lambda_0^{-1/p} < x/\lambda_0$. However, $h(x) \leq a_y^{1-p}y + a_y\lambda_0$. Indeed, for $x = a_y\lambda_0$ we have equality here and $h'(x) = -p\gamma\,(x/a_y)^{p-1} \leq 0$. It remains to observe that $a_y^{1-p}y + a_y\lambda_0 = p(p-1)^{1/p-1}\lambda_0^{1-1/p}y^{1/p}$. This gives (1.1). Its sharpness follows at once from the fact that $\mathbb{B}_p(x,y) = \Gamma\left(\frac{2p-1}{p-1}\right)^{1-1/p} y^{1/p}$ on a part of the domain of $\mathbb{B}_p$.

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