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Sharp weak-type inequalities for Fourier multipliers and second-order Riesz transforms

Research Article

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Abstract: We study sharp weak-type inequalities for a wide class of Fourier multipliers resulting from modulation of the jumps of Lévy processes. In particular, we obtain optimal estimates for second-order Riesz transforms, which lead to interesting a priori bounds for smooth functions on $\mathbb{R}^d$. The proofs rest on probabilistic methods: we deduce the above inequalities from the corresponding estimates for martingales. To obtain the lower bounds, we exploit the properties of laminates, important probability measures on the space of matrices of dimension 2×2, and some transference-type arguments.

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1. Introduction

Computing the exact norm of a singular integral operator is in general a very difficult task. Probably the first result in this direction is that of Pichorides [24], where the value of the L^p norm of the Hilbert transform was determined. Recall that the Hilbert transform on the line is the operator acting on $f \in L^1(\mathbb{R})$, given by the formula

$$\mathcal{H}^{\mathbb{R}} f(x) = \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{f(t)}{x-t} \mathrm{d}t.$$

Pichorides proved that if $1 < p < \infty$, then $||\mathcal{H}^{\mathbb{R}}||_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} = \cot(\pi/2p^*)$, where $p^* = \max\{p, p/(p-1)\}$. This result was generalized to the higher dimensional setting by Iwaniec and Martin [15] and Bañuelos and Wang [5]. If $d \geq 1$ is a fixed integer, then the collection of Riesz transforms (cf. [25]) is given by

$$R_j f(x) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{(d+1)/2}} \text{p.v.} \int_{\mathbb{R}^d} \frac{x_j - y_j}{|x-y|^{d+1}} f(y) \mathrm{d}y, \qquad j = 1, 2, \ldots, d.$$

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For $d = 1$, the family contains only one element, the Hilbert transform. The aforementioned result of Iwaniec, Martin, Bañuelos and Wang asserts that

$$||R_j||_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} = \cot\left(\frac{\pi}{2p^*}\right)$$

for all d, all $j \in \{1, 2, \ldots, d\}$ and all $1 < p < \infty$. This result can be formulated in the language of Fourier multipliers. Recall that for any bounded function $m : \mathbb{R}^d \to \mathbb{C}$, there is a unique bounded linear operator T_m on $L^2(\mathbb{R}^d)$, called *the Fourier multiplier with the symbol* m, given by the following relation between Fourier transforms: $\widehat{T_m f} = m\hat{f}$. The norm of T_m on $L^2(\mathbb{R}^d)$ is equal to $||m||_{L^\infty(\mathbb{R}^d)}$ and it has been long of interest to study those m, for which the corresponding Fourier multiplier extends to a bounded linear operator on $L^p(\mathbb{R}^d)$, $1 < p < \infty$. In general, the characterization of such symbols seems to be hopeless, but one can study various examples and their properties. For instance, one can consider the above collection of Riesz transforms on $\mathbb{R}^d$: it can be shown that R_j is a Fourier multiplier with the symbol $i\xi_j/|\xi|$, $j = 1, 2, \ldots, d$. See Stein [25] for the detailed computation and related discussion.

In the present paper we consider a class of symbols which can be obtained with the use of probabilistic methods: more precisely, by the modulation of jumps of certain Lévy processes. This class has been introduced and studied by Bañuelos and Bogdan [3] and Bañuelos, Bielaszewski and Bogdan [2]. Let ν be a Lévy measure on $\mathbb{R}^d$, i.e., a nonnegative Borel measure on $\mathbb{R}^d$ such that $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d} \min\{|x|^2, 1\}\nu(\mathrm{d}x) < \infty.$$

Assume further that μ is a finite Borel measure on the unit sphere $\mathbb{S}$ of $\mathbb{R}^d$ and fix two Borel functions ϕ on $\mathbb{R}^d$ and ψ on $\mathbb{S}$ which take values in the unit ball of $\mathbb{C}$. We define the associated multiplier $m = m_{\phi,\psi,\mu,\nu}$ on $\mathbb{R}^d$ by

$$m(\xi) = \frac{\frac{1}{2}\int_{\mathbb{S}}\langle\xi,\theta\rangle^2\psi(\theta)\mu(\mathrm{d}\theta) + \int_{\mathbb{R}^d}[1-\cos\langle\xi,x\rangle]\phi(x)\nu(\mathrm{d}x)}{\frac{1}{2}\int_{\mathbb{S}}\langle\xi,\theta\rangle^2\mu(\mathrm{d}\theta) + \int_{\mathbb{R}^d}[1-\cos\langle\xi,x\rangle]\nu(\mathrm{d}x)} \tag{1.1}$$

if the denominator is not 0, and $m(\xi) = 0$ otherwise. Here $\langle\cdot,\cdot\rangle$ stands for the scalar product in $\mathbb{R}^d$. This class includes many important examples (cf. [2], [3]). Let $\mathcal{BA}$ be the Beurling-Ahlfors transform on the plane, i.e., a Fourier multiplier with the symbol $m(\xi) = \overline{\xi}/\xi$, $\xi \in \mathbb{C}$. This operator is of fundamental importance in the study of partial differential equations and quasiconformal mappings, since it changes the complex derivative $\overline{\partial}$ to ∂. It turns out that the real and imaginary parts of $\mathcal{BA}$ can be represented as the Fourier multipliers with the symbols of the form (1.1). For example, the choice $d = 2$, $\mu = \delta_{(1,0)} + \delta_{(0,1)}$, $\psi(1, 0) = 1 = -\psi(0, 1)$ and $\nu = 0$ gives rise to $T_m = \mathrm{Re}\,(\mathcal{BA})$; furthermore, $d = 2$, $\mu = \delta_{(1/\sqrt{2},1/\sqrt{2})} + \delta_{(1/\sqrt{2},-1/\sqrt{2})}$, $\psi(1/\sqrt{2}, 1/\sqrt{2}) = -1 = -\psi(1/\sqrt{2}, -1/\sqrt{2})$ and $\nu = 0$ leads to $T_m = \mathrm{Im}\,(\mathcal{BA})$. A similar choice of the parameters leads to $\frac{1}{2}\mathcal{BA}$. For a higher-dimensional example, pick a proper subset J of $\{1, 2, \ldots, d\}$ and take $\mu = \delta_{e_1} + \delta_{e_2} + \ldots + \delta_{e_d}$, $\nu = 0$ and $\psi(e_j) = \chi_J(j)$, $j = 1, 2, \ldots, d$, where $e_1, e_2, \ldots, e_d$ are the versors in $\mathbb{R}^d$. This yields the operator $\sum_{j\in J} R_j^2$ on $\mathbb{R}^d$.

One of the principal results of [2] is the following L^p estimate.

Theorem 1.1.
Let $1 < p < \infty$ and let $m = m_{\phi,\psi,\mu,\nu}$ be given by (1.1). Then for any $f \in L^p(\mathbb{R}^d)$ we have

$$||T_m f||_{L^p(\mathbb{R}^d)} \leq (p^* - 1)||f||_{L^p(\mathbb{R}^d)}. \tag{1.2}$$

It turns out that the constant $p^* - 1$ is the best possible: see Geiss, Montgomery-Smith and Saksman [14] or Bañuelos and Osękowski [4] for details.

A very interesting phenomenon is that, essentially, all the sharp estimates mentioned above are proved with the use of probabilistic methods (Pichorides exploits certain special superharmonic functions, but these, in fact, lead to more general inequalities for orthogonal martingales: see Bañuelos and Wang [5]). It turns out that martingale methods lead to more general results for Fourier multipliers (see e.g. [23]). The purpose of this paper is to explore further this fruitful connection. Our motivation comes from a natural question about the best constant in the weak-type estimate

$$||R_j^2 f||_{p,\infty} \leq C_p ||f||_p, \qquad 1 \leq p < \infty,$$

where

$$||g||_{p,\infty} = \sup_{\lambda > 0} \left\{ \lambda^p \big| \{x \in \mathbb{R}^d : |g(x)| \geq \lambda\} \big| \right\}^{1/p}$$

denotes the weak p-th norm of g. In fact, our argumentation will yield the inequality for a larger class of multipliers. For $2 < p < \infty$, let $c = c(p)$ be the unique positive solution to the equation

$$c^{p-1} = 2c + 1 \tag{1.3}$$

and put

$$C_p = \begin{cases} \dfrac{1}{2} \left[\dfrac{(2c+p-1)^{p-1}}{c+1} \right]^{1/p} & \text{if } 2 < p < \infty, \\ 1 & \text{if } p = 2. \end{cases} \tag{1.4}$$

Theorem 1.2.
Suppose that m is a symbol given by (1.1), where ϕ and ψ are assumed to take values in $[0,1]$. Then for any $2 \leq p < \infty$ we have

$$||T_m f||_{p,\infty} \leq C_p ||f||_p. \tag{1.5}$$

The inequality is sharp. More precisely, for any $2 \leq p < \infty$, any $C < C_p$, any $d \geq 2$ and any proper subset J of $\{1, 2, \ldots, d\}$ there is $f \in L^p(\mathbb{R}^d)$ such that

$$\left|\left| \sum_{j \in J} R_j^2 f \right|\right|_{p,\infty} > C ||f||_{L^p(\mathbb{R}^d)}.$$

To the best of our knowledge, this is one of the very few results in the literature when the exact weak-type norm of a Fourier multiplier has been determined (Davis [12] and Janakiraman [16] identified the weak-type constants for the Hilbert transform on the line). Unfortunately, we have managed to push the calculations through only in the case $p \geq 2$. The reason is that for $1 \leq p < 2$ we did not succeed in showing an appropriate martingale bound (see Section 2 below).

Before we proceed, let us mention here the following application of Theorem 1.2, which can be of interest in the theory of elliptic differential operators and potential theory. Let u be a C^∞ compactly supported function on $\mathbb{R}^d$. By a direct comparison of Fourier transforms, we check that $R_j^2 \Delta u = -\frac{\partial^2 u}{\partial x_j^2}$ for all $1 \leq j \leq d$ and thus we obtain the sharp bound

$$\left|\left|\sum_{j\in J} \frac{\partial^2 u}{\partial x_j^2}\right|\right|_{L^p(\mathbb{R}^d)} \leq C_p ||\Delta u||_{L^p(\mathbb{R}^d)}, \qquad 2 \leq p < \infty,$$

whenever J is a proper subset of $\{1, 2, \ldots, d\}$.

A few words about the organization of the paper is in order. In the next section we study a martingale inequality, which can be regarded as a probabilistic counterpart of (1.5). In Section 3 we combine this estimate with the representation of Fourier multipliers (1.1) in terms of Lévy processes, and provide the proof of the inequality (1.5). Finally, in Section 4 we show that the constant C_p cannot be replaced by a smaller number, even if we restrict ourselves to the operators $\sum_{j\in J} R_j^2$. This will be based on the technique of laminates, an important object in the convex integration theory.

2. A martingale inequality

The key ingredient of the proof of the announced estimate (1.5) is an appropriate inequality for differentially subordinated martingales. We begin with introducing the necessary probabilistic background and notation. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, equipped with $(\mathcal{F}_t)_{t\geq 0}$, a nondecreasing family of sub-σ-fields of $\mathcal{F}$, such that $\mathcal{F}_0$ contains all the events of probability 0. Let X, Y be two adapted martingales taking values in a certain separable Hilbert space $(\mathcal{H}, |\cdot|)$; with no loss of generality, we may put $\mathcal{H} = \ell^2$. As usual, we assume that the processes have right-continuous trajectories with the limits from the left. The symbol $[X, Y]$ will stand for the quadratic covariance process of X and Y. See e.g. Dellacherie and Meyer [13] for details in the case when the processes are real-valued, and extend the definition to the vector setting by $[X, Y] = \sum_{k=0}^\infty [X^k, Y^k]$, where X^k, Y^k are the k-th coordinates of X, Y, respectively. We will say that Y is non-symmetrically differentially subordinate to X, if the process $([X, Y]_t - [Y, Y]_t)_{t\geq 0}$ is nonnegative and nondecreasing as a function of t (see Bañuelos and Wang [5], Osękowski [22] and Wang [28]). The nonsymmetric differential subordination implies many interesting inequalities comparing the sizes of X and Y; see e.g. Choi [10] and Burkholder [8]. We mention here only one result by the author [23], which will be of importance to our further considerations. For $p \geq 2$, let C_p be given by (1.4). We use the notation $||X||_p = \sup_{t\geq 0} ||X_t||_p$.

Theorem 2.1.
Suppose that X, Y are $\mathcal{H}$-valued martingales such that Y is non-symmetrically differentially subordinate to X. Then for $p \geq 2$ we have

$$\mathbb{P}\left(\sup_{t\geq 0} |Y_t| \geq 1\right) \leq C_p^p ||X||_p^p \tag{2.1}$$

and the inequality is sharp.

Now we are ready to formulate our main probabilistic result. As we will see, it can be regarded as a dual estimate to (2.1).

Theorem 2.2.
Assume that X, Y are $\mathcal{H}$-valued martingales such that Y is non-symmetrically differentially subordinate to X. Then for any $1 < q \leq 2$,

$$||Y||_q^q \leq C_{q/(q-1)}^q ||X||_1 ||X||_\infty^{q-1}. \tag{2.2}$$

For each q, the constant $C_{q/(q-1)}$ is the best possible.

The proof rests on Burkholder's method: we shall deduce the inequality (2.2) from the existence of a family $\{V_q\}_{q\in(1,\infty)}$ of certain special functions defined on the set $S = \{(x,y) \in \mathcal{H}\times\mathcal{H} : |x| \leq 1\}$. In order to simplify the technicalities, we shall combine the technique with an "integration argument", invented in [20] (see also [21]): first we introduce two simple functions v_1, $v_\infty : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$, for which the calculations are relatively easy; then define V_q by integrating these two objects against appropriate nonnegative kernels. Let

$$v_1(x,y) = \begin{cases} |2y-x|^2 - |x|^2 & \text{if } |x| + |2y-x| \leq 1, \\ 1 - 2|x| & \text{if } |x| + |2y-x| > 1 \end{cases}$$

and

$$v_\infty(x,y) = \begin{cases} 0 & \text{if } |x| + |2y-x| < 1, \\ (|2y-x|-1)^2 - |x|^2 & \text{if } |x| + |2y-x| \geq 1. \end{cases}$$

We have the following fact, which appears as Lemma 2.2 and Lemma 3.2 in [22].

Lemma 2.1.
For all $\mathcal{H}$-valued martingales X, Y such that Y is non-symmetrically differentially subordinate to X, we have

$$\mathbb{E}v_1(X_t, Y_t) \leq 0 \qquad \textit{for all } t \geq 0.$$

In addition, if X, Y are bounded in L^2, then $\mathbb{E}v_\infty(X_t,Y_t) \leq 0$ for all $t \geq 0$.

Recall that $S = \{(x,y) \in \mathcal{H}\times\mathcal{H} : |x| \leq 1\}$. For $1 < q < 2$, let $c = c(q/(q-1))$ be given by (1.3) and put $b = (c+1)^{-1}$. Define $V_q : S \to \mathbb{R}$ by

$$V_q(x,y) = \frac{q(2-q)}{2^{q+1}} \int_0^b r^{q-1} v_1(x/r, y/r)\mathrm{d}r + \frac{qb^{q-2}}{2^{q+1}}(|2y-x|^2 - |x|^2).$$

A little calculation shows that if $|x| + |2y-x| \leq b$, then

$$V_q(x,y) = \frac{1}{2^q(q-1)}(|x| + |2y-x|)^{q-1}((q-1)|2y-x| - |x|),$$

while for $|x| + |2y - x| > b$,

$$V_q(x,y) = \frac{q(2-q)b^{q-1}}{2^{q+1}}\left(\frac{b}{q} - \frac{2|x|}{q-1}\right) + \frac{qb^{q-2}}{2^{q+1}}\left(|2y-x|^2 - |x|^2\right).$$

We also define V_2 by the formula $V_2(x,y) = \frac{1}{2}(|2y-x|^2 - |x|^2)$; it is not difficult to check that V_2 is the pointwise limit of V_q as q to 2 (indeed: directly from (1.3), we infer that $b \to 0$ and $b^{q-2} \to 2$).

We shall need the following majorization property of these functions.

Lemma 2.2.
For any $1 < q \leq 2$ we have

$$V_q(x,y) \geq |y|^q - C^q_{q/(q-1)}|x| \qquad \textit{for all } (x,y) \in S. \tag{2.3}$$

Proof. It is convenient to split the reasoning into a few parts.

Step 1: $q = 2$. When $q = 2$, the inequality reads

$$\frac{1}{2}(|2y-x|^2 - |x|^2) \geq |y|^2 - |x|,$$

or, equivalently, $|y-x|^2 + |x| - |x|^2 \geq 0$, which is obvious.

Step 2: some reductions. Now, assume that $q \neq 2$. Note that it suffices to establish the majorization for $\mathcal{H} = \mathbb{R}$ and for x, y satisfying $0 \leq x \leq 2y$. To see this, let us, for a moment, write $V_q^{\mathcal{H}}$ to indicate the Hilbert space in which we work. Pick $x, y \in \mathcal{H}$, put $x' = |x|$ and $y' = |x/2| + |x/2 - y|$. Then $0 \leq x' \leq 2y'$, $2y' - x' = |2y - x|$ and $y' \geq |y|$, so

$$V_q^{\mathcal{H}}(x,y) - |y|^q - C^q_{q/(q-1)}|x| \geq V_q^{\mathbb{R}}(x',y') - |y'|^q - C^q_{q/(q-1)}|x'|,$$

since the dependence of $V_q^{\mathcal{H}}$ on x and y is through x and $2y - x$ only.

Next, observe that for any $y \geq 0$ the functions $x \mapsto v_1(x,y)$ and $x \mapsto |2y-x|^2 - |x|^2$ are concave on $[0,1]$. Therefore, V_q also has this property and since the right-hand side of (2.3) is linear in x, we will be done if we prove the majorization for $x \in \{0,1\}$.

Step 3: the case $x = 0$. If $y \leq b/2$, then both sides of (2.3) are equal. If $x = 0$ and $y > b/2$, then the majorization can be rewritten in the equivalent form

$$(y^2)^{q/2} - \left(\frac{b^2}{4}\right)^{q/2} \leq \frac{q}{2}\left(\frac{b^2}{4}\right)^{q/2-1}\left(y^2 - \frac{b^2}{4}\right),$$

which follows immediately from the mean value property.

Step 4: the case $x = 1$. We restrict ourselves to $y \geq 1/2$ (see Step 2 above). This time, (2.3) takes the form

$$F(y) := \frac{q(2-q)b^{q-1}}{2^{q+1}}\left(\frac{b}{q} - \frac{2}{q-1}\right) + \frac{qb^{q-2}}{2^{q+1}}(4y^2 - 4y) - y^q + C^q_{q/(q-1)} \geq 0.$$

We easily check that $F(1-b/2)=F'(1-b/2)=0$ and

$$F''(y)=\frac{8qb^{q-2}}{2^{q+1}}-q(q-1)y^{q-2}\geq \frac{8qb^{q-2}}{2^{q+1}}-\frac{q(q-1)}{2^{q-2}}=\frac{q}{2^{q-2}}(b^{q-2}-(q-1)).$$

However, the latter expression is positive: we have $b^{q-2}\geq 1$ and $q-1<1$. This proves that F is convex on $[1/2,\infty)$ and hence it is nonnegative on this interval. This completes the proof. □

Now we are ready to establish Theorem 2.2.

Proof of (2.2). It suffices to show that $||Y||_q^q\leq C^q_{q/(q-1)}||X||_1$ for any X, Y as in the statement satisfying the additional condition $||X||_\infty\leq 1$. Fix $t\geq 0$. The non-symmetric differential subordination implies that the process $([X,X]_t-[2Y-X,2Y-X]_t)_{t\geq 0}$ is nonnegative and nondecreasing as a function of t; consequently, for any t we have

$$\mathbb{E}|2Y_t-X_t|^2=\mathbb{E}[2Y-X,2Y-X]_t\leq \mathbb{E}[X,X]_t=\mathbb{E}|X_t|^2\leq 1,$$

because of the boundedness of X. Therefore, Lemma 2.1 and Fubini's theorem imply

$$\mathbb{E}V_q(X_t,Y_t)\leq \frac{q(2-q)}{2^{q+1}}\int_0^b r^{q-1}\mathbb{E}v_1(X_t/r,Y_t/r)\mathrm{d}r\leq 0. \tag{2.4}$$

To see that Fubini's theorem is applicable, note that $|v_1(x,y)|\leq c(|x|+|y|+1)$ for all x, $y\in\mathcal{H}$ and some absolute constant c; thus

$$\mathbb{E}\int_0^b r^{q-1}|v_1(X_t/r,Y_t/r)|\mathrm{d}r\leq \tilde{c}\mathbb{E}(|X_t|+|Y_t|+1)<\infty,$$

where $\tilde{c}$ is another universal constant. Combining (2.4) with (2.3) yields $\mathbb{E}|Y_t|^q\leq C^q_{q/(q-1)}\mathbb{E}|X_t|$ and it suffices to let $t\to\infty$ to get the claim. □

The optimality of the constant $C^q_{q/(q-1)}$ will be dealt with in Section 4. It will follow immediately from the sharpness of (1.5).

Remark 2.3. Unfortunately, we have been unable to find an appropriate majorant V_q in the case $q\geq 2$. Though the "integration argument" is available and we have the dual simple function v_∞ ready to use, all the special V_q we managed to construct have led us to non-optimal $C_{q/(q-1)}$.

3. Proof of Theorem 1.2

Let $m=m_{\phi,\psi,\mu,\nu}$ be a multiplier as in (1.1). By the results in [2], we may assume that the Lévy measure ν satisfies the symmetry condition $\nu(B)=\nu(-B)$ for all Borel subsets B of $\mathbb{R}^d$. More precisely, there are $\bar{\mu}$, $\bar{\nu}$, $\bar{\phi}$, $\bar{\psi}$ such that $\bar{\nu}$ is symmetric and $m_{\phi,\psi,\mu,\nu}=m_{\bar{\phi},\bar{\psi},\bar{\mu},\bar{\nu}}$. Assume in addition that $|\nu|=\nu(\mathbb{R}^d)$ is finite and nonzero, and define $\tilde{\nu}=\nu/|\nu|$. Consider the independent random variables T_{-1}, T_{-2}, $\ldots$, Z_{-1}, Z_{-2}, $\ldots$ such that for each

$n = -1, -2, \ldots, T_n$ has exponential distribution with parameter $|\nu|$ and Z_n takes values in $\mathbb{R}^d$ and has $\tilde{\nu}$ as the distribution. Next, put $S_n = -(T_{-1} + T_{-2} + \ldots + T_n)$ for $n = -1, -2, \ldots$ and let

$$X_{s,t} = \sum_{s<S_j\leq t} Z_j, \qquad X_{s,t-} = \sum_{s<S_j<t} Z_j, \qquad \Delta X_{s,t} = X_{s,t} - X_{s,t-},$$

for $-\infty < s \leq t \leq 0$. For a given $f \in L^\infty(\mathbb{R}^d)$, define its parabolic extension $\mathcal{U}_f$ to $(-\infty, 0] \times \mathbb{R}^d$ by

$$\mathcal{U}_f(s,x) = \mathbb{E}f(x + X_{s,0}).$$

Next, fix $x \in \mathbb{R}^d$, $s < 0$ and $f \in L^\infty(\mathbb{R}^d)$. We introduce the processes $F = (F_t^{x,s,f})_{t\in[s,0]}$ and $G = (G_t^{x,s,f,\phi})_{t\in[s,0]}$ by

$$\begin{aligned} F_t &= \mathcal{U}_f(t, x + X_{s,t}), \\ G_t &= \sum_{s<u\leq t} \left[\Delta F_u \cdot \phi(\Delta X_{s,u})\right] - \int_s^t \int_{\mathbb{R}^d} \left[\mathcal{U}_f(v, x + X_{s,v-} + z) - \mathcal{U}_f(v, x + X_{s,v-})\right]\phi(z)\nu(\mathrm{d}z)\mathrm{d}v. \end{aligned} \tag{3.1}$$

Note that the sum in the definition of G can be seen as the result of modulating of the jumps of F by ϕ, and the subsequent double integral can be regarded as an appropriate compensator. We have the following statement, proved in [3].

Lemma 3.1.
For any fixed x, s, f as above, the processes $F^{x,s,f}$, $G^{x,s,f,\phi}$ are martingales with respect to $(\sigma(\{X_u : u \leq t\}))_{t\in[s,0]}$. Furthermore, if $||\phi||_\infty \leq 1$, then $G^{x,s,f,\phi}$ is differentially subordinate to $F^{x,s,f}$.

Now, fix $s < 0$ and define the operator $\mathcal{S} = \mathcal{S}^{s,\phi,\nu}$ by the bilinear form

$$\int_{\mathbb{R}^d} \mathcal{S}f(x)g(x)\mathrm{d}x = \int_{\mathbb{R}^d} \mathbb{E}\left[G_0^{x,s,f,\phi} g(x + X_{s,0})\right]\mathrm{d}x, \tag{3.2}$$

where f, $g \in C_0^\infty(\mathbb{R}^d)$. We have the following fact, proved in [3]. It constitutes the crucial part of the aforementioned representation of Fourier multipliers in terms of Lévy processes.

Lemma 3.2.
Let $1 < p < \infty$ and $d \geq 2$. The operator $\mathcal{S}^{s,\phi,\nu}$ is well defined and extends to a bounded operator on $L^p(\mathbb{R}^d)$, which can be expressed as a Fourier multiplier with the symbol

$$M(\xi) = M_{s,\phi,\nu}(\xi) = \left[1 - \exp\left(2s \int_{\mathbb{R}^d} (1 - \cos\langle\xi, z\rangle)\nu(dz)\right)\right] \frac{\int_{\mathbb{R}^d}(1 - \cos\langle\xi, z\rangle)\phi(z)\nu(dz)}{\int_{\mathbb{R}^d}(1 - \cos\langle\xi, z\rangle)\nu(dz)}$$

if $\int_{\mathbb{R}^d}(1 - \cos\langle\xi, z\rangle)\nu(dz) \neq 0$, and $M(\xi) = 0$ otherwise.

We are ready to establish the following dual version of (1.5).

Theorem 3.1.
Assume that $1 < q \leq 2$ and let $m : \mathbb{R}^d \to \mathbb{C}$ be a symbol given by (1.1), where ϕ and ψ are assumed to take values in $[0,1]$. Then for any function $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ we have

$$||T_m f||^q_{L^q(\mathbb{R}^d)} \leq C^q_{q/(q-1)} ||f||_{L^1(\mathbb{R}^d)} ||f||^{q-1}_{L^\infty(\mathbb{R}^d)}. \tag{3.3}$$

Proof. By homogeneity, it suffices to establish the bound for f bounded by 1. Furthermore, we may and do assume that at least one of the measures μ, ν is nonzero. It is convenient to split the reasoning into two parts.

Step 1. First we show the estimate for the multipliers of the form

$$M_{\phi,\nu}(\xi) = \frac{\int_{\mathbb{R}^d} (1 - \cos\langle \xi, z\rangle)\phi(z)\nu(\mathrm{d}z)}{\int_{\mathbb{R}^d} (1 - \cos\langle \xi, z\rangle)\nu(\mathrm{d}z)}. \tag{3.4}$$

Assume that $0 < \nu(\mathbb{R}^d) < \infty$, so that the above machinery using Lévy processes is applicable. Fix $s < 0$ and functions f, $g \in C_0^\infty(\mathbb{R}^d)$ such that f is bounded by 1; of course, then the martingale $F^{x,s,f}$ also takes values in the unit ball of $\mathbb{C}$. By Hölder's inequality, Fubini's theorem and (2.2), we have

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \mathbb{E}\big[G_0^{x,s,f,\phi} g(x + X_{s,0})\big] \mathrm{d}x \right| &\leq \left(\int_{\mathbb{R}^d} \mathbb{E}|G_0^{x,s,f,\phi}|^q \mathrm{d}x \right)^{1/q} \left(\int_{\mathbb{R}^d} \mathbb{E}|g(x + X_{s,0})|^p \mathrm{d}x \right)^{1/p} \\ &= \left(\int_{\mathbb{R}^d} \mathbb{E}|G_0^{x,s,f,\phi}|^q \mathrm{d}x \right)^{1/q} ||g||_{L^p(\mathbb{R}^d)} \\ &\leq \left(C^q_{q/(q-1)} \int_{\mathbb{R}^d} \mathbb{E}|F_0^{x,s,f}| \mathrm{d}x \right)^{1/q} ||g||_{L^p(\mathbb{R}^d)} \\ &= \left(C^q_{q/(q-1)} ||f||_1 \right)^{1/q} ||g||_{L^p(\mathbb{R}^d)}. \end{aligned} \tag{3.5}$$

Here $p = q/(q-1)$ is the harmonic conjugate to q. Plugging this into the definition of $\mathcal{S}$, we obtain

$$||\mathcal{S}^{s,\phi,\nu} f||^q_{L^q(\mathbb{R}^d)} \leq C^q_{q/(q-1)} ||f||_{L^1(\mathbb{R}^d)}.$$

Now if we let $s \to -\infty$, then $M_{s,\phi,\nu}$ converges pointwise to $M_{\phi,\nu}$ given by (3.4). The symbols are bounded in absolute value by 1, so, by Lebesgue's dominated convergence theorem, we have $\widehat{S^{s,\phi,\nu}} f \to \widehat{T_{M_{\phi,\nu}}} f$ in $L^2(\mathbb{R}^d)$. By Plancherel's theorem, we also have $\mathcal{S}^{s,\phi,\nu} f \to T_{M_{\phi,\nu}} f$ in $L^2(\mathbb{R}^d)$ and hence there is a sequence $(s_n)_{n=1}^\infty$ converging to $-\infty$ such that $\lim_{n\to\infty} \mathcal{S}^{s_n,\phi,\nu} f \to T_{M_{\phi,\nu}} f$ almost everywhere. Thus Fatou's lemma yields the desired bound for the multiplier $T_{M_{\phi,\nu}}$.

Step 2. Now we deduce the result for the general multipliers as in (1.1) and drop the assumption $0 < \nu(\mathbb{R}^d) < \infty$. For a given $\varepsilon > 0$, define a Lévy measure ν_ε in polar coordinates $(r, \theta) \in (0, \infty) \times \mathbb{S}$ by

$$\nu_\varepsilon(\mathrm{d}r\mathrm{d}\theta) = \varepsilon^{-2} \delta_\varepsilon(\mathrm{d}r)\mu(d\theta).$$

Here δ_ε denotes Dirac measure on $\{\varepsilon\}$. Next, consider a multiplier $M_{\varepsilon,\phi,\psi,\mu,\nu}$ as in (3.4), in which the Lévy measure is $1_{\{|x|>\varepsilon\}}\nu + \nu_\varepsilon$ and the jump modulator is given by $1_{\{|x|>\varepsilon\}}\phi(x) + 1_{\{|x|=\varepsilon\}}\psi(x/|x|)$. Note that this Lévy measure is finite and nonzero, at least for sufficiently small ε. If we let $\varepsilon \to 0$, we see that

$$\int_{\mathbb{R}^d} [1 - \cos\langle \xi, x\rangle]\psi(x/|x|)\nu_\varepsilon(\mathrm{d}x) = \int_{\mathbb{S}} \langle \xi, \theta\rangle^2 \psi(\theta) \frac{1 - \cos\langle \xi, \varepsilon\theta\rangle}{\langle \xi, \varepsilon\theta\rangle^2} \mu(d\theta) \to \frac{1}{2} \int_{\mathbb{S}} \langle \xi, \theta\rangle^2 \psi(\theta)\mu(\mathrm{d}\theta)$$

and, consequently, $M_{\varepsilon,\phi,\psi,\mu,\nu} \to m_{\phi,\psi,\mu,\nu}$ pointwise. This yields the claim by the similar argument as above, using Plancherel's theorem and the passage to the subsequence which converges almost everywhere. □

Now we shall apply duality to deduce (1.5).

Proof of Theorem 1.2. Observe that the class (1.1) is closed under the complex conjugation: we have $\bar{m} = m_{\bar{\phi},\bar{\psi},\mu,\nu}$. Fix $f \in L^p(\mathbb{R}^d)$ and put

$$g = \frac{T_m f}{|T_m f|} 1_{\{x\in\mathbb{R}^d : |T_m f(x)|\geq 1\}}.$$

By Hölder's inequality and Parseval's identity,

$$\begin{aligned} |\{x \in \mathbb{R}^d : |T_m f(x)| \geq 1\}| &\leq \int_{\mathbb{R}^d} T_m f(x)\overline{g(x)}\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \widehat{T_m f}(x)\overline{\widehat{g}(x)}\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \widehat{f}(x)\overline{\widehat{T_{\bar{m}} g}(x)}\mathrm{d}x \\ &= \int_{\mathbb{R}^d} f(x)\overline{T_{\bar{m}} g(x)}\mathrm{d}x \\ &\leq ||f||_{L^p(\mathbb{R}^d)}||T_{\bar{m}} g||_{L^q(\mathbb{R}^d)} \\ &\leq ||f||_{L^p(\mathbb{R}^d)}\left(C_{q/(q-1)}^q ||g||_{L^1(\mathbb{R}^d)}\right)^{1/q}. \end{aligned} \tag{3.6}$$

Here in the latter passage we have used (3.3) and the fact that g takes values in the unit ball of $\mathbb{C}$. However, $||g||_{L^1(\mathbb{R}^d)} = |\{x \in \mathbb{R}^d : |T_m f(x)| \geq 1\}|$ and $C_{q/(q-1)} = C_p$. This completes the proof of the weak type estimate. □

In the remainder of this section we discuss the possibility of extending the assertion of Theorem 1.2 to the vector-valued multipliers. For any bounded function $m = (m_1, m_2, \ldots, m_n) : \mathbb{R}^d \to \mathbb{C}^n$, we may define the associated Fourier multiplier acting on complex valued functions on $\mathbb{R}^d$ by the formula $T_m f = (T_{m_1} f, T_{m_2} f, \ldots, T_{m_n} f)$. As we shall see, the reasoning presented above can be easily modified to yield the following statement.

Theorem 3.2.

Let ν, μ be two measures on $\mathbb{R}^d$ and $\mathbb{S}$, respectively, satisfying the assumptions of Theorem 1.2. Assume further that ϕ, ψ are two Borel functions on $\mathbb{R}^d$ taking values in the unit ball of $\mathbb{C}^n$ and let $m : \mathbb{R}^d \to \mathbb{C}^n$ be the associated symbol given by (1.1). Then for any Borel function $f : \mathbb{R}^d \to \mathbb{C}$ we have

$$||T_m f||_{L^q(\mathbb{R}^d;\mathbb{C}^n)}^q \leq C_{q/(q-1)}^q ||f||_{L^1(\mathbb{R}^d)}||f||_{L^\infty(\mathbb{R}^d)}^{q-1}, \qquad 1 < q \leq 2,$$

and

$$||T_m f||_{L^{p,\infty}(\mathbb{R}^d;\mathbb{C}^n)} \leq C_p ||f||_{L^p(\mathbb{R}^d)}, \qquad p \geq 2.$$

Proof. Suppose first that ν is finite. For a given function $f \in C_0^\infty(\mathbb{R}^d)$ bounded by 1, we introduce the martingales F and $G = (G^1, G^2, \ldots, G^n)$ by (3.1). It is not difficult to check that Lemma 3.1 is also valid in the vector-valued setting (repeat the reasoning from [3]). Applying the representation (3.2) to each coordinate of G separately, we obtain the associated multiplier $\mathcal{S} = (\mathcal{S}^1, \mathcal{S}^2, \ldots, \mathcal{S}^n)$, where S^j has symbol $M_{\phi_j,\nu}$ defined in (3.4). Now we repeat the reasoning from (3.5), with a vector-valued function $g : \mathbb{R}^d \to \mathbb{C}^n$ (the expression $G_0^{x,s,f,\phi} g(x + X_{s,0})$ under the first integral is replaced with the corresponding scalar product). An application of (2.2) gives

$$||S^{s,\phi,\nu} f||^q_{L^q(\mathbb{R}^d;\mathbb{C}^n)} \leq C_q ||f||_{L^1(\mathbb{R}^d)},$$

which extends to general f by standard density arguments. The passage to general m as in (1.1) is carried over in the same manner as in the scalar case; this yields the vector version of Theorem 3.1. The duality argument explained in (3.6) extends to the vector-valued setting with no difficulty (one only has to replace appropriate multiplications by scalar products) and thus Theorem 1.2 holds true for the multipliers on $\mathbb{C}^n$. □

4. Sharpness

In the final part of the paper we show the second half of Theorem 1.2: the constant C_p in (1.5) is the best possible, even for the special multipliers $\sum_{j \in J} R_j^2$, where J is a proper subset of $\{1, 2, \ldots, d\}$. This, of course, will immediately imply that the constant in (2.2) is also optimal (otherwise, its improvement would lead to a smaller constant in (1.5)). Our approach will be based on the properties of certain special probability measures, the so-called laminates. For the sake of convenience and clarity, we have decided to split this section into a few separate parts.

4.1. Necessary definitions

Let $\mathbb{R}^{m \times n}$ denote the space of all real matrices of dimension $m \times n$ and let $\mathbb{R}^{n \times n}_{sym}$ be the class of all real symmetric $n \times n$ matrices.

Definition 4.1. A function $f : \mathbb{R}^{m \times n} \to \mathbb{R}$ is said to be *rank-one convex*, if $t \mapsto f(A + tB)$ is convex for all $A, B \in \mathbb{R}^{m \times n}$ with rank $B = 1$.

Next, let $\mathcal{P} = \mathcal{P}(\mathbb{R}^{m \times n})$ stand for the class of all compactly supported probability measures on $\mathbb{R}^{m \times n}$. For $\nu \in \mathcal{P}$ we denote by $\overline{\nu} = \int_{\mathbb{R}^{m \times n}} X d\nu(X)$ the center of mass or *barycenter* of ν.

Definition 4.2. We say that a measure $\nu \in \mathcal{P}$ is a *laminate* (and denote it by $\nu \in \mathcal{L}$), if

$$f(\overline{\nu}) \leq \int_{\mathbb{R}^{m \times n}} f d\nu \tag{4.1}$$

for all rank-one convex functions f. The set of laminates with barycenter 0 is denoted by $\mathcal{L}_0(\mathbb{R}^{m \times n})$.

Laminates arise naturally in several applications of convex integration: they can be used to produce interesting counterexamples, see e.g. [1],[11], [18], [19], [27]. We will be particularly interested in the case of 2×2 symmetric matrices. The important fact is that laminates can be regarded as probability measures that record the distribution of the gradients of smooth maps, see Corollary 4.4 below. Let us briefly explain this; detailed proofs of the statements below can be found for example in [17], [19], [27].

Definition 4.3. Let $U \subset \mathbb{R}^{2\times 2}$ be a given set. Then $\mathcal{PL}(U)$ denotes the class of *prelaminates* generated in U, i.e., the smallest class of probability measures on U, contained in $\mathcal{L}$, which

(i) contains all measures of the form $\lambda\delta_A + (1-\lambda)\delta_B$ with $\lambda \in [0,1]$ and $\operatorname{rank}(A-B) = 1$ (here δ_A, δ_B stand for the Dirac measures concentrated on A and B);

(ii) is closed under splitting in the following sense: if $\lambda\delta_A + (1-\lambda)\tilde{\nu}$ belongs to $\mathcal{PL}(U)$ for some $\tilde{\nu} \in \mathcal{P}(\mathbb{R}^{2\times 2})$ and $\lambda \in [0,1]$, and μ belongs to $\mathcal{PL}(U)$ with $\overline{\mu} = A$, then $\lambda\mu + (1-\lambda)\tilde{\nu}$ also belongs to $\mathcal{PL}(U)$.

It might be helpful to provide here an alternative, inductive definition of the class $\mathcal{PL}(U)$. We start from Dirac measures concentrated on the elements of U, and then allow the following modification of these. Namely, if a measure $\mu = \lambda\delta_A + (1-\lambda)\tilde{\nu}$ (with $\lambda \in [0,1]$) is a laminate, we replace it with $\lambda_-\delta_{A_-} + \lambda_+\delta_{A_+} + (1-\lambda)\tilde{\nu}$, where $\lambda_\pm \geq 0$, $\lambda_- + \lambda_+ = \lambda$, $\lambda_- A_- + \lambda_+ A_+ = \lambda A$ and $\operatorname{rank}(A_+ - A_-) = 1$. Then one easily checks that the new object is also a laminate; actually, $\mathcal{PL}$ consists of all measures which can be obtained from Dirac measures after a finite number of the above splittings. It follows immediately from the above inductive definition that the class $\mathcal{PL}(U)$ contains atomic measures only. Also, by a successive application of Jensen's inequality, we have the inclusion $\mathcal{PL} \subset \mathcal{L}$. Let us state two well-known facts (see [1], [17], [19], [27]). In what follows, $\mathcal{B}$ denotes the unit ball of $\mathbb{R}^2$.

Lemma 4.1.
Let $\nu = \sum_{i=1}^N \lambda_i \delta_{A_i} \in \mathcal{PL}(\mathbb{R}^{2\times 2}_{sym})$ with $\overline{\nu} = 0$. Moreover, let $0 < r < \frac{1}{2}\min|A_i - A_j|$ and $\delta > 0$. For any bounded domain $\Omega \subset \mathbb{R}^2$ there exists $u \in W_0^{2,\infty}(\Omega)$ such that $\|u\|_{C^1} < \delta$ and for all $i = 1, 2, \ldots, N$,

$$\left|\{x \in \Omega :\, |D^2u(x) - A_i| < r\}\right| = \lambda_i|\Omega|.$$

Lemma 4.2.
Let $K \subset \mathbb{R}^{2\times 2}_{sym}$ be a compact convex set and $\nu \in \mathcal{L}(\mathbb{R}^{2\times 2}_{sym})$ with $\operatorname{supp}\nu \subset K$. For any set $U \subset \mathbb{R}^{2\times 2}_{sym}$, relatively open with respect to $\mathbb{R}^{2\times 2}_{sym}$ and satisfying $K \subset U$, there exists a sequence $\nu_j \in \mathcal{PL}(U)$ of prelaminates with $\overline{\nu}_j = \overline{\nu}$ and $\nu_j \overset{*}{\rightharpoonup} \nu$.

Combining these two lemmas and using a simple mollification, we obtain the following statement, proved by Boros, Shékelyhidi Jr. and Volberg [6]. It links laminates supported on symmetric matrices with second derivatives of functions, and will play a crucial role in our argumentation below.

Corollary 4.4.
Let $\nu \in \mathcal{L}_0(\mathbb{R}^{2\times 2}_{sym})$. Then there exists a sequence $u_j \in C_0^\infty(\mathcal{B})$ with uniformly bounded second derivatives, such that

$$\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} \phi(D^2u_j(x))\,dx \to \int_{\mathbb{R}^{2\times 2}_{sym}} \phi\,d\nu$$

for all continuous $\phi : \mathbb{R}^{2\times 2}_{sym} \to \mathbb{R}$.

This corollary reveals the idea behind the proof of the sharpness of (1.5), at least in the two-dimensional case. Roughly speaking, the reasoning is as follows. Suppose that $u \in C_0^\infty(\mathbb{R}^2)$ is given and put $f = \Delta u$. If ϕ_1, ϕ_2 are appropriately chosen continuous functions on $\mathbb{R}^{2\times 2}_{sym}$ (see below), then

$$\frac{|\{x\in\mathbb{R}^2:|R_2^2f(x)|\geq 1\}|}{\int_{\mathbb{R}^2}|f(x)|^p\mathrm{d}x}=\frac{|\{x\in\mathbb{R}^2:|\partial^2_{22}u(x)|\geq 1\}|}{\int_{\mathbb{R}^2}|\partial^2_{11}u(x)+\partial^2_{22}u(x)|^p\mathrm{d}x}\approx\frac{\int_{\mathbb{R}^2}\phi_1(D^2u(x))\mathrm{d}x}{\int_{\mathbb{R}^2}\phi_2(D^2u(x))\mathrm{d}x}.$$

Hence, by Corollary 4.4, it is enough to construct, for any $\varepsilon>0$, a laminate $\nu\in\mathcal{L}_0(\mathbb{R}^{2\times 2}_{sym})$ for which the ratio $\int\phi_1\mathrm{d}\nu/\int\phi_2\mathrm{d}\nu$ is larger than $C_p^p-\varepsilon$.

4.2. Biconvex functions and a special laminate

Let us start by introducing an auxiliary notion. A function $\zeta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is said to be biconvex if for any fixed $z\in\mathbb{R}$, the functions $x\mapsto\zeta(x,z)$ and $y\mapsto\zeta(z,y)$ are convex. First we show the following inequality for biconvex functions in the plane. This estimate may seem unexpected, some arguments which suggest its use are presented in §4.5.

Lemma 4.3.
Suppose that $\zeta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is biconvex. Then for any numbers κ_0, κ_1 satisfying $0<\kappa_0<\kappa_1<1$, we have

$$\begin{aligned}\zeta(-\kappa_0,\kappa_0)\leq\kappa_0^{p-1}\Bigg\{&\frac{2\kappa_1^{2-p}}{(p-1)(1+2\kappa_0)+(3-p)\kappa_1}\zeta(-\kappa_1,1+2\kappa_0)\\&+\frac{(1+2\kappa_0-\kappa_1)(p-1)\kappa_1^{1-p}}{(p-1)(1+2\kappa_0)+(3-p)\kappa_1}\zeta\left(-\kappa_1,\frac{p-3}{p-1}\kappa_1\right)\\&+\frac{p-1}{2}\int_{\kappa_0}^{\kappa_1}\left[\zeta\left(\frac{3-p}{p-1}s,s\right)+\zeta\left(-s,\frac{p-3}{p-1}s\right)\right]\frac{ds}{s^p}\Bigg\}.\end{aligned}\tag{4.2}$$

Proof. By a standard regularization argument, it suffices to show the inequality for $\zeta\in C^1(\mathbb{R}^2)$. Fix $s\in(\kappa_0,\kappa_1)$ and a small positive δ. Using biconvexity of ζ, we may write

$$\zeta(-s,s)\leq\frac{\delta(p-1)}{2+\delta(p-1)}\zeta\left(-s,\frac{p-3}{p-1}s\right)+\frac{2}{2+\delta(p-1)}\zeta(-s,s+\delta s)$$

and

$$\zeta(-s,s+\delta s)\leq\frac{2(\delta+1)-\delta(p-1)}{2(\delta+1)}\zeta(-s-\delta s,s+\delta s)+\frac{\delta(p-1)}{2(\delta+1)}\zeta\left(\frac{3-p}{p-1}(s+\delta s),s+\delta s\right).$$

Plugging the latter estimate into the former, subtracting $\zeta(-s-\delta s,s+\delta s)$ from both sides and dividing throughout by δ gives

$$\begin{aligned}\frac{\zeta(-s,s)-\zeta(-s-\delta s,s+\delta s)}{\delta}\leq&\left(-\frac{p-1}{2+(p-1)\delta}-\frac{p-1}{2(\delta+1)}\right)\zeta(-s-\delta s,s+\delta s)\\&+\frac{p-1}{(2+(p-1)\delta)(\delta+1)}\zeta\left(\frac{3-p}{p-1}(s+\delta s),s+\delta s\right)\\&+\frac{p-1}{2+(p-1)\delta}\zeta\left(-s,\frac{p-3}{p-1}s\right).\end{aligned}$$

Therefore, letting $\delta \to 0$ implies

$$-s\frac{\mathrm{d}}{\mathrm{d}s}\zeta(-s,s) \leq (1-p)\zeta(-s,s) + \frac{p-1}{2}\left[\zeta\left(\frac{3-p}{p-1}s, s\right) + \zeta\left(-s, \frac{p-3}{p-1}s\right)\right].$$

Multiply both sides by s^{-p} to get the equivalent bound

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[s^{1-p}\zeta(-s,s)\right] \geq \frac{1-p}{2}s^{-p}\left[\zeta\left(\frac{3-p}{p-1}s, s\right) + \zeta\left(-s, \frac{p-3}{p-1}s\right)\right].$$

Integrating over s from κ_0 to κ_1 gives

$$\kappa_1^{1-p}\zeta(-\kappa_1,\kappa_1) - \kappa_0^{1-p}\zeta(-\kappa_0,\kappa_0) \geq \frac{1-p}{2}\int_{\kappa_0}^{\kappa_1} s^{-p}\left[\zeta\left(\frac{3-p}{p-1}s, s\right) + \zeta\left(-s, \frac{p-3}{p-1}s\right)\right]\mathrm{d}s.$$

It suffices to combine this inequality with the following consequence of the biconvexity of ζ:

$$\begin{aligned}\zeta(-\kappa_1,\kappa_1) \leq\ & \frac{2\kappa_1}{(1+2\kappa_0)(p-1)+(3-p)\kappa_1}\zeta(-\kappa_1, 1+2\kappa_0) \\ & + \frac{(p-1)(1+2\kappa_0-\kappa_1)}{(p-1)(1+2\kappa_0)+(3-p)\kappa_1}\zeta\left(-\kappa_1, \frac{p-3}{p-1}\kappa_1\right),\end{aligned}$$

and the claim is established. □

Let $\mu = \mu_{\kappa_0,\kappa_1} \in \mathcal{P}(\mathbb{R}^{2\times 2})$ be defined by the right-hand side of (4.2); that is, for any $f \in C(\mathbb{R}^{2\times 2})$, let

$$\begin{aligned}\int_{\mathbb{R}^{2\times 2}} f\mathrm{d}\mu_{\kappa_0,\kappa_1} = \kappa_0^{p-1}\Bigg\{ & \frac{2\kappa_1^{2-p}}{(p-1)(1+2\kappa_0)+(3-p)\kappa_1} f(\mathrm{diag}(-\kappa_1, 1+2\kappa_0)) \\ & + \frac{(1+2\kappa_0-\kappa_1)(p-1)\kappa_1^{1-p}}{(p-1)(1+2\kappa_0)+(3-p)\kappa_1} f\left(\mathrm{diag}\left(-\kappa_1, \frac{p-3}{p-1}\kappa_1\right)\right) \\ & + \frac{p-1}{2}\int_{\kappa_0}^{\kappa_1}\left[f\left(\mathrm{diag}\left(\frac{3-p}{p-1}s, s\right)\right) + f\left(\mathrm{diag}\left(-s, \frac{p-3}{p-1}s\right)\right)\right]\frac{\mathrm{d}s}{s^p}\Bigg\}.\end{aligned}$$

Then μ_{κ_0,κ_1} is a probability with barycenter $\overline{\mu}_{\kappa_0,\kappa_1} = \mathrm{diag}(-\kappa_0,\kappa_0)$. Observe that if f is rank-one convex, then $(x,y) \mapsto f(\mathrm{diag}(x,y))$ is biconvex. Therefore, using Lemma 4.3 we see that μ_{κ_0,κ_1} is a laminate. Hence, if we introduce a probability measure ν by putting $\nu(A) = \mu(A + \mathrm{diag}(-\kappa_0,\kappa_0))$ for any Borel subset A of $\mathbb{R}^{2\times 2}$, then $\nu \in \mathcal{L}_0(\mathbb{R}^{2\times 2}_{sym})$. Next, consider a continuous function $\phi_1 : \mathbb{R}^{2\times 2}_{sym} \to [0,\infty)$, which satisfies $\phi_1(A) = 0$ when $A_{22} \leq 1$, $\phi_1(A) = 1$ when $A_{22} \geq 1+\kappa_0$ and $\phi_1(A) \in [0,1]$ for remaining A. In addition, let $\phi_2 : \mathbb{R}^{2\times 2}_{sym} \to [0,\infty)$ be given by $\phi_2(A) = |A_{11}+A_{22}|^p$. Directly from the definition of ν, we have

$$\int \phi_1 \mathrm{d}\nu = \kappa_0^{p-1}\cdot\frac{2\kappa_1^{2-p}}{(p-1)(1+2\kappa_0)+(3-p)\kappa_1}$$

and

$$\begin{aligned}\int \phi_2\mathrm{d}\nu = \int \phi_2\mathrm{d}\mu = \kappa_0^{p-1}\Bigg\{ & \frac{2\kappa_1^{2-p}}{(p-1)(1+2\kappa_0)+(3-p)\kappa_1}(1+2\kappa_0-\kappa_1)^p \\ & + \frac{(1+2\kappa_0-\kappa_1)(p-1)\kappa_1^{1-p}}{(p-1)(1+2\kappa_0)+(3-p)\kappa_1}\left(\frac{2}{p-1}\kappa_1\right)^p + (p-1)\int_{\kappa_0}^{\kappa_1}\left(\frac{2}{p-1}s\right)^p\frac{\mathrm{d}s}{s^p}\Bigg\}.\end{aligned}$$

Now let κ_0 go to 0. Then

$$\frac{\int \phi_2 \mathrm{d}\nu}{\int \phi_1 \mathrm{d}\nu} \to (1-\kappa_1)^p + \left(\frac{2}{p-1}\right)^{p-1} \kappa_1^{p-1}(p+(2-p)\kappa_1).$$

If we put $\kappa_1 = (p-1)/(2c+p-1)$, then some straightforward computations show that the latter limit is equal to C_p^{-p}.

4.3. Sharpness, $d = 2$

Fix $p \geq 2$. The weak p-th norms of R_1^2 and R_2^2 are the same, so it suffices to prove that $||R_2^2||_{L^p(\mathbb{R}^2) \to L^{p,\infty}(\mathbb{R}^2)} \geq C_p$. By the above reasoning, if $\varepsilon > 0$ is a given number, then we can pick $\kappa_0 > 0$ such that the ratio $\int \phi_2 \mathrm{d}\nu / \int \phi_1 \mathrm{d}\nu$ is smaller than $C_p^{-p} + \varepsilon$. Therefore, an application of Corollary 4.4 yields the existence of a C^∞ function u, supported on $\mathcal{B}$, such that

$$\frac{\int_{\mathcal{B}} \phi_2(D^2 u(x)) \mathrm{d}x}{\int_{\mathcal{B}} \phi_1(D^2 u(x)) \mathrm{d}x} \leq C_p^{-p} + 2\varepsilon.$$

However, by the very definition, we have $\phi_1(A) \leq \chi_{\{A_{22} \geq 1\}}$. Thus, the above inequality implies

$$\left|\{x \in \mathbb{R}^2 : \partial_{22}^2 u(x) \geq 1\}\right| \geq \frac{1}{C_p^{-p} + 2\varepsilon} \int_{\mathcal{B}} |\partial_{11}^2 u(x) + \partial_{22}^2 u(x)|^p \mathrm{d}x.$$

Therefore, if we put $f = \Delta u$, we obtain

$$||R_2^2 f||_{L^{p,\infty}(\mathbb{R}^2)} \geq (C_p^{-p} + 2\varepsilon)^{-1/p} ||f||_{L^p(\mathbb{R}^2)}.$$

Since ε was arbitrary, this proves the desired sharpness of (1.5).

4.4. The case $d \geq 3$.

Fix $p \geq 2$. Let J be a proper subset of $\{1, 2, \ldots, d\}$ and write $T = \sum_{j \in J} R_j^2$. We must prove that $||T||_{L^p(\mathbb{R}^d) \to L^{p,\infty}(\mathbb{R}^d)} \geq C_p$. It suffices to consider only those J, which satisfy $1 \notin J$ and $2 \in J$: for any $J' \in \{1, 2, \ldots, d\}$ of the same cardinality as J, the weak norms of T and $\sum_{j \in J'} R_j^2$ are the same. So, suppose that T is of that special form and assume that for some positive constant C we have

$$\left|\{x \in \mathbb{R}^d : |Tf(x)| \geq 1\}\right| \leq C ||f||_{L^p(\mathbb{R}^d)}^p \tag{4.3}$$

for all $f \in L^p(\mathbb{R}^d)$. For $t > 0$, define the dilation operator δ_t as follows: for any function $g : \mathbb{R}^2 \times \mathbb{R}^{d-2} \to \mathbb{R}$, we let $\delta_t g(\xi, \zeta) = g(\xi, t\zeta)$; for any $A \subset \mathbb{R}^2 \times \mathbb{R}^{d-2}$, let $\delta_t A = \{(\xi, t\zeta) : (\xi, \zeta) \in A\}$. By (4.3), the operator $T_t := \delta_t^{-1} \circ T \circ \delta_t$ satisfies

$$\begin{aligned} \left|\{x \in \mathbb{R}^d : |T_t f(x)| \geq 1\}\right| &= t^{d-2} \left|\{x \in \mathbb{R}^d : |T \circ \delta_t f(x)| \geq 1\}\right| \\ &\leq C t^{d-2} ||\delta_t f||_{L^p(\mathbb{R}^d)}^p \\ &= C ||f||_{L^p(\mathbb{R}^d)}. \end{aligned} \tag{4.4}$$

Now fix $f \in L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. It is straightforward to check that the Fourier transform $\mathcal{F}$ satisfies the identity $\mathcal{F} = t^{d-2}\delta_t \circ \mathcal{F} \circ \delta_t$; since $2 \notin J$, the operator T_t has the property that

$$\widehat{T_t f}(\xi,\zeta) = -\frac{\xi_2^2 + t^2 \sum_{j\in K} \zeta_j^2}{|\xi|^2 + t^2|\zeta|^2} \widehat{f}(\xi,\zeta), \qquad (\xi,\zeta) \in \mathbb{R}^2 \times \mathbb{R}^{d-2},$$

where the set K is defined by the requirement that $k \in K$ if and only if $k+2 \in J$. By Lebesgue's dominated convergence theorem, we have

$$\lim_{t\to 0} \widehat{T_t f}(\xi,\zeta) = \widehat{T_0 f}(\xi,\zeta)$$

in $L^2(\mathbb{R}^d)$, where $\widehat{T_0 f}(\xi,\zeta) = -\xi_2^2 \widehat{f}(\xi,\zeta)/|\xi|^2$. By Plancherel's theorem, the passage to a subsequence which converges almost everywhere and Fatou's lemma, we see that (4.4) implies

$$\left|\{x \in \mathbb{R}^d : |T_0 f(x)| > 1\}\right| \leq C||f||^p_{L^p(\mathbb{R}^d)}. \tag{4.5}$$

Now pick an arbitrary function $g : \mathbb{R}^2 \to \mathbb{R}$ and define $f : \mathbb{R}^2 \times \mathbb{R}^{d-2} \to \mathbb{R}$ by $f(\xi,\zeta) = g(\xi)1_{[0,1]^{d-2}}(\zeta)$. Denoting by $\mathcal{R}_2$ the *planar* Riesz transform, we have $T_0 f(\xi,\zeta) = \mathcal{R}_2^2 g(\xi) 1_{[0,1]^{d-2}}(\zeta)$, because of the identity

$$\widehat{T_0 f}(\xi,\zeta) = -\frac{\xi_2^2}{|\xi|^2}\widehat{g}(\xi)\widehat{1_{[0,1]^{d-2}}}(\zeta).$$

Plugging this into (4.5) gives

$$|\{x \in \mathbb{R}^2 : |\mathcal{R}_2^2 g(x)| > 1\}| \leq C||g||^p_{L^p(\mathbb{R}^2)}.$$

Fix $\varepsilon > 0$ and apply the above inequality to the function $(1+\varepsilon)\cdot g$. We obtain

$$|\{x \in \mathbb{R}^2 : |\mathcal{R}_2^2 g(x)| \geq 1\}| \leq |\{x \in \mathbb{R}^2 : |\mathcal{R}_2^2 g(x)| > (1+\varepsilon)^{-1}\}| \leq C(1+\varepsilon)||g||^p_{L^p(\mathbb{R}^2)}.$$

Since ε was arbitrary, the reasoning from the previous subsection gives $C \geq C_p^p$. The proof is complete.

4.5. On the search of an appropriate laminate

Let us present here some heuristic arguments which lead to the laminate studied in §4.3. It is strictly related to the extremal example in the martingale inequality (2.1). Suppose that $d = 2$ and let us look at the estimate

$$|\{x \in \mathbb{R}^2 : |R_2^2 f(x)| \geq 1\}| \leq C_p^p \int_{\mathbb{R}^2} |f(x)|^p \mathrm{d}x$$

or, since $R_1^2 + R_2^2 = Id$,

$$|\{x \in \mathbb{R}^2 : |R_2^2 f(x)| \geq 1\}| \leq C_p^p \int_{\mathbb{R}^2} |R_1^2 f(x) + R_2^2 f(x)|^p \mathrm{d}x.$$

On the other hand, a slightly weaker form of the inequality (2.1) can be rewritten as $\mathbb{P}(|G_t| \geq 1) \leq C_p^p \mathbb{E}|F_t + G_t|^p$, or, more or less,

$$\mathbb{E}\phi_1(\text{diag}(F_t, G_t)) - C_p^p \phi_2(\text{diag}(F_t, G_t)) \leq 0, \tag{4.6}$$

where ϕ_1, ϕ_2 are as in §4.3, and the martingale G is non-symetrically differentially subordinate to $F + G$. Thus Corollary 4.4 suggests the following approach: find the extremal martingale pair (F, G) (for which the equality in (4.6) is attained, or almost attained); then the distribution of the random variable $\text{diag}(F_t, G_t)$ is the desired laminate.

The sharpness of (4.6) can be obtained by the use of the following example (the argument in [22] is slightly different and exploits the properties of the underlying boundary value problem). Fix a small number $\delta > 0$ such that

$$\delta(1+\delta)^N = (p-1)/(2c+p-1) \qquad \text{for some integer } N \tag{4.7}$$

(where $c = c(p)$ is given by (1.3)). Consider the discrete-time Markov martingale (f, g) whose transition function is uniquely determined by the following conditions:

(i) (f, g) starts from $(-\delta, \delta)$.

(ii) For $\delta \leq s < (p-1)/(2c+p-1)$, the state $(-s, s)$ leads to $\left(-s, \frac{p-3}{p-1}s\right)$ or to $(-s, s + \delta s)$.

(iii) for $\delta \leq s < (p-1)/(2c+p-1)$, the state $(-s, s+\delta s)$ leads to $(-s-\delta s, s+\delta s)$ or to $\left(\frac{p-1}{3-p}(s+\delta s), s+\delta s\right)$.

(iv) the state $(-(p-1)/(2c+p-1), (p-1)/(2c+p-1))$ leads to $(-(p-1)/(2c+p-1), 1)$ or to $(-(p-1)/(2c+p-1), (p-3)/(2c+p-1))$.

(v) all the remaining states are absorbing.

The technical assumption (4.7) guarantees that the Markov process reaches the state studied in (iv). It is not difficult to check that if we let $\delta \to 0$, then the distribution of $\text{diag}(f_\infty, g_\infty)$ is close to the laminate ν exploited in §5.4. This explains the use of this particular probability measure. See also [6] for a similar discussion.

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