SHARP WEAK TYPE INEQUALITIES FOR FRACTIONAL INTEGRAL OPERATORS

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ABSTRACT. Suppose that $d \geq 1$ is an integer, $\alpha \in (0, d)$ is a fixed parameter and let I_α be the fractional integral operator associated with d-dimensional Walsh-Fourier series on $(0, 1]^d$. Let p, q be arbitrary numbers satisfying the conditions $1 \leq p < d/\alpha$ and $1/q = 1/p - \alpha/d$. We determine the optimal constant $K(\alpha, \beta, p)$ depending only on the parameters indicated such that for any $f \in L^p((0, 1]^d)$ we have

$$||I_\alpha f||_{L^{q,\infty}((0,1]^d)} \leq K(\alpha, \beta, p)||f||_{L^p((0,1]^d)}.$$

Actually, we study this inequality in a more general context of probability spaces equipped with a regular tree-like structures. This allows us to obtain this result also for non-integer dimension. The proof exploits a certain modification of the so-called Bellman function method and appropriate interpolation-type arguments. We also present a sharp weighted weak-type bound for I_α, which can be regarded as a version of the Muckenhoupt-Wheeden conjecture for fractional integral operators.

1. INTRODUCTION

Let d be a positive integer. For $0 < \alpha < d$, the fractional integral (or Riesz potential) I_α is defined by the formula

$$I_\alpha f(x) = \frac{4^{(d-\alpha)/2}}{(4\pi)^{d/2}} \frac{\Gamma\big((d-\alpha)/2\big)}{\Gamma(\alpha/2)} \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} \mathrm{d}y. \tag{1.1}$$

Recalling that the that Fourier transform of the Laplacian $-\Delta$ is given by $\widehat{(-\Delta)}f(\xi) = 4\pi^2|\xi|^2\widehat{f}(\xi)$, the constant in front of the integral is chosen so

$$I_\alpha(f)(x) = (-\Delta)^{-\alpha/2} f(x). \tag{1.2}$$

The above operators play an important role in analysis. For example, as evidenced in the monographs of Stein [33] and Grafakos [14], they can be used in the study of differentiability or smoothness properties of functions. It is well known that if $1 < p < d/\alpha$ and q satisfies the relation $1/q = 1/p - \alpha/d$, then I_α maps L^p into L^q; furthermore, in the limit cases $(p, q) = \{(1, d/(d-\alpha)), (d/\alpha, \infty)\}$ the corresponding $L^p \to L^q$ estimates do not hold. The boundedness has many important applications, e.g. it leads to Sobolev embedding theorems and related comparisons between sizes of functions and their derivatives.

A simple Fubini theorem argument can be used to show that these operators also have a representation in terms of the heat semigroup of Brownian motion in $\mathbb{R}^d$. More precisely, with our normalization given above we have

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$$I_\alpha(f)(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} P_t f(x)\, dt, \tag{1.3}$$

where $P_t f(x)$ is the convolution of f with the Gaussian kernel $\frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/(4t)}$. From the representation in (1.3), it was shown in [1] that $I_\alpha(f)(x)$ also has a representation as the projection of a martingale transform of a stochastic integral (with unbounded multiplier) in the style of the representation given for Riesz transforms and many other singular integrals and Fourier multipliers; see [2] for an overview of such results. Such a representation for the fractional integrals raises many questions about the possibility of obtaining sharp inequalities for these operators using the martingale transform techniques of Burkholder [4] and their many extensions and refinements (as presented in, for example, [24]), which have been so effectively used to obtain bounds for the classical Riesz transforms and other Fourier multipliers.

The purpose of this paper is to study the properties of closely related operators arising in the context of the d-dimensional Walsh system where the martingale tools can be brought to bear. Consider the unit cube $(0, 1]^d$ in $\mathbb{R}^d$ equipped with the lattice of its dyadic subcubes. That is, consider sets of the form $\left(\frac{a_1}{2^n}, \frac{a_1+1}{2^n}\right] \times \left(\frac{a_2}{2^n}, \frac{a_2+1}{2^n}\right] \times \ldots \left(\frac{a_d}{2^n}, \frac{a_d+1}{2^n}\right]$ for some nonnegative integer n and some $a_1, a_2, \ldots, a_d \in \{0, 1, \ldots, 2^n - 1\}$. Recall that the Rademacher system $\{r_n\}_{n\geq 0}$ of functions on $(0, 1]$ is given by

$$r_n(t) = \operatorname{sgn}\left(\sin(2^{n+1}\pi t)\right).$$

Then $\{w_n\}_{n\geq 0}$, the Walsh system on $(0, 1]$, is defined as follows: $w_0 \equiv 1$ and if n is a positive integer with $n = 2^{n_1} + 2^{n_2} + \ldots + 2^{n_k}$ and $n_1 > n_2 > \ldots > n_k$, then

$$w_n(t) = r_{n_1}(t) r_{n_2}(t) \ldots r_{n_k}(t).$$

The extension of the Walsh system to the d-dimensional setting is the collection of all functions on $(0, 1]^d$ which are of the form

$$x = (x_1, x_2, \ldots, x_d) \mapsto w_{j_1}(x_1) w_{j_2}(x_2) \ldots w_{j_d}(x_d),$$

where $j_1, j_2, \ldots, j_d$ are nonnegative integers.

Now, assume that f is a Lebesgue-integrable function on the cube $(0, 1]^d$. We define the associated rectangular partial sums of d-dimensional Walsh-Fourier series by the formula

$$S_{n_1,n_2,\ldots,n_d}(f)(x) = \sum_{j_1=0}^{n_1-1} \sum_{j_2=0}^{n_2-1} \ldots \sum_{j_d=0}^{n_d-1} \hat{f}(j_1, j_2, \ldots, j_d) \prod_{k=1}^d w_{j_k}(x_k).$$

Here $x = (x_1, x_2, \ldots, x_d) \in (0, 1]^d$ and

$$\hat{f}(j_1, j_2, \ldots, j_d) = \int_{[0,1)^d} f(x) \prod_{k=1}^d w_{j_k}(x_k) \mathrm{d}x$$

is the $(j_1, j_2, \ldots, j_d)$th Walsh-Fourier coefficient of f. There has been a considerable interest in the relation between the size of f and the behavior of the partial sum $S_{n,n,\ldots,n}(f)$. We refer the reader to the works of Goginava [11], [12], Goginava and Weisz [13], Nagy [20], Simon [29], [30] and Weisz [39], [40], [41]. Our contribution is the study of properties of fractional integral operators which arise naturally in this setting. The definition is the following. Given a parameter $\alpha \in (0, d)$, the associated fractional integral operator I_α

is given by

$$I_\alpha f = S_{0,0,\ldots,0}(f) + \sum_{k=1}^{\infty} 2^{-k\alpha}\big(S_{k,k,\ldots,k}(f) - S_{k-1,k-1,\ldots,k-1}(f)\big). \tag{1.4}$$

This is the discrete and localized version of the usual fractional integral operator in $\mathbb{R}^d$, discussed at the beginning. In the literature, one may encounter an alternative definition

$$\mathcal{I}_\alpha f = \sum_{k=0}^{\infty} 2^{-k\alpha} S_{k,k,\ldots,k}(f),$$

which differs from the preceding one by the multiplicative factor only: we have $I_\alpha = (1-2^{-\alpha})\mathcal{I}_\alpha$. The operator (1.4) was introduced and studied by Watari [38]. A convenient reference, which presents a probabilistic approach, is the paper of Chao and Ombe [7]. For more recent works, we refer the interested reader to the works of Lacey et. al. [16], Cruz-Uribe and Moen [8] and Osękowski [25]. The arguments presented in the first two papers can be used to prove that the fractional integral operator is bounded as an operator from $L_p((0,1]^d)$ to $L_q((0,1]^d)$, where, as in the classical case, $1 < p \leq d/\alpha$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$. Furthermore (cf. [25]), for the limit values $p = 1$, $q = d/(d-\alpha)$, we have the sharp weak-type estimate

$$|||I_\alpha f|||_{L^{q,\infty}((0,1]^d)} = (1-2^{-\alpha})|||\mathcal{I}_\alpha f|||_{L^{q,\infty}((0,1]^d)} \leq \frac{2^{d-\alpha}-2^{-\alpha}}{2^{d-\alpha}-1}||f||_{L^p((0,1]^d)}, \tag{1.5}$$

where

$$|||f|||_{L^{q,\infty}((0,1]^d)} = \sup_{\lambda>0} \lambda |\{x \in (0,1]^d : |f(x)| \geq \lambda\}|^{1/q}$$

is the usual weak-type quasi-norm. These results are in perfect correspondence with those for the classical Riesz potentials.

The principal goal of this paper is to extend (1.5) to other values of p and q. It will be convenient for us to work under a different (but equivalent) norming of the weak spaces:

$$||f||_{L^{q,\infty}((0,1]^d)} = \sup\left\{ \frac{1}{|E|^{1-1/q}} \int_E |f| \mathrm{d}x \,:\, E \subset (0,1]^d, |E| > 0 \right\}.$$

We will prove the following fact.

Theorem 1.1. *Let p, q, α be arbitrary numbers satisfying $0 < \alpha < d$, $1 \leq p < d/\alpha$ and $1/q = 1/p - \alpha/d$. Then*

$$\begin{aligned} &||I_\alpha||_{L^p((0,1]^d)\to L^{q,\infty}((0,1]^d)} \\ &= \frac{2^{d-\alpha}-2^{-\alpha}}{2^{d-\alpha}-1}\left(1 + \frac{(1-2^{-\alpha})^{p'}}{(1-2^{-d})^{p'-1}(2^{p'(d-\alpha)-d}-1)}\right)^{1/p'}, \end{aligned} \tag{1.6}$$

where p' is the harmonic conjugate to p.

Actually, we will work in a more general setting of probability spaces equipped with regular tree-like structures (for the definitions, see Section 2 below). In particular, this will enable us to obtain a version of the above theorem for non-integer values of d as well.

Let us say a few words about the structure of our approach. The main point, presented in Theorem 3.1, is a restricted-type estimate for I_α. More specifically, for an arbitrary set A, we provide a sharp upper bound for the essential supremum of $I_\alpha \chi_A$ in terms of the measure of A. Then, using some interpolation-type arguments, we are able to extend this result to a class of appropriate sharp upper bounds for p-th norms of $I_\alpha\chi_A$, and establish the desired weak-type estimates using duality arguments.

The proof of Theorem 3.1 rests on Bellman function method, a powerful technique used widely in analysis and probability theory. Roughly speaking, the method enables one to deduce a given inequality from the existence of a certain special function, which enjoys some majorization and convexity-type properties. This type of approach originates from the theory of stochastic optimal control, and its connection with other areas of mathematics was first observed by Burkholder in [4], who studied sharp inequalities for martingale transforms. Since then, the method has been intensively developed in subsequent works of Burkholder and his students; a convenient reference on the subject is the monograph [24] by the second author. Furthermore, in the late 90's, Nazarov, Treil and Volberg showed that the method can be applied in the much wider setting of harmonic analysis. Since the seminal papers [22], [23], the technique has been used in numerous settings; see for example [9], [31], [32], [36], [37], and references therein. At this point we should mention that some attempts to introduce a general Bellman setup for the study of dyadic fractional integral operators can be found in [25]. However, in comparison to that paper, we investigate below the less restrictive setting of probability spaces equipped with tree-like structures. Furthermore, which is even more interesting, the results investigated here require several novel reductions and much more delicate analysis; this in turn leads to much more complicated Bellman functions than those appearing in [25].

This paper is organized as follows. In the next section we introduce the necessary probabilistic background and make the connection to martingales. In Section 3, we establish a sharp upper bound for $||I_\alpha\chi_A||_\infty$, using the Bellman function method. Section 4 contains the proof of Theorem 1.1 and its generalized, probabilistic version. In the final part of the paper, we show how our approach can lead to certain *weighted* weak-type bounds, which can be regarded as a version of Muckenhoupt-Wheeden conjecture for fractional integral operators.

2. PROBABILISTIC SETUP

Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a fixed, nonatomic probability space, equipped with the following tree-like structure $\mathcal{T}$.

Definition 2.1. A set $\mathcal{T}$ of measurable subsets of Ω will be called a tree if the following conditions are satisfied:

(i) $\Omega \in \mathcal{T}$ and for every $J \in \mathcal{T}$, we have $\mathbb{P}(J) > 0$.

(ii) For every $J \in \mathcal{T}$ there is a finite subset $C(J) \subset \mathcal{T}$ containing at least two elements such that

  (a) the elements of $C(J)$ are pairwise disjoint subsets of J and

  (b) $J = \bigcup C(J)$.

(iii) $\mathcal{T} = \bigcup_{m\geq 0} \mathcal{T}^m$, where $\mathcal{T}^0 = \{\Omega\}$ and $T^{m+1} = \bigcup_{J\in\mathcal{T}^m} C(J)$.

In what follows, we will need to work with trees satisfying certain regularity-type property. Our β below plays the role of the dimension d. We emphasize that, as in the Varopoulos' analysis of "finite dimensional" semigroups [34, 35], this "dimension" may not be an integer.

Definition 2.2. Let $\beta \geq 1$ be a given number. A tree $\mathcal{T}$ is called β-regular, if for any nonnegative integer n and any $J_1 \in \mathcal{T}^n$, $J_2 \in C(J_1)$ we have $\mathbb{P}(J_2)/\mathbb{P}(J_1) \in [2^{-\beta}, 1 - 2^{-\beta}]$.

Any tree-like structure gives rise to the corresponding filtration $(\mathcal{F}_n)_{n\geq 0}$, given by $\mathcal{F}_n = \sigma(J : J \in \mathcal{T}^n)$. Given an integrable random variable f, one can consider the associated martingale given by $(\mathbb{E}(f|\mathcal{F}_n))_{n\geq 0}$; this sequence will be denoted by $(f_n)_{n\geq 0}$ or, with a slight abuse of notation, again by the letter f. Note that such martingales are simple, i.e., for any nonnegative integer n, the random variable f_n takes only a finite number of values. This follows at once from the fact that $\mathcal{F}_n$ consists of finite number of sets. For a given martingale $f = (f_n)_{n\geq 0}$, we will denote the associated difference sequence by $df = (df_n)_{n\geq 0}$: $df_0 = f_0$ and $df_n = f_n - f_{n-1}$ for $n \geq 1$.

Any tree-like structure $\mathcal{T}$ also gives rise to the corresponding class of fractional integral operators. Given $\alpha \geq 0$ and an integrable random variable f, we define $g = (g_n)_{n\geq 0}$, the associated α-transform of f, by the the identity

$$g_n = \sum_{k=0}^{n} 2^{-k\alpha} df_k, \qquad n = 0,\, 1,\, 2,\, \ldots .$$

Of course, this is equivalent to saying that $dg_n = 2^{-n\alpha} df_n$ for each n. It follows from the escape inequalities of Burkholder [5] that the martingale g converges almost surely. Its pointwise limit will be denoted by $I_\alpha f$ and called the fractional integral of f. We would like to stress here that the operator I_α depend on the underlying tree and sometimes we will indicate this dependence by adding an appropriate superscript such as $I_\alpha^{\mathcal{T}}$. We will also often write $(I_\alpha f)_n$ instead of g_n.

Here is our main example which exploits all the concepts introduced above.

Example 2.1. *Suppose that the underlying probability space is the d-dimensional unit cube $(0,1]^d$ equipped with its Borel subsets and Lebesgue measure. Let $\mathcal{T}^k$ be a collection of all dyadic cubes of volume 2^{-kd}, contained in $(0,1]^d$. Then $\mathcal{T} = (\mathcal{T}^n)_{n\geq 0}$ is a d-regular tree. For a given integrable function $f : (0,1]^d \to \mathbb{R}$, one easily checks that the associated martingale $(f_n)_{n\geq 0}$ is just the sequence of appropriate partial sums associated with the Walsh-Fourier series: $f_n = \mathbb{E}(f|\mathcal{F}_n) = S_{n,n,\ldots,n}(f)$. Hence the corresponding fractional integral operator is given by*

$$I_\alpha f = \sum_{n=0}^{\infty} 2^{-n\alpha}\big(S_{n,n,\ldots,n}(f) - S_{n-1,n-1,\ldots,n-1}(f)\big)$$

(we use the convention $S_{-1,-1,\ldots,-1}(f) = 0$), which is precisely the fractional integral operator I_α defined in (1.4).

In our further considerations we will exploit the fact that the operator I_α is self-adjoint. That is, if f, g are bounded random variables, then

$$\mathbb{E}(I_\alpha f)g = \mathbb{E}f(I_\alpha g). \tag{2.1}$$

This is straightforward. By the martingale property we have $\mathbb{E}df_n dg_m = 0$ whenever $n \neq m$, and hence

$$\begin{aligned}\mathbb{E}(I_\alpha f)g &= \mathbb{E}\sum_{m,n=0}^{\infty} 2^{-n\alpha} df_n dg_m \\ &= \mathbb{E}\sum_{n=0}^{\infty} 2^{-n\alpha} df_n dg_n = \mathbb{E}\sum_{m,n=0}^{\infty} 2^{-m\alpha} df_n dg_m = \mathbb{E}f(I_\alpha g).\end{aligned}$$

We conclude this section by recalling some classical notions from real analysis. Given a random variable f, we define its decreasing rearrangement $f^*:(0,1]\to[0,\infty)$ by

$$f^*(t)=\inf\big\{s:\mathbb{P}(|f|>s)\leq t\big\}.$$

Then $f^{**}:(0,1]\to[0,\infty)$, the maximal function of f^*, is given by the formula

$$f^{**}(t)=\frac{1}{t}\int_0^t f^*(s)\mathrm{d}s,\qquad t\in(0,1].$$

One easily verifies that f^{**} can alternatively be defined by

$$f^{**}(t)=\frac{1}{t}\sup\left\{\int_E|f|\mathrm{d}\mathbb{P}\,:\,\mathbb{P}(E)=t\right\}.$$

3. Restricted bounds for martingales

Now we will establish a sharp martingale bound which can be regarded as the fundamental "building block" for our further considerations. For the sake of convenience, we will split this section into two parts.

3.1. **A special function.** Suppose that $\beta\geq 1$ and $\alpha\in(0,\beta)$ are given and fixed. For an arbitrary $x\in(0,1]$, introduce the parameter $n(x)=\sup\{k:2^{\beta k}x<1\}$. The main object of this subsection is the function $B_{\alpha,\beta}:[0,1]\times\mathbb{R}\to\mathbb{R}$ given by the formula

$$B_{\alpha,\beta}(x,y)=y+\frac{2^{\beta-\alpha}-2^{-\alpha}}{2^{\beta-\alpha}-1}x\big((2^{(\beta-\alpha)n(x)}-1\big)+2^{-(n(x)+1)\alpha}(1-2^{\beta n(x)}x)$$

if $x>0$, and $B_{\alpha,\beta}(0,y)=y$ for all $y\in\mathbb{R}$. In most instances it will be clear from the context which α and β we are working with, so we will skip the lower indices and simply write B instead of $B_{\alpha,\beta}$.

One checks easily that B is a continuous function. We will need the following further properties of B.

Lemma 3.1. *(i) We have $B(x,y)\geq y$ for all $(x,y)\in[0,1]\times\mathbb{R}$.*
(ii) For any $y\in\mathbb{R}$, the function $\xi(x)=B(x,y+x)$ is nondecreasing on $[0,1]$.

Proof. The majorization (i) is evident: $B(x,y)$ is a sum of y and two additional terms, both of which are nonnegative. We turn our attention to (ii). We have $B(x,y+x)=y+B(x,x)$, so it suffices to show the claim for $y=0$. Furthermore, by the continuity of B, it is enough to establish the monotonicity on the interval $(2^{-(n+1)\beta},2^{-n\beta})$, where n is an arbitrary nonnegative integer. On such an interval we have $n(x)=n$. Differentiating gives

$$\begin{aligned}\frac{\partial B(x,x)}{\partial x}&=1+\frac{2^{\beta-\alpha}-2^{-\alpha}}{2^{\beta-\alpha}-1}(2^{(\beta-\alpha)n}-1)-2^{(\beta-\alpha)n-\alpha}\\&=\frac{(2^{(\beta-\alpha)(n+1)}-1)(1-2^{-\alpha})}{2^{\beta-\alpha}-1}\geq 0,\end{aligned}$$

as needed. □

The main property of B is studied in a the next lemma below. It can be regarded as a concavity-type condition.

Lemma 3.2. *Fix $(x,y)\in[0,1]\times\mathbb{R}$. Suppose that a finite collection of points $(x_j,y_j)\in[0,1]\times\mathbb{R}$ have the following properties:*

(a) *All the points lie on a line of slope $2^{-\alpha}$ passing through the point (x,y).*

(b) *There are numbers* $\lambda_j \in [2^{-\beta}, 1-2^{-\beta}]$ *summing up to* 1*, such that*

$$(x,y)=\sum_j \lambda_j (x_j,y_j).$$

Then we have the inequality

$$B(x,y) \geq \max_j \left\{2^{-\alpha}B(x_j,2^\alpha y_j)\right\}. \tag{3.1}$$

Proof. With no loss of generality, we may assume that the sequence (x_j) is nonincreasing. Consider the function $\xi:[-x,1-x]\to\mathbb{R}$ given by

$$\xi(t)=B(x+t,2^\alpha y+t).$$

By the second part of the preceding lemma, ξ is nondecreasing. In addition, since all (x_j,y_j) lie on a line of slope $2^{-\alpha}$ passing through (x,y), we have

$$B(x_j,2^\alpha y_j)=B\big(x+(x_j-x),2^\alpha y+(x_j-x)\big)=\xi(x_j-x).$$

By the monotonicity of (x_j) which we have assumed at the beginning, the right-hand side of (3.1) is equal to $2^\alpha B(x_1,2^\alpha y_1)$. In addition, by (ii), we may write

$$x_1=\lambda_1^{-1}x-\sum_{j\neq 1}\lambda_j\lambda_1^{-1}x_j \leq \lambda_1^{-1}x \leq 2^\beta x,$$

which implies

$$B(x_1,2^\alpha y_1)=\xi(x_1-x)\leq \xi\big(\min\{2^\beta x,1\}-x\big).$$

It remains to observe that the latter expression is equal to $2^\alpha B(x,y)$. Let us briefly check this: if $2^\beta x<1$, then $n(2^\beta x)=n(x)-1$ and

$$\begin{aligned}
&\xi(2^\beta x-x)\\
&=B(2^\beta x,2^\alpha y+2^\beta x-x)\\
&=2^\alpha y+2^\beta x-x+2^{\beta-\alpha}(1-2^{-\beta})2^\beta x\frac{(2^{\beta-\alpha})^{n(x)-1}-1}{2^{\beta-\alpha}-1}+2^{-n(x)\alpha}\big(1-2^{\beta n(x)}x\big)\\
&=2^\alpha y+2^\beta x-x+2^\beta(1-2^{-\beta})x\frac{(2^{\beta-\alpha})^{n(x)}-2^{\beta-\alpha}}{2^{\beta-\alpha}-1}+2^{-n(x)\alpha}\big(1-2^{\beta n(x)}x\big)\\
&=2^\alpha y+2^\beta(1-2^{-\beta})x\frac{(2^{\beta-\alpha})^{n(x)}-1}{2^{\beta-\alpha}-1}+2^{-n(x)\alpha}\big(1-2^{\beta n(x)}x\big)\\
&=2^\alpha B(x,y).
\end{aligned}$$

On the other hand, if $2^\beta x\geq 1$, then $n(x)=0$ and thus

$$\xi\big(\min\{2^\beta x,1\}-x\big)=\xi(1-x)=B(1,2^\alpha y+1-x)=2^\alpha y+1-x=2^\alpha B(x,y). \qquad \square$$

3.2. **Restricted bound for fractional integral operators.** We are ready to formulate and prove the main result of this section.

Theorem 3.1. *Suppose that* f *is an* $[0,1]$*-valued random variable on a probability space equipped with a* β*-regular tree* $\mathcal{T}$*. Then for any* $0<\alpha<\beta$ *we have the inequality*

$$||I_\alpha f||_\infty \leq B\left(\mathbb{E}f,\mathbb{E}f\right). \tag{3.2}$$

This bound is sharp: for any $x\in[0,1]$ *there is a* β*-regular tree* $\mathcal{T}$ *and a random variable* f *taking values in* $[0,1]$ *such that* $\mathbb{E}f=x$ *and both sides of* (3.2) *are equal.*

Proof of (3.2). The main step in the proof of the inequality is to show that the sequence $(||2^{-n\alpha}B(f_n, 2^{n\alpha}(I_\alpha f)_n)||_\infty)_{n\geq 0}$ is nonincreasing. To see that this is true, fix a nonnegative integer n and pick $J \in \mathcal{T}^n$. Let $J_1, J_2, \ldots, J_k$ be the collection of all pairwise disjoint children of J belonging to $\mathcal{T}^{n+1}$. Set $x = f_n|_J$, $y = 2^{n\alpha}(I_\alpha f)_n|_J$ and put $x_j = f_{n+1}|_{J_j}$, $y_j = 2^{n\alpha}(I_\alpha f)_{n+1}|_{J_j}$, $j = 1, 2, \ldots, k$. On the set J, we have $f_{n+1} = f_n + df_{n+1} = x + df_{n+1}$ and $2^{n\alpha}(I_\alpha f)_{n+1} = 2^{n\alpha}(I_\alpha f)_n + 2^{-\alpha}df_{n+1} = y + 2^{-\alpha}df_{n+1}$. Consequently, the property (a) formulated in the statement of Lemma 3.2 is satisfied. Next, the tree $\mathcal{T}$ is β-regular, so $\mathbb{P}(J_j)/\mathbb{P}(J) \in [2^{-\beta}, 1 - 2^{-\beta}]$ and hence, by the martingale property of $(f_n)_{n\geq 0}$, the condition (b) also holds true. Therefore, from (3.1) it follows that on J,

$$\begin{aligned}||2^{-n\alpha}B(f_n, 2^{n\alpha}(I_\alpha f)_n)||_\infty &\geq 2^{-n\alpha}B(f_n, 2^{n\alpha}(I_\alpha f)_n)\\ &\geq \operatorname*{ess\,sup}_J 2^{-(n+1)\alpha}B(f_{n+1}, 2^{(n+1)\alpha}(I_\alpha f)_{n+1}).\end{aligned}$$

Thus, taking the supremum over J, we get that

$$||2^{-n\alpha}B(f_n, 2^{n\alpha}(I_\alpha f)_n)||_\infty \geq ||2^{-(n+1)\alpha}B(f_{n+1}, 2^{(n+1)\alpha}(I_\alpha f)_{n+1})||_\infty.$$

Now, combining this with the first part of Lemma 3.1, we obtain

$$||(I_\alpha f)_n||_\infty \leq ||2^{-n\alpha}B(f_n, 2^{n\alpha}(I_\alpha f)_n)||_\infty \leq B(f_0, (I_\alpha f)_0) = B(\mathbb{E}f, \mathbb{E}f).$$

It remains to take the supremum over all n to complete the proof. □

Sharpness of (3.2). Fix $x \in [0,1]$. If $x = 0$, we consider the constant martingale $f = (0, 0, \ldots)$; then $I_\alpha f = (0, 0, \ldots)$ as well and $||I_\alpha f||_\infty = 0 = B(0, 0)$.

Suppose next that $x > 0$ and recall that $n(x) = \sup\{k : 2^{\beta k}x < 1\}$. Consider the tree $\mathcal{T}$ satisfying the following requirements:

(i) We have $\mathcal{T}^0 = \Omega$;
(ii) For any n, the collection $\mathcal{T}^{n+1}$ is obtained from $\mathcal{T}^n$ by splitting each $J \in \mathcal{T}^n$ into two sets J_-, J_+ satisfying $\mathbb{P}(J_-)/\mathbb{P}(J) = 2^{-\beta}$, $\mathbb{P}(J_+)/\mathbb{P}(J) = 1 - 2^{-\beta}$.

Of course, the tree $\mathcal{T}$ is β-regular. Furthermore, there is a sequence $J_0 \supset J_1 \supset J_2 \supset \ldots$ with $J_n \in \mathcal{T}^n$ such that $\mathbb{P}(J_n) = 2^{-\beta n}$ for all n. Pick a set $A \in \mathcal{F}$ satisfying $\mathbb{P}(A) = x$ and $J_{n(x)+1} \subset A \subset J_{n(x)}$, and put $f = \chi_A$. Then f takes values in $[0,1]$ and $\mathbb{E}f = x$. Furthermore, directly from the definition, if $1 \leq k \leq n(x)$, then $f_k = 0$ outside J_k; on the other hand, for $\omega \in J_k$ we have

$$f_k(\omega) = \frac{1}{\mathbb{P}(J_k)}\int_{J_k}\chi_A \mathrm{d}\mathbb{P} = \frac{\mathbb{P}(A)}{\mathbb{P}(J_k)} = 2^{\beta k}x, \qquad k = 0, 1, \ldots, n(x).$$

A similar argument shows that on $J_{n(x)+1}$ we have

$$f_{n(x)+1} = \frac{1}{\mathbb{P}(J_{n(x)+1})}\int_{J_{n(x)+1}}\chi_A \mathrm{d}\mathbb{P} = 1.$$

This implies that for $\omega \in J_{n(x)+1}$ we have $df_0 = x$, $df_k = (2^{\beta k} - 2^{\beta(k-1)})x$ ($k = 1, 2, \ldots, n(x)$) and $df_{n+1} = 1 - 2^{\beta n(x)}x$. Consequently,

$$\begin{aligned}(I_\alpha f)_{n(x)+1}(\omega) &= \sum_{k=0}^{n+1} 2^{-k\alpha}df_k(\omega)\\ &= x + 2^\beta(1 - 2^{-\beta}x\frac{(2^{\beta-\alpha})^{n(x)} - 1}{2^{\beta-\alpha} - 1} + 2^{-(n(x)+1)\alpha}(1 - 2^{\beta n(x)}x)\\ &= B(x, x).\end{aligned}$$

Since $J_{n(x)+1}$ has positive measure, we see that the bound (3.2) can be attained. This completes the proof of the theorem. □

Remark 3.1. The inequality (3.2) is still sharp if the underlying probability space with the tree structure is the unit cube $(0,1]^d$ equipped with its dyadic lattice. More precisely, for any $x \in [0,1]$ there is a Borel subset A of the cube such that $|A| = x$ and both sides of (3.2) are equal. This is trivial for $x = 0$. When $x > 0$, the same construction as above works as well. The only modification which is needed is to let $\mathcal{T}^n$ be the class of all dyadic subcubes of $(0,1]^d$ of measure 2^{-nd}. The whole analysis of the difference sequence of the corresponding martingale $f = (f_n)_{n\geq 0}$ can be repeated word-by-word. We leave these details to the reader.

As an application, we obtain the following result for fractional integral operator, which can be regarded as a restricted $L^\infty \to L^{\beta/(\beta-\alpha),\infty}$ estimate. The further extensions of this result will be taken up in the next section.

Corollary 3.1. *Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space equipped with a β-regular tree $\mathcal{T}$ and A is an arbitrary element of $\mathcal{F}$. Then for any $0 < \alpha < \beta$ we have the estimate*

$$||I_\alpha \chi_A||_\infty \leq C_{\alpha,\beta} \mathbb{P}(A)^{\alpha/\beta}, \tag{3.3}$$

where

$$C_{\alpha,\beta} = \frac{2^{\beta-\alpha} - 2^{-\alpha}}{2^{\beta-\alpha} - 1}.$$

The constant is the best possible.

Proof. Since χ_A takes values in $[0,1]$, the inequality (3.2) implies that

$$||I_\alpha \chi_A||_\infty \leq B(\mathbb{P}(A), \mathbb{P}(A)).$$

Furthermore, recall that in the proof of the sharpness of this bound, the constructed extremals were characteristic functions of some measurable sets. So, to complete the proof of (3.3) (as well as its sharpness), it suffices to show that $B(x,x) \leq C_{\alpha,\beta} x^{\alpha/\beta}$, and that the constant $C_{\alpha,\beta}$ cannot be decreased. To do this, let us take a look at the function $\xi : x \mapsto B(x,x) - C_{\alpha,\beta} x^{\alpha/\beta}$. It is continuous and concave on each interval of the form $[2^{-(n+1)\beta}, 2^{-n\beta}]$; thus, to show that ξ is nonpositive, it suffices to verify that $\xi(2^{-n\beta}) \leq 0$ for each n. However, we easily check that

$$\xi(2^{-n\beta}) = 2^{-n\beta}(1 - C_{\alpha,\beta}) = 2^{-n\beta} \frac{2^{-\alpha} - 1}{2^{\beta-\alpha} - 2^{-\alpha}} \leq 0,$$

so the bound $B(x,x) \leq C_{\alpha,\beta} x^{\alpha/\beta}$ holds true. To see that $C_{\alpha,\beta}$ is optimal, we observe that

$$\frac{B(2^{-n\beta}, 2^{-n\beta})}{(2^{-n\beta})^{\alpha/\beta}} = 2^{-n(\beta-\alpha)}(1 - C_{\alpha,\beta}) + C_{\alpha,\beta} \to C_{\alpha,\beta},$$

as $n \to \infty$ and this completes the proof. □

The inequality (3.3) remains sharp for the fractional integral operator associated with the Walsh-Fourier series on $(0,1]^d$. This follows at once from Remark 3.1.

4. Weak type inequalities for fractional integral operators

We have split this section into two parts. The main step in the proof of the weak type inequality rests on providing an efficient upper bound for $||I_\alpha \chi_A||_p$. In order to achieve this goal, we will construct a tree $\mathcal{T}'$ and a set $\mathfrak{A}$ such that $I_\alpha^{\mathcal{T}}\chi_A$ is appropriately controlled by $I_\alpha^{\mathcal{T}'}\chi_{\mathfrak{A}}$. The construction is presented in the first part of this section, and the remainder is devoted to the weak-type estimate.

4.1. **A special tree and a special random variable.** The construction is a slight modification of the example used in the proof of the sharpness of (3.2). Fix $x \in (0,1]$ and $\beta \geq 1$. We consider essentially the same tree $\mathcal{T}$: $\mathcal{T}^0 = \Omega$ and for each $n \geq 0$, the class $\mathcal{T}^{n+1}$ is obtained from $\mathcal{T}^n$ by splitting each $J \in \mathcal{T}^n$ into two sets J_-, J_+ satisfying $\mathbb{P}(J_-)/\mathbb{P}(J) = 2^{-\beta}$, $\mathbb{P}(J_+)/\mathbb{P}(J) = 1 - 2^{-\beta}$. The only exception of this inductive rule is the following. Pick a set $J \in \mathcal{T}^{n(x)}$ satisfying $\mathbb{P}(J) = 2^{-\beta n(x)}$ (if there are several sets of this type, just fix one of them). Then we split it into two sets $\mathfrak{A}$, $\mathfrak{A}'$ such that $\mathbb{P}(\mathfrak{A}) = x$ and $\mathbb{P}(\mathfrak{A}') = 2^{-\beta n(x)} - x$. The children of $\mathfrak{A}$ and $\mathfrak{A}'$ are created according to the inductive pattern described above. Of course, this tree may or may not be β-regular. This will depends on the split of J into $\mathfrak{A}$ and $\mathfrak{A}'$. More precisely, one easily checks that we have the β-regularity if and only if $2^{\beta n(x)}x \in [2^{-\beta}, 1 - 2^{-\beta}]$.

Now, let $f = \chi_{\mathfrak{A}}$ and let us analyze the random variable $I_\alpha f$. First, note that $\mathfrak{A}$ is $\mathcal{F}_{n(x)+1}$-measurable, which implies that $f_{n(x)+1} = f_{n(x)+2} = \ldots = \chi_{\mathfrak{A}}$ and hence we also have $(I_\alpha f)_{n(x)+1} = (I_\alpha f)_{n(x)+2} = \ldots = I_\alpha f$. For any $0 \leq k \leq n(x)$, let J_k be the unique element of $\mathcal{T}^k$ which contains $\mathfrak{A}$. A similar analysis to that from the preceding section reveals the following structure of the martingale difference sequence. We have $df_0 = x$ and for $1 \leq k \leq n(x)$, we have $df_k = \left(2^{k\beta} - 2^{(k-1)\beta}\right)x$ on J_k, $df_k = -2^{(k-1)\beta}$ on $J_{k-1} \setminus J_k$ and $df_k = 0$ elsewhere. Finally, $df_{n(x)+1} = 1 - 2^{\beta n(x)}x$ on $\mathfrak{A}$, $df_{n(x)+1} = -2^{\beta n(x)}x$ on $\mathfrak{A}'$ and $df_{n(x)+1} = 0$ elsewhere. With these observation we can compute explicitly the distribution of $I_\alpha f$. In fact, it is not difficult to see that we have the following identities. For $0 \leq k \leq n(x) - 1$, and

$$\mathbb{P}\left(I_\alpha f = x\left(1 + C_{\alpha,\beta}(2^{n(x)(\beta-\alpha)} - 1) - 2^{n(x)(\beta-\alpha)-\alpha}\right)\right) = 2^{-n(x)\beta}(1 - 2^{\beta n(x)}x)$$

and

$$\mathbb{P}\left(I_\alpha f = x\left(1 + C_{d,\alpha}(2^{n(x)(\beta-\alpha)} - 1) + 2^{-(n(x)+1)\alpha}(1 - 2^{\beta n(x)}x)\right)\right) = x. \tag{4.1}$$

Finally, we have $\mathbb{P}(I_\alpha f = a) = 0$ for any point a different from these mentioned above. Directly from this description one can identify the explicit formula for the function $(I_\alpha f)^{**}$. We state it as a separate lemma.

Lemma 4.1. *We have*

$$(I_\alpha f)^{**}(s) = \begin{cases} B(x,x), & \text{if } s \in (0,x] \\ \frac{x}{s}B(s,s), & \text{if } s \in (x,1]. \end{cases}$$

Proof. Let $s_0 = 0$, $s_1 = x$ and $s_k = 2^{-(n(x)+2-k)\beta}$, $k = 2, 3, \ldots, n(x)+2$. It follows from the above considerations that

$$(I_\alpha f)^*(s) = x\left(1 + C_{\alpha,\beta}(2^{n(x)(\beta-\alpha)} - 1)) + 2^{-(n(x)+1)\alpha}(1 - 2^{-\beta n(x)}x)\right) = B(x,x),$$

if $s \in (0,x]$. Moreover,

$$(I_\alpha f)^*(s) = x\left(1 + C_{\alpha,\beta}(2^{(n(x)+1-k)(\beta-\alpha)} - 1) - 2^{(n(x)+1-k)(\beta-\alpha)-\alpha}\right),$$

if $s \in (s_k, s_{k+1})$, $k = 1, 2, \ldots, n(x)+1$. Therefore, if $s \in (0, x]$, then

$$(I_\alpha f)^{**}(s) = \frac{1}{s}\int_0^s (I_\alpha f)^*(u)\mathrm{d}u = B(x,x).$$

Next, if $s \in (s_1, s_2) = (x, 2^{-n\beta})$, then $n(s) = n(x)$ and hence

$$\begin{aligned}(I_\alpha f)^{**}(s) &= \frac{1}{s}\left(xs\left(1 + C_{\alpha,\beta}(2^{n(x)(\beta-\alpha)} - 1)) - 2^{n(x)(\beta-\alpha)-\alpha}\right) + 2^{-(n(x)+1)\alpha}x\right)\\ &= \frac{x}{s}B(s,s).\end{aligned}$$

Finally, if $s \in (s_k, s_{k+1}]$ for some $k = 2, 3, \ldots, n(x)+1$, then $n(s) = n(x)+1-k$ and

$$\begin{aligned}s(I_\alpha f)^{**}(s) &= 2^{-n\beta}x\big(1 + C_{\alpha,\beta}(2^{n(x)(\beta-\alpha)} - 1)\big)\\ &\quad + \sum_{j=2}^{k-1}(s_{j+1} - s_j)x\left(1 + C_{\alpha,\beta}(2^{(n(x)+1-j)(\beta-\alpha)} - 1) - 2^{(n(x)+1-j)(\beta-\alpha)-\alpha}\right)\\ &\quad + (s - s_k)\left(1 + C_{\alpha,\beta}(2^{(n(x)+1-k)(\beta-\alpha)} - 1) - 2^{(n(x)+1-k)(\beta-\alpha)-\alpha}\right)\\ &= xB(s,s) + 2^{(n(x)-1)\alpha}x(1-2^{-\beta})(C_{\alpha,\beta} - 2^{-\alpha})\frac{2^{(k-2)\alpha} - 1}{2^\alpha - 1}\\ &\quad + C_{\alpha,\beta}2^{-n(x)\alpha} - 2^{-\beta-(n(x)+1-k)\alpha}(C_{\alpha,\beta} - 2^{-\alpha}).\end{aligned}$$

Using the identities

$$C_{\alpha,\beta} = \frac{1-2^{-\beta}}{1-2^{-\alpha}}(C_{\alpha,\beta} - 2^{-\alpha}), \qquad 2^{\alpha-\beta}(C_{\alpha,\beta} - 2^{-\alpha}) + 1 = C_{\alpha,\beta},$$

we verify that the latter expression is equal to $xB(s,s)$. This completes the proof. □

Remark 4.1. Suppose that the probability space is the unit cube $(0,1]^d$ and that the tree structure is given by the dyadic lattice. If $x = 2^{-(n+1)d}$ for some integer n, then there is a dyadic cube $\mathfrak{A}$ of measure x for which all the properties listed above hold true. To see this it suffices to repeat word-by-word the above reasoning, replacing the classes $\mathcal{T}^n$ by the appropriate families of dyadic subcubes of $(0,1]^d$.

4.2. **Weak-type estimates.** We turn our attention to the main results of this paper. We start with the following intermediate fact.

Lemma 4.2. *Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is equipped with a β-regular tree $\mathcal{T}$ and let $A \in \mathcal{F}$. Then for any $E \in \mathcal{F}$ we have*

$$\int_E I_\alpha \chi_A d\mathbb{P} \leq \begin{cases}\mathbb{P}(E)B(\mathbb{P}(A), \mathbb{P}(A)) & \text{if } \mathbb{P}(E) \leq \mathbb{P}(A),\\ \mathbb{P}(A)B(\mathbb{P}(E), \mathbb{P}(E)) & \text{if } \mathbb{P}(E) > \mathbb{P}(A).\end{cases} \tag{4.2}$$

Proof. This is straightforward. If $\mathbb{P}(E) \leq \mathbb{P}(A)$, then

$$\int_E I_\alpha \chi_A \mathrm{d}\mathbb{P} \leq \mathbb{P}(E)||I_\alpha \chi_A||_\infty \leq \mathbb{P}(E)B(\mathbb{P}(A), \mathbb{P}(A)),$$

in view of (3.2). If $\mathbb{P}(E) > \mathbb{P}(A)$, then we use the fact that I_α is a self-adjoint operator (see (2.1)) to obtain

$$\int_E I_\alpha \chi_A \mathrm{d}\mathbb{P} = \mathbb{E}\chi_E I_\alpha \chi_A = \mathbb{E}\chi_A I_\alpha \chi_E \leq \mathbb{P}(A)||I_\alpha \chi_E||_\infty \leq \mathbb{P}(A)B(\mathbb{P}(E), \mathbb{P}(E)),$$

where the latter bound follows again from (3.2). □

The next statement is a significant extension of Corollary 3.1 and can be regarded as a dual to the weak-type bound.

Theorem 4.1. *Suppose that* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space equipped with a* β*-regular tree* $\mathcal{T}$*. Then for any* $A \in \mathcal{F}$ *and any* $1 \leq p \leq \infty$ *we have*

$$||I_\alpha \chi_A||_p \leq K_{\alpha,\beta,p} \mathbb{P}(A)^{1/p+\alpha/\beta}, \tag{4.3}$$

where

$$K_{\alpha,\beta,p} = \left(C_{\alpha,\beta}^p + \frac{(1-2^{-\beta})(C_{\alpha,\beta} - 2^{-\alpha})^p}{2^{p(\beta-\alpha)-\beta} - 1} \right)^{1/p}$$

(when $p = \infty$*, then* $K_{\alpha,\beta,p} = C_{\alpha,\beta}$*). The constant is the best possible.*

Proof of (4.3). The case $p = \infty$ has already been studied in Corollary 3.1, so we may assume that p is finite. If $\mathbb{P}(A) = 0$, then the claim is obvious. Therefore, from now on, we assume that $x = \mathbb{P}(A) > 0$. Fix $s \in (0, 1]$ and suppose that $\mathbb{P}(E) = s$. If we divide both sides of (4.2) by $\mathbb{P}(E)$ and apply Lemma 4.1, we obtain the bound

$$\frac{1}{\mathbb{P}(E)} \int_E I_\alpha \chi_A \mathrm{d}\mathbb{P} \leq (I_\alpha \chi_{\mathfrak{A}})^{**}(\mathbb{P}(E)).$$

Here $\mathfrak{A}$ is the set of probability x constructed in §4.1. Thus, taking the supremum over all E satisfying $\mathbb{P}(E) = s \in (0, 1]$, we get

$$(I_\alpha \chi_A)^{**}(s) \leq (I_\alpha \chi_{\mathfrak{A}})^{**}(s), \qquad s \in (0, 1].$$

This is the classical domination relation introduced by Hardy, Littlewood and Pólya. In particular (see [15, §249]), it implies that for any convex, nondecreasing function Φ on $[0, \infty)$ we have

$$\mathbb{E}\Phi(I_\alpha \chi_A) \leq \mathbb{E}\Phi(I_\alpha \chi_{\mathfrak{A}})$$

and hence, in particular,

$$||I_\alpha \chi_A||_p^p \leq ||I_\alpha \chi_{\mathfrak{A}}||_p^p.$$

Since we know explicitly the distribution of $I_\alpha \chi_{\mathfrak{A}}$ (see the reasoning preceding the statement of Lemma 4.1), we easily compute the right-hand side above: it equals

$$\begin{aligned} x^p &\sum_{k=0}^{n(x)-1} 2^{-k\beta}(1-2^{-\beta}) \left[1 + C_{\alpha,\beta}(2^{k(\beta-\alpha)} - 1) - 2^{k(\beta-\alpha)-\alpha} \right]^p \\ &+ x^p 2^{-n\beta}(1 - 2^{n\beta} x) \left[1 + C_{\alpha,\beta}(2^{n(x)(\beta-\alpha)} - 1) - 2^{n(x)(\beta-\alpha)-\alpha} \right]^p \\ &+ x \left[x + C_{\alpha,\beta} x (2^{n(x)(\beta-\alpha)} - 1) + 2^{-(n(x)+1)\alpha}(1 - 2^{n(x)\beta} x) \right]^p. \end{aligned}$$

Now, we have $C_{\alpha,\beta} \geq 1$, so

$$1 + C_{\alpha,\beta}(2^{k(\beta-\alpha)} - 1) - 2^{k(\beta-\alpha)-\alpha} \leq (C_{\alpha,\beta} - 2^{-\alpha}) 2^{k(\beta-\alpha)}.$$

Furthermore, the expression in the third square bracket is equal to $B(x, x)$, and we have shown in the proof of Corollary 3.1 that this number is not larger than $C_{\alpha,\beta} x^{\alpha/\beta}$. Consequently, we may write

$$\begin{aligned} ||I_\alpha \chi_{\mathfrak{A}}||_p^p \leq & x^p (1 - 2^{-\beta})(C_{\alpha,\beta} - 2^{-\alpha})^p \frac{2^{n(x)((\beta-\alpha)p-\beta)} - 1}{2^{(\beta-\alpha)p-\beta} - 1} \\ & + x^p (1 - 2^{\beta n(x)} x)(C_{\alpha,\beta} - 2^{-\alpha})^p 2^{n(x)((\beta-\alpha)p-\beta} + C_{\alpha,\beta}^p x^{1+p\alpha/\beta}. \end{aligned}$$

Thus, the claim will proved if we manage to show that

(4.4)
$$(1-2^{-\beta})(C_{\alpha,\beta}-2^{-\alpha})^p\frac{2^{n(x)((\beta-\alpha)p-\beta)}-1}{2^{(\beta-\alpha)p-\beta}-1} + (1-2^{\beta n(x)}x)(C_{\alpha,\beta}-2^{-\alpha})^p 2^{n(x)((\beta-\alpha)p-\beta)} - (K^p_{\alpha,\beta,p}-C^p_{\alpha,\beta})x^{1+p\alpha/d-p}\leq 0.$$

It is easy to check that the left-hand side is a continuous function of $x\in(0,1]$. Furthermore, if $x\in(2^{-(n+1)\beta},2^{-n\beta})$ (so that $n(x)=n$), then the left-hand side is convex in x. Therefore, it is enough to verify (4.4) for $x=2^{-n\beta}$ only. Plugging this particular x, we see that the bound becomes

$$(K^p_{\alpha,\beta,p}-C^p_{\alpha,\beta})(2^{n((\beta-\alpha)p-\beta)}-1)\leq (K^p_{\alpha,\beta,p}-C^p_{\alpha,\beta})2^{n((\beta-\alpha)p-\beta)},$$

which is evident. □

Sharpness of (4.3). Now we will show that the constant $K_{\alpha,\beta,p}$ cannot be improved. Again, it suffices to deal with finite p only. Fix a large positive integer n and consider the example of §4.1 corresponding to $x=2^{-\beta(n+1)}$. Then, as we have observed there, the tree is β-regular. By the above computations, $||I_\alpha\chi_{\mathfrak{A}}||_p^p/|\mathfrak{A}|^{1+p\alpha/\beta}$ equals

$$\begin{aligned}&2^{(1-p)\beta(n+1)+(n+1)p\alpha}\sum_{k=0}^n 2^{-k\beta}(1-2^{-\beta})\Big[1+C_{\alpha,\beta}(2^{k(\beta-\alpha)}-1)-2^{k(\beta-\alpha)-\alpha}\Big]^p\\&+2^{(n+1)p\alpha}\Big[2^{-\beta(n+1)}+C_{\alpha,\beta}2^{-\beta(n+1)}(2^{n(\beta-\alpha)}-1)+2^{-(n+1)\alpha}(1-2^{-\beta})\Big]^p\\&=M_n+N_n.\end{aligned}$$

Now, we easily see that

$$\lim_{n\to\infty}N_n=(C_{\alpha,\beta}2^{\alpha-\beta}+1-2^\beta)^p=C^p_{\alpha,\beta}.$$

To handle the limit of M_n, we use Stolz-Cesáro theorem: we have

$$\begin{aligned}\lim_{n\to\infty}M_n&=\lim_{n\to\infty}\frac{2^{-n\beta}(1-2^{-\beta})\left[1+C_{\alpha,\beta}(2^{n(\beta-\alpha)}-1)-2^{n(\beta-\alpha)-\alpha}\right]^p}{2^{(p-1)\beta(n+1)-(n+1)p\alpha}-2^{(p-1)\beta n-np\alpha}}\\&=\frac{(1-2^{-\beta})(C_{\alpha,\beta}-2^{-\alpha})^p}{2^{p(\beta-\alpha)-\beta}-1}.\end{aligned}$$

This yields $\lim_{n\to\infty}(M_n+N_n)=K^p_{\alpha,\beta,p}$ and hence the inequality (4.3) is indeed sharp. □

Now we will establish the main result of this paper. In what follows, p' will stand for the harmonic conjugate to p.

Theorem 4.2. *Suppose that* $(\Omega,\mathcal{F},\mathbb{P})$ *is a probability space equipped with a* β*-regular tree. Let* p, q *and* α *be arbitrary numbers satisfying* $0<\alpha<\beta$, $1\leq p<\beta/\alpha$ *and* $1/q=1/p-\alpha/\beta$. *Then for any random variable* $f\in L^p$ *we have the sharp bound*

$$||I_\alpha f||_{q,\infty}\leq K_{\alpha,\beta,p'}||f||_p. \tag{4.5}$$

Proof. Without loss of generality, we may assume that the random variable f is nonnegative since the passage from f to $|f|$ does not affect the norm of f, and does not decrease the weak norm of $I_\alpha f$. Pick an arbitrary set $A\in\mathcal{F}$ of positive probability. Using the fact that I_α is a self-adjoint operator, we may write

$$\int_E I_\alpha f\,\mathrm{d}\mathbb{P}=\mathbb{E}fI_\alpha\chi_E\leq||f||_p||\chi_E||_{p'}.$$

Consequently, by the previous theorem, we obtain

$$\int_E I_\alpha f \mathrm{d}\mathbb{P} \le K_{\alpha,\beta,p'} ||f||_p \mathbb{P}(E)^{1/p'+\alpha/\beta} = K_{\alpha,\beta,p'} ||f||_p \mathbb{P}(E)^{1-1/q}.$$

It suffices to divide both sides by $\mathbb{P}(E)^{1-1/q}$ and take the supremum over all E on the left to obtain (4.5).

To see that the constant $K_{\alpha,\beta,p'}$ cannot be replaced by a smaller number, assume first that $p > 1$ (so that $p' < \infty$). Pick $\varepsilon > 0$ and a set $\mathfrak{A} \in \mathcal{F}$ such that $||I_\alpha \chi_{\mathfrak{A}}||_{p'} > (K_{\alpha,\beta,p'} - \varepsilon)\mathbb{P}(\mathfrak{A})^{1/p'+\alpha/\beta}$. The existence of such a set is guaranteed by Theorem 4.1. We take $f = (I_\alpha \chi_{\mathfrak{A}})^{p'-1}$ and repeat the above calculations to get

$$\begin{aligned} ||I_\alpha f||_{q,\infty} &\ge \frac{1}{\mathbb{P}(\mathfrak{A})^{1-1/q}} \int_{\mathfrak{A}} I_\alpha f \, \mathrm{d}\mathbb{P} \\ &= \frac{1}{\mathbb{P}(\mathfrak{A})^{1-1/q}} \mathbb{E} f I_\alpha \chi_{\mathfrak{A}} \\ &= \frac{1}{\mathbb{P}(\mathfrak{A})^{1-1/q}} ||f||_p ||I_\alpha \chi_{\mathfrak{A}}||_{p'} \ge (K_{\alpha,\beta,p'} - \varepsilon)||f||_p. \end{aligned}$$

To deal with the case $p = 1$, we look once again at the set $\mathfrak{A}$ of probability $x = 2^{-(n+1)\beta}$, constructed in §4.1. As we have proved there (see (4.1)), we have

$$\mathbb{P}\left(I_\alpha f = 2^{-(n+1)\beta}\left(1 + C_{d,\alpha}(2^{(n+1)(\beta-\alpha)} - 1)\right)\right) = 2^{-(n+1)\beta}.$$

Hence, if we take $E = \left\{I_\alpha f = x\left(1 + C_{\alpha,\beta}(2^{(n+1)(\beta-\alpha)} - 1)\right)\right\}$, we get

$$\frac{||I_\alpha \chi_{\mathfrak{A}}||_{q,\infty}}{||\chi_{\mathfrak{A}}||_1} \ge \frac{1}{\mathbb{P}(\mathfrak{A})\mathbb{P}(E)^{1-1/q}} \int_E I_\alpha f \mathrm{d}\mathbb{P} = \frac{1 + C_{\alpha,\beta}(2^{(n+1)(\beta-\alpha)} - 1)}{2^{(n+1)(\beta-\alpha)}} \to C_{\alpha,\beta}$$

as $n \to \infty$. This proves the desired sharpness. □

Here is the analogue of Remark 3.1

Remark 4.2. The inequalities (3.3), (4.3) and (4.5) are still sharp for the fractional integral operators associated with the Walsh-Fourier series. To see this, note that all the extremal functions we considered above were characteristic functions of certain sets $\mathfrak{A}$ of probability $2^{-(n+1)\beta}$. Thus, the claim follows from Remark 4.1.

5. Muckenhoupt-Wheeden inequality for fractional integral operators

Let w be a weight (i.e., a nonnegative, locally integrable function) on $\mathbb{R}^d$ and let M be the Hardy-Littlewood maximal operator. In 1971, Fefferman and Stein [10] proved the existence of a universal constant c such that

$$\lambda w\left(\{x \in \mathbb{R}^d : Mf(x) \ge \lambda\}\right) \le c||f||_{L^1(Mw)}, \qquad \lambda > 0$$

(we use the notation $w(E) = \int_E w(x)\mathrm{d}x$ and $||f||_{L^1(Mw)} = \int_{\mathbb{R}^d} |f(x)|Mw(x)\mathrm{d}x$). This gave rise to the following natural question, formulated by Muckenhoupt and Wheeden in the seventies. Suppose that T is a Calderón-Zygmund singular integral operator. Is there a constant c, depending only on T, such that

$$\lambda w\left(\{x \in \mathbb{R}^d : Tf(x) \ge \lambda\}\right) \le c||f||_{L^1(Mw)}\,? \tag{5.1}$$

This problem, called the Muckenhoupt-Wheeden conjecture, remained open for a long time, and many mathematicians contributed to some partial results in this direction. In particular, Chanillo and Wheeden [6] proved that the estimate holds true for the square function. Buckley [3] showed that the conjecture is true for weights of the form $w_\delta(x) = |x|^{-d(1-\delta)}$, $0<\delta<1$. The best known result in this direction is that of Pérez, who showed that if M^2 denotes the second iteration of M, then

$$\lambda w\left(\left\{x\in\mathbb{R}^d : Tf(x)\geq\lambda\right\}\right)\leq c||f||_{L^1(M^2w)}.$$

In fact, he proved a stronger statement in which the operator M^2 is replaced by the smaller object $M_{L(\log L)^\varepsilon}$; see [26] for details. We also refer the interested reader to the recent works of Lerner, Ombrosi and Pérez [17, 18, 19] for further results concerning weaker forms of (5.1). In 2010, the Muckenhoupt-Wheeden conjecture was finally shown to be false. See the counterexamples by Reguera, Thiele, Nazarov, Reznikov, Vasyunin and Volberg in [21, 27, 28].

The purpose of this section is to establish a sharp version of the Muckenhoupt-Wheeden inequality for fractional integral operators in the martingale setting. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space equipped with a β-regular tree $\mathcal{T}$. Given an adapted martingale $f=(f_n)_{n\geq 0}$, its maximal function Mf is defined by $Mf=\sup_{n\geq 0}|f_n|$. Let w be a weight, i.e., a nonnegative, integrable variable. In analogy to the previous setting, we will write $w(E)=\mathbb{E}\chi_E w$; we will also use the notation $||f||_{L^1(w)}=\mathbb{E}|f|w$ and

$$||f||_{L^{p,\infty}(w)}=\sup\left\{\frac{1}{w(E)^{1-1/p}}\int_E|f|w\,\mathrm{d}\mathbb{P} \,:\, E\in\mathcal{F},\, w(E)>0\right\}.$$

Our result can be stated as follows. For an analogous statement concerning classical Riesz potentials on $\mathbb{R}^d$, consult the work of Lacey et. al. [16].

Theorem 5.1. *For any weight w and any $0<\alpha<\beta$ we have*

$$||I_\alpha f||_{L^{\beta/(\beta-\alpha),\infty}(w)}\leq C_{\alpha,\beta}||f||_{L^1((Mw)^{1-\alpha/\beta})}. \tag{5.2}$$

The constant is the best possible.

In the proof of the above theorem, we will need the following auxiliary fact.

Lemma 5.1. *Suppose that g is a nonnegative function and $(g_n)_{n\geq 0}$ is the associated martingale. Then*

$$\left|\left|\frac{I_\alpha g}{(Mg)^{1-\alpha/\beta}g_0^{\alpha/\beta}}\right|\right|_\infty\leq C_{\alpha,\beta} \tag{5.3}$$

and the constant $C_{\alpha,\beta}$ is the best possible.

Proof. By homogeneity, we may assume that $Mg(\omega)=1$; in particular, this implies $g_n(\omega)\leq 1$ for all n. Let A be a set of probability $g_0(\omega)$, constructed in §2 in the proof of the sharpness of (3.2) and pick $\omega'\in A$. A key observation is that we have the pointwise bound $I_\alpha g(\omega)\leq I_\alpha\chi_A(\omega')$. To see this, note that by the β-regularity of $\mathcal{T}$, for any $n\geq 1$ we have $g_n(\omega)\leq 2^\beta g_{n-1}(\omega)$, and hence

$$g_n(\omega)\leq\min\left\{2^{n\beta}g_0,1\right\}=(I_\alpha\chi_A)_n(\omega').$$

This implies

$$I_\alpha g(\omega)=(1-2^{-\alpha})\sum_{n=0}^\infty 2^{-n\alpha}g_n(\omega)\leq(1-2^{-\alpha})\sum_{n=0}^\infty 2^{-n\alpha}(I_\alpha\chi_A)_n(\omega')=I_\alpha\chi_A(\omega').$$

Therefore, by (3.3),

$$\left|\left|\frac{I_\alpha g}{(Mg)^{1-\alpha/\beta}g_0^{\alpha/\beta}}\right|\right|_\infty\leq\frac{||I_\alpha\chi_A||_\infty}{\mathbb{P}(A)^{\alpha/\beta}}\leq C_{\alpha,\beta}.$$

This is precisely (5.3). To see that this bound is sharp, pick $A \in \mathcal{F}$ of positive probability and let $g = \chi_A$. Then

$$\left|\left|\frac{I_\alpha g}{(Mg)^{1-\alpha/\beta} g_0^{\alpha/\beta}}\right|\right|_\infty = \frac{||I_\alpha \chi_A||_\infty}{\mathbb{P}(A)^{\alpha/\beta}},$$

which, by Corollary 3.1, can be made arbitrarily close to $C_{\alpha,\beta}$ by an appropriate choice of A. The proof is complete. □

We turn our attention to the main result.

Proof of Theorem 5.1. With no loss of generality, we may restrict ourselves to nonnegative functions f. Pick an arbitrary event $E \in \mathcal{F}$ of positive measure. We have

$$\begin{aligned}\int_E I_\alpha f\, w\, \mathrm{d}\mathbb{P} = \mathbb{E} f I_\alpha(\chi_E w) &\leq ||f||_{L^1((Mw)^{1-\alpha/\beta})} \left|\left|(Mw)^{\alpha/\beta-1} I_\alpha(\chi_E w)\right|\right|_\infty \\ &\leq ||f||_{L^1((Mw)^{1-\alpha/\beta})} \left|\left|(M(\chi_E w))^{\alpha/\beta-1} I_\alpha(\chi_E w)\right|\right|_\infty.\end{aligned}$$

Applying (5.3) to the function $g = \chi_E w$, we obtain

$$\int_E I_\alpha f\, w\, \mathrm{d}\mathbb{P} \leq C_\alpha ||f||_{L^1((Mw)^{1-\alpha/\beta})} \mathbb{P}(A)^{\alpha/\beta} = C_\alpha ||f||_{L^1((Mw))^{1-\alpha/\beta}} w(E)^{\alpha/\beta},$$

since $\mathbb{P}(A) = g_0 = \mathbb{E}g = w(E)$. This is the desired bound (5.2). To see the sharpness, take $w = 1$. The estimate reduces to the weak-type bound (4.5) with $p = 1$ in which the constant $K_{\alpha,\beta,\infty} = C_{\alpha,\beta}$ is the best possible. □

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