TWO-WEIGHT INEQUALITIES FOR GEOMETRIC MAXIMAL OPERATORS

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Abstract. We study one- and two-weight inequalities for the geometric maximal operator on probability spaces equipped with a tree-like structure. We provide a characterization of weights, in terms of Muckenhoupt and Sawyer-type conditions, for which the appropriate strong-type estimates hold. Our approach rests on Bellman function method, which allows us to identify sharp constants involved in the estimates.

1. Introduction

Hardy-Littlewood maximal operator $\mathcal{M}$ on $\mathbb{R}^d$ is an operator acting on measurable functions $f:\mathbb{R}^d\to\mathbb{R}$ by the formula

$$\mathcal{M}f(x)=\sup\left\{\frac{1}{|Q|}\int_Q|f(u)|\mathrm{d}u\right\},\qquad x\in\mathbb{R}^d.$$

Here the supremum is taken over all cubes $Q\subset\mathbb{R}^d$ containing x, with sides parallel to the axis, and $|Q|$ is the Lebesgue measure of Q. A related object, the so-called geometric maximal operator, is given by

$$\mathcal{G}f(x)=\sup\,\exp\left\{\frac{1}{|Q|}\int_Q\log(|f(u)|)\mathrm{d}u\right\},\qquad x\in\mathbb{R}^d,$$

the supremum being taken over the same parameters as above. The purpose of this paper is to investigate sharp versions of one- and two-weight inequalities for the operator $\mathcal{G}$. Here and below, the word "weight" stands for a nonnegative and integrable function on $\mathbb{R}^d$. Let us discuss several related results from the literature. The paper [7] by Shi contains the characterization of weights w such that $\mathcal{G}$ is bounded as an operator on $L^p(w)$.

Theorem 1.1. *Given a weight w, the following conditions are equivalent.*

(i) $w\in A_\infty$: there exists a finite constant C such that for all cubes Q,

$$\left(\frac{1}{|Q|}\int_Q w\mathrm{d}x\right)\exp\left(\frac{1}{|Q|}\int_Q\log(w^{-1})\mathrm{d}x\right)\leq C.$$

(ii) For $0<p<\infty$, there is a finite C_p such that the inequality

$$||\mathcal{G}f||_{L^p(w)}\leq C_p||f||_{L^p(w)}$$

holds for all $f\in L^p(w)$.

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Here we have used the notation

$$||f||_{L^p(w)} = \left(\int_{\mathbb{R}^d} |f(x)|^p w(x) \mathrm{d}x \right)^{1/p}$$

for the usual weighted p-th norm of f. A two-weight version of the result above, involving a Sawyer-type testing condition, was established by Yin and Muckenhoupt [14] in the one-dimensional setting. The statement can be formulated as follows.

Theorem 1.2. *Given a pair of weights (w, v) on $\mathbb{R}$, the following are equivalent:*
(i) $(w, v) \in W_\infty$: there exists a constant C such that for all dyadic intervals I,

$$\int_I \mathcal{G}(v^{-1}\chi_I) w dx \leq C|I|.$$

(ii) For $0 < p < \infty$, there is a constant $C_p < \infty$ for which the estimate

$$||\mathcal{G}f||_{L^p(w)} \leq C_p ||f||_{L^p(v)}$$

holds for all $f \in L^p(v)$.

Two remarks are in order. First, the two theorems above imply that $(w, w) \in W_\infty$ if and only if $w \in A_\infty$ (possibly with different constants C on the right). Furthermore, we would also like to mention here that when studying the L^p estimate of Theorem 1.2, it is enough to consider the case $p = 1$ only: this follows directly from the identity $\mathcal{G}|f|^p = (\mathcal{G}|f|)^p$.

For further results in this direction, see e.g. Cruz-Uribe [1], Cruz-Uribe and Neugebauer [2], Ortega Salvador and Ramírez Torreblanca [6].

We will study versions of the two theorems above in the dyadic setting, putting the particular emphasis on the size of the constants involved. Let us first handle the local context. Suppose that $(0, 1]^d$ is the unit cube equipped with the family $\mathcal{D}$ of all the dyadic subcubes $Q \subseteq (0, 1]^d$. The dyadic maximal operator $\mathcal{M}_{\mathcal{D}}$ and the geometric maximal operator $\mathcal{G}_{\mathcal{D}}$ on $(0, 1]^d$ act on integrable functions $f : (0, 1]^d \to \mathbb{R}$ by the formulae

$$\mathcal{M}_{\mathcal{D}} f(x) = \sup \left\{ \frac{1}{|I|} \int_I |f(u)| \mathrm{d}u : x \in I, I \in \mathcal{D} \right\}$$

and

$$\mathcal{G}_{\mathcal{D}} f(x) = \sup \left\{ \exp \left(\frac{1}{|I|} \int_I \log(|f(u)|) \mathrm{d}u \right) : x \in I, I \in \mathcal{D} \right\}.$$

We start the analysis with the one-weight inequalities. The question is: for a given $0 < p < \infty$, characterize those weights w such that for all f,

$$||\mathcal{G}_{\mathcal{D}} f||_{L^p(w)} \leq C ||f||_{L^p(w)},$$

with C independent of f. Suppose that w has this property, fix $\varepsilon > 0$ and plug $f = (w \vee \varepsilon)^{-1/p} \chi_Q$ for a fixed $Q \in \mathcal{D}$. Here and below, we use the notation

$$a \vee b = \max\{a, b\}.$$

It is easy to compute that then

$$\mathcal{G}_{\mathcal{D}} f \geq \exp \left(\frac{1}{|Q|} \int_Q \log(w \vee \varepsilon)^{-1/p} \mathrm{d}x \right)$$

on Q and $\mathcal{G}_\mathcal{D} f=0$ on the complement of Q. Consequently, the assumed L^p estimate gives

$$C^p|Q| \geq C^p||f||^p_{L^p(w)} \geq ||\mathcal{G}_\mathcal{D} f||^p_{L^p(w)} \geq \exp\left(\frac{1}{|Q|}\int_Q \log(w\vee\varepsilon)^{-1}\mathrm{d}x\right)\cdot\int_Q w\mathrm{d}x.$$

Letting $\varepsilon\to 0$ and using the fact that Q is arbitrary, we see that w must satisfy the dyadic A_∞ condition

$$[w]_{A_\infty} := \sup_{Q\in\mathcal{D}}\left(\frac{1}{|Q|}\int_Q w\mathrm{d}x\right)\exp\left(\frac{1}{|Q|}\int_Q \log(w^{-1})\mathrm{d}x\right) < \infty.$$

This condition is also sufficient, as we will prove now. Furthermore, we will identify the best constant involved in the L^p estimate. Here is the precise statement.

Theorem 1.3. *Let w be a weight on $(0,1]^d$ satisfying the dyadic A_∞ condition. Then for any $0<p<\infty$ we have*

$$||\mathcal{G}_\mathcal{D} f||_{L^p(w)} \leq C_{p,[w]_{A_\infty}}||f||_{L^p(w)}, \tag{1.1}$$

where $C_{p,r}$ is the largest positive root of the equation

$$C^p = erp\log C. \tag{1.2}$$

The result is sharp in the sense that for any $0<p<\infty$, any $1\leq r<\infty$ and any $C<C_{p,r}$ there are a function f and a weight w with $[w]_{A_\infty}\leq r$ such that

$$||\mathcal{G}_\mathcal{D} f||_{L^p(w)} > C||f||_{L^p(w)}.$$

Actually, we will prove a more general statement involving a mixture of two weights. Consider the two-weight A_∞ condition

$$[w,v]_{A_\infty} := \sup_{Q\in\mathcal{D}}\left(\frac{1}{|Q|}\int_Q w\mathrm{d}x\right)\exp\left(\frac{1}{|Q|}\int_Q \log(v^{-1})\mathrm{d}x\right) < \infty. \tag{1.3}$$

It is well-known that (1.3) is not sufficient for the validity of the $L^p(w)\to L^p(v)$ estimate for $\mathcal{G}_\mathcal{D}$ (cf. [14]). However, we will manage to establish the following generalization of (1.1).

Theorem 1.4. *Let w, v be two weights on $(0,1]^d$. If the condition (1.3) is satisfied, then for any $0<q<p<\infty$ and any $f\in L^p(v)$ we have*

$$||\mathcal{G}_\mathcal{D} f||_{L^p(w)} \leq \left([w,v]_{A_\infty}\frac{p}{p-q}\right)^{1/q}||f||_{L^p(v^{p/q}w^{(q-p)/q})}. \tag{1.4}$$

The factor $p/(p-q)$ is the best possible: for any $0<q<p<\infty$ and any $\eta< p/(p-q)$ there are weights w, v and a function f for which

$$||\mathcal{G}_\mathcal{D} f||_{L^p(w)} > \left([w,v]_{A_\infty}\eta\right)^{1/q}||f||_{L^p(v^{p/q}w^{(q-p)/q})}.$$

We turn our attention to two-weight L^p bounds. Pick two weights w, v on $(0,1]^d$, a parameter $0<p<\infty$ and assume that there is a finite constant C such that

$$\int_{(0,1]^d}(\mathcal{G}_\mathcal{D} f)^p w\mathrm{d}x \leq C\int_{(0,1]^d}|f|^p v\mathrm{d}x$$

for all functions f on $(0,1]^d$. Testing this inequality on the functions $f=v^{-1/p}\chi_Q$, where Q is a fixed element of $\mathcal{D}$, we see that w and v must enjoy the bound

$$\int_{(0,1]^d}\mathcal{G}_\mathcal{D}(v^{-1}\chi_Q)w\mathrm{d}x = \int_{(0,1]^d}\left(\mathcal{G}_\mathcal{D}(v^{-1/p}\chi_Q)\right)^p w\mathrm{d}x \leq C|Q|,$$

or $|Q|^{-1}\int_Q \mathcal{G}_\mathcal{D}(v^{-1}\chi_Q)w\mathrm{d}x \leq C$. In particular, this implies the Sawyer-type condition (see Theorem 1.2 above)

$$S_{w,v} := \sup_{Q\in\mathcal{D}} \frac{1}{|Q|}\int_Q \mathcal{G}_\mathcal{D}(v^{-1}\chi_Q)w\mathrm{d}x < \infty. \tag{1.5}$$

We will show that this condition is sufficient for the validity of the L^p inequality, at the cost of additional multiplicative factor $e^{1/p}$. Here is the precise statement.

Theorem 1.5. *Suppose that w, v are weights on $(0,1]^d$ such that $S_{w,v} < \infty$. Then for any $0 < p < \infty$ and any f on $(0,1]^d$ we have*

$$||\mathcal{G}_\mathcal{D} f||_{L^p(w)} \leq e^{1/p} S_{w,v}^{1/p} ||f||_{L^p(v)}. \tag{1.6}$$

The constant $e^{1/p}$ is the best possible: for any $0 < p < \infty$ and any $c < e^{1/p}$ there are weights w, v satisfying (1.5) and a function f on $(0,1]^d$ such that

$$||\mathcal{G}_\mathcal{D} f||_{L^p(w)} > c S_{w,v}^{1/p} ||f||_{L^p(v)}.$$

Though the above statements are formulated in the localized setting (i.e., when the functions and weights are defined on $(0,1]^d$), a straightforward dilation argument immediately generalizes the results to the case when the underlying space is equal to $\mathbb{R}^d$ (and when the suprema defining A_∞ and Sawyer testing conditions are taken over all dyadic cubes in $\mathbb{R}^d$).

Actually, all the above results remain valid in the context of probability spaces equipped with a tree-like structure. Here is the appropriate definition.

Definition 1.6. Suppose that (X,μ) is a nonatomic probability space. A set $\mathcal{T}$ of measurable subsets of X will be called a tree if the following conditions are satisfied:

- (i) $X \in \mathcal{T}$ and for every $I \in \mathcal{T}$ we have $\mu(I) > 0$.
- (ii) For every $I \in \mathcal{T}$ there is a finite subset $Ch(I) \subset \mathcal{T}$ containing at least two elements such that
  - (a) the elements of $Ch(I)$ are pairwise disjoint subsets of I,
  - (b) $I = \bigcup_{J\in Ch(I)} J$.
- (iii) $\mathcal{T} = \bigcup_{m\geq 0} \mathcal{T}^m$, where $\mathcal{T}^0 = \{X\}$ and $T^{m+1} = \bigcup_{I\in\mathcal{T}^m} Ch(I)$.
- (iv) We have $\lim_{m\to\infty} \sup_{I\in\mathcal{T}^m} \mu(I) = 0$.

It is easy to see that the cube $(0,1]^d$ endowed with Lebesgue measure and the tree of its dyadic subcubes has the properties listed above. Any probability space equipped with a tree gives rise to the corresponding maximal operator $\mathcal{M}_\mathcal{T}$ and the geometric maximal operator $\mathcal{G}_\mathcal{T}$, given by

$$\mathcal{M}_\mathcal{T} f(x) = \sup\left\{\frac{1}{\mu(I)}\int_I |f(u)|\mathrm{d}\mu(u) : x \in I, I \in \mathcal{T}\right\}$$

and

$$\mathcal{G}_\mathcal{T} f(x) = \sup\left\{\exp\left(\frac{1}{\mu(I)}\int_I \log(|f(u)|)\mathrm{d}\mu(u)\right) : x \in I, I \in \mathcal{T}\right\}.$$

Furthermore, one easily defines the corresponding one- and two-weight A_∞ conditions, simply by requiring that appropriate suprema are taken over all $Q \in \mathcal{T}$. We will prove below that the assertions of Theorems 1.3-1.5 hold true in this more general context as well.

In a sense, the main "bulding blocks" of this paper are Theorems 1.4 and 1.5. These results will be proved with the use of the so-called Bellman function technique: roughly speaking, the estimates (1.4) and (1.6) will be deduced from the existence of certain special functions, enjoying appropriate majorization and concavity. This type of approach has gathered a lot of interest in the literature: see e.g. [5], [8], [9], [10], [11], [13] and the references therein.

The next section is devoted to the weighted estimates which follow from the assumption (1.3), i.e., to the proofs of Theorems 1.3 and 1.4. The final part of the paper studies the condition (1.5) and its consequence, Theorem 1.5.

2. Mixed-weight estimates

Fix a positive constant c, two exponents $0 < q < p$ and consider the function $b = b_{c,p,q} : \mathbb{R} \times \mathbb{R} \times [0,\infty) \times \mathbb{R} \to \mathbb{R}$, given by

$$b(x,y,r,s) = e^{py}\left(\frac{r}{p} - \frac{c}{p-q}e^{q(x-y)-s}\right).$$

Let us start the properties of this object.

Lemma 2.1. *(i) If x, r, $s \in \mathbb{R}$ satisfy $re^s \leq c$, then*

$$b(x,x,r,s) \leq 0. \tag{2.1}$$

(ii) For any x, y, r, $s \in \mathbb{R}$ we have

$$b(x,y,r,s) \geq \frac{q}{p^2}\left(e^{py}r - \left(\frac{pc}{p-q}\right)^{p/q} e^{px-ps/q}r^{1-p/q}\right). \tag{2.2}$$

Proof. (i) We have $b(x,x,r,s) = e^{px}\big(r/p - ce^{-s}/(p-q)\big) \leq re^{px}(p^{-1}-(p-q)^{-1}) \leq 0$.

(ii) The majorization is equivalent to the estimate

$$\begin{aligned}\frac{p-q}{p}\cdot\frac{r}{p} + \frac{p}{q}&\left[p^{-q/p}\frac{pc}{p-q}e^{q(x-y)-s}r^{q/p-1}\right]^{p/q}\\ &\geq \left(\frac{r}{p}\right)^{(p-q)/p}\left[p^{-q/p}\frac{pc}{p-q}e^{q(x-y)-s}r^{q/p-1}\right],\end{aligned}$$

which follows directly from Young's inequality. □

The key property of b is the following concavity. In what follows, the symbols b_x, b_y, b_r and b_s will stand for the partial derivatives of b with respect to x, y, r and s.

Lemma 2.2. *Fix $(x,y,r,s) \in \mathbb{R} \times \mathbb{R} \times [0,\infty) \times \mathbb{R}$ and h, t and $u \in \mathbb{R}$ such that $x \leq y$, $re^s \leq c$ and $(r+t)e^{s+u} \leq c$. Then we have*

$$\begin{aligned}b(x+h, y\vee(x+h), r+t, s+u) \leq b(x,y,r,s) + b_x(x,y,r,s)h&\\ + b_r(x,y,r,s)t + b_s(x,y,r,s)u&.\end{aligned} \tag{2.3}$$

Proof. Note first that if $x' > y$, then $b_y(x',y,r,s) = e^{py}(r - ce^{q(x'-y)}) \leq 0$. This implies

$$b(x+h, y\vee(x+h), r+t, s+u) \leq b(x+h, y, r+t, s+u), \tag{2.4}$$

regardless of whether $x + h \geq y$ or not. However,

$$\frac{r+t}{p}e^{py} = \frac{r}{p}e^{py} + b_r(x,y,r,s)t$$

and, using the convexity of the exponential function,

$$-\frac{c}{p-q}e^{py+q(x+h-y)-(s+u)} \le -\frac{c}{p-q}e^{py+q(x-y)-s} + b_x(x,y,r,s)h + b_s(x,y,r,s)u.$$

Adding these two facts gives

$$\begin{aligned} b(x+h,y,r+t,s+u) \le b(x,y,r,s) &+ b_x(x,y,r,s)h \\ &+ b_r(x,y,r,s)t + b_s(x,y,r,s)u, \end{aligned}$$

which combined with (2.4) yields the assertion. □

Proof of (1.4). Fix weights w, v as in the statement and a function f on X. We may assume that $f \ge 0$, replacing f with $|f|$ if necessary. Consider functional sequences $(f_n)_{n\ge 0}$, $(w_n)_{n\ge 0}$ and $(v_n)_{n\ge 0}$ given as follows. For any $n \ge 0$ and $x \in X$, set

$$f_n(x) = \frac{1}{\mu(Q_n(x))}\int_{Q_n(x)} \log f \mathrm{d}\mu, \qquad w_n(x) = \frac{1}{\mu(Q_n(x))}\int_{Q_n(x)} w \mathrm{d}\mu \tag{2.5}$$

and

$$v_n(x) = \frac{1}{\mu(Q_n(x))}\int_{Q_n(x)} \log(v^{-1}) \mathrm{d}\mu,$$

where $Q_n(x)$ is the unique element of $\mathcal{T}^n$ which contains x. Furthermore, set $g_n = \sup_{0\le k\le n} f_k$ and let $b = b_{[w,v]_{A_\infty},p,q}$ be the special function corresponding to the parameters $[w,v]_{A_\infty}$, p and q (the number $[w,v]_{A_\infty}$ is defined in (1.3)). Then, as we will prove now, the sequence

$$\left(\int_X b(f_n,g_n,w_n,v_n)\mathrm{d}\mu\right)_{n\ge 0}$$

is nonincreasing. To show this, fix $n \ge 0$, $Q \in \mathcal{T}^n$ and let Q_1, Q_2, $\ldots$, Q_m be the pairwise disjoint elements of $\mathcal{T}^{n+1}$ whose union is Q. By the definition, the functions f_n, g_n, w_n and v_n are constant on Q: let us denote the corresponding values by x, y, r and s. Similarly, f_{n+1}, g_{n+1}, w_{n+1} and w_{n+1} are constant on each Q_j, and hence there exist real numbers h_j, t_j and u_j such that

$$x + h_j = f_{n+1}|_{Q_j}, \quad r + t_j = w_{n+1}|_{Q_j}, \quad \text{and} \quad s + u_j = v_{n+1}|_{Q_j}.$$

Note that $g_{n+1}|_{Q_j} = \max\{g_n|_{Q_j}, f_{n+1}|_{Q_j}\} = y \vee (x+h_j)$. Furthermore, the parameters introduced above satisfy the following conditions. First, we have $x \le y$, since $g_n \ge f_n$ on X. Furthermore, we have $re^s \le [w,v]_{A_\infty}$ and $(r+t_j)e^{s+u_j} \le [w,v]_{A_\infty}$, which is a direct consequence of the assumption (1.3) (applied to the sets Q and Q_j). Finally, observe that the obvious identity

$$\frac{1}{\mu(Q)}\int_Q f\mathrm{d}\mu = \sum_{j=1}^m \frac{\mu(Q_j)}{\mu(Q)} \cdot \frac{1}{\mu(Q_j)}\int_{Q_j} f\mathrm{d}\mu,$$

with similar versions for w and v, implies that

$$\sum_{j=1}^m \frac{\mu(Q_j)}{\mu(Q)}h_j = 0, \quad \sum_{j=1}^m \frac{\mu(Q_j)}{\mu(Q)}t_j = 0 \quad \text{and} \quad \sum_{j=1}^m \frac{\mu(Q_j)}{\mu(Q)}u_j = 0. \tag{2.6}$$

Let us apply the estimate (2.3) to $h = h_j$, $t = t_j$ and $u = u_j$, multiply both sides by $\mu(Q_j)/\mu(Q)$ and sum over j to obtain

$$\sum_{j=1}^{m} \frac{\mu(Q_j)}{\mu(Q)} b(x + h_j, y \vee (x + h_j), r + t_j, s + u_j) \le b(x, y, r, s)$$

(here we have exploited the identities (2.6)), or, equivalently,

$$\int_Q b(f_{n+1}, g_{n+1}, w_{n+1}, v_{n+1})\mathrm{d}\mu \le \int_Q b(f_n, g_n, w_n, v_n)\mathrm{d}\mu.$$

Summing over all $Q \in \mathcal{T}^n$, we get the aforementioned monotonicity of the sequence $(\int_X b(f_n, g_n, w_n, v_n)\mathrm{d}\mu)_{n\ge 0}$. Therefore, for any n we have

$$\int_X b(f_n, g_n, w_n, v_n)\mathrm{d}\mu \le \int_X b(f_0, g_0, w_0, v_0)\mathrm{d}\mu = b(f_0, g_0, w_0, v_0) \le 0, \tag{2.7}$$

where the latter inequality follows from (2.1). Now we carry out a (partial) limiting procedure. By the very definition, the function w_n is a conditional expectation of w with respect to the σ-algebra generated by $\mathcal{T}^n$ (see (2.5)), so

$$\frac{1}{p}\int_X e^{pg_n} w_n \mathrm{d}\mu = \frac{1}{p}\int_X e^{pg_n} w \mathrm{d}\mu.$$

Furthermore, f_n, v_n are conditional expectations of w and $\log(v^{-1})$, so by Jensen's inequality,

$$-\frac{c}{p-q}\int_X e^{qf_n - v_n + (p-q)g_n}\mathrm{d}\mu \ge -\frac{c}{p-q}\int_X e^{q\log f - \log(v^{-1}) + (p-q)g_n}\mathrm{d}\mu.$$

If we add the two statements above, we get

$$\int_X b(f_n, g_n, w_n, v_n)\mathrm{d}\mu \ge \int_X b(\log f, g_n, w, \log(v^{-1}))\mathrm{d}\mu.$$

Combining this with (2.2) and (2.7) yields

$$\int_X \left[e^{pg_n} w - \left(\frac{pc}{p-q}\right)^{p/q} e^{p\log f - p\log(v^{-1})/q} w^{1-p/q} \right] \mathrm{d}\mu \le 0.$$

However, note that $g_n \uparrow \sup_{k\ge 0} f_k$ and hence $e^{pg_n} \uparrow (\mathcal{G}_\mathcal{T} f)^p$. Therefore, the claim follows from Lebesgue's monotone convergence theorem. □

The sharpness of the above estimate is a more delicate issue. We start with the following lemma, which can be found in [4].

Lemma 2.3. *For every $I \in \mathcal{T}$ and every $\alpha \in (0, 1)$ there is a subfamily $F(I) \subset \mathcal{T}$ consisting of pairwise disjoint subsets of I such that*

$$\mu\left(\bigcup_{J\in F(I)} J\right) = \sum_{J\in F(I)} \mu(J) = \alpha\mu(I).$$

We will also need the following simple geometric fact (see Figure 1 below). In what follows, we will use the following notation: for a given positive number a, set

$$\gamma_a = \{(x, y) \in \mathbb{R}^2 : xe^y = a\}.$$

Lemma 2.4. *Let $a<b$ be two positive numbers. For $(x_0,y_0)\in\gamma_a$, let ℓ_{x_0,y_0} denote the line passing through (x_0,y_0) and tangent to the curve γ_b at some point (x_1,y_1) with $x_1>x_0$. Then there is a constant $\kappa=\kappa_{a,b}\in(0,1)$ depending only on a and b such that slope of the line ℓ_{x_0,y_0} is equal to $-\kappa_{a,b}/x_0$.*

Proof. Suppose that the line $\ell_{1,\log a}$ is given by the equation $y=-\kappa x+B$. Clearly, $\kappa<1$: the slope of $\ell_{1,\log a}$ must be bigger than the slope of the line tangent to γ_a (which is equal to -1). It remains to check that for any $(x_0,y_0)\in\gamma_a$, the equation for ℓ_{x_0,y_0} is given by $y=-\frac{\kappa}{x_0}x+B-\log x_0$. $\square$

Sharpness. Fix $0<q<p$, a parameter $\varepsilon>0$ and let L be the optimal constant in the inequality

$$\int_X(\mathcal{G}_{\mathcal{T}}f)^p w\mathrm{d}\mu\le (L[w,v]_{A_\infty})^{p/q}\int_X f^pv^{p/q}w^{(q-p)/q}\mathrm{d}\mu. \tag{2.8}$$

We will construct an example showing that $L\ge\frac{p}{(p-q)(1+2\varepsilon)}$; since ε has been chosen arbitrarily, this will complete the proof. For the sake of clarity, we have decided to split the reasoning into several separate parts.

Step 1. Auxiliary parameters. It is straightforward to check that

$$\lim_{K\to p/(p-q)}\lim_{\delta\to0}\frac{\delta}{K+\delta}\left(1-\frac{Ke^{-q\delta/p}}{K+\delta}\right)^{-1}=\frac{p-q}{p}$$

and

$$\lim_{K\to p/(p-q)}\lim_{\delta\to0}\left[\frac{q(K+\delta)}{\frac{p\delta}{K+\delta}\left(1-\frac{Ke^{-q\delta/p}}{K+\delta}\right)^{-1}-pe^{q\delta/p}}+\frac{K+\delta}{\delta}\left(1-\frac{Ke^{-q\delta/p}}{K+\delta}\right)\right]=0.$$

Therefore there are $K<p/(p-q)$ and $\delta>0$ such that

$$\frac{p-q}{p}(1+\varepsilon)\le\frac{\delta}{K+\delta}\left(1-\frac{Ke^{-q\delta/p}}{K+\delta}\right)^{-1}\le\frac{p-q}{p}(1+\varepsilon) \tag{2.9}$$

and

$$\frac{q(K+\delta)}{\frac{p\delta}{K+\delta}\left(1-\frac{Ke^{-q\delta/p}}{K+\delta}\right)^{-1}-pe^{q\delta/p}}+\kappa\cdot\frac{K+\delta}{\delta}\left(1-\frac{Ke^{-q\delta/p}}{K+\delta}\right)<0. \tag{2.10}$$

Finally, let $a=e^{Kq/p}(p-q)(1+\varepsilon)/p$, $b=e^{Kq/p}(p-q)(1+2\varepsilon)/p$ and let $\kappa=\kappa_{a,b}\in(0,1)$ be the number guaranteed by the preceding lemma.

Step 2. Construction of f, w and v. First we introduce an appropriate family $(A_n)_{n\ge0}$ of subsets of X, such that each A_n is a union of at most countable collection of pairwise disjoint elements of $\mathcal{T}$ (called the *atoms* of A_n): $A_n=\bigcup_{Q\in F_n}Q$. We proceed by induction: to start, put $A_0=X$. Suppose that we have successfully defined $A_n=\bigcup_{Q\in F_n}Q$. Then, by Lemma 2.3, for each $Q\in F_n$ there is a family $F(Q)\subset\mathcal{T}$ of subsets of Q such that $\mu(\bigcup_{R\in F(Q)}R)=K\mu(Q)/(K+\delta)$; we set $F_{n+1}=\bigcup_{Q\in F_n}F(Q)$. Directly from this definition, we see that $A_0\supset A_1\supset A_2\supset\dots$ and $\mu(A_n)=(K/(K+\delta))^n$. Next, introduce $f:X\to\mathbb{R}$ by

$$f=\exp((-K+n\delta)/p)\qquad\text{on } A_n\setminus A_{n+1},\qquad n=0,\,1,\,2,\,\dots. \tag{2.11}$$

Finally, consider the weights $v=w=f^{-q}$.

Step 3. Integral properties of f. Let $k \geq 0$ and let $Q \in F_k$ be an atom of A_k. By the above construction, we have

$$\begin{aligned}\frac{1}{\mu(Q)}\int_Q \log f \mathrm{d}\mu &= \sum_{n=k}^{\infty} \frac{-K+n\delta}{p}\left[\left(\frac{K}{K+\delta}\right)^{n-k} - \left(\frac{K}{K+\delta}\right)^{n-k+1}\right] \\ &= \frac{-K+k\delta}{p} + \sum_{m=0}^{\infty}\frac{m\delta}{p}\left(\frac{K}{K+\delta}\right)^m \frac{\delta}{K+\delta} \\ &= \frac{-K+k\delta}{p} + \frac{K}{p} = \frac{k\delta}{p},\end{aligned} \tag{2.12}$$

where in the second passage we have used the substitution $m = n-k$. Consequently, we have $(\mathcal{G}_{\mathcal{T}} f)^p \geq e^{-k\delta}$ on A_k; since $f^p = e^{K-k\delta}$ on $A_k \setminus A_{k+1}$ and k was arbitrary, we conclude that

$$(\mathcal{G}_{\mathcal{T}} f)^p \geq e^K f^p \qquad \text{on } X. \tag{2.13}$$

A similar calculation shows that for any atom Q of A_k and any $r \leq p-q$ we have

$$\begin{aligned}\frac{1}{\mu(Q)}\int_Q f^r \mathrm{d}\mu &= \sum_{m=0}^{\infty} \exp\left(\frac{r(-K+(m+k)\delta)}{p}\right)\left(\frac{K}{K+\delta}\right)^m \frac{\delta}{K+\delta} \\ &= \frac{\delta}{K+\delta}\exp\left(\frac{r(-K+k\delta)}{p}\right)\sum_{m=0}^{\infty}\left(\frac{Ke^{r\delta/q}}{K+\delta}\right)^m.\end{aligned}$$

If δ is sufficiently small, then the above geometric series converges; this is due to

$$\frac{Ke^{r\delta/p}}{K+\delta} = 1 + \left(\frac{r}{p} - \frac{1}{K}\right)\delta + o(\delta) \leq 1 + \left(\frac{p-q}{p} - \frac{1}{K}\right)\delta + o(\delta) < 1,$$

since $K < p/(p-q)$, as we have assumed at the beginning. So, we obtain

$$\frac{1}{\mu(Q)}\int_Q f^r \mathrm{d}\mu = \frac{\delta}{K+\delta}\exp\left(\frac{r(-K+k\delta)}{p}\right)\left(1 - \frac{Ke^{r\delta/p}}{K+\delta}\right)^{-1} < \infty. \tag{2.14}$$

Step 4. Back to the weighted estimate. Using (2.13) and (2.14), we obtain

$$\int_X f^p v^{p/q} w^{(q-p)/q} \mathrm{d}\mu = \int_X f^{p-q}\mathrm{d}\mu \leq e^{-K}\int_X \mathcal{G}_{\mathcal{T}} f^p w \mathrm{d}\mu.$$

Plugging these facts into (2.8) and dividing throughout by the finite quantity $\int_X f^{p-q}\mathrm{d}\mu$, we see that $L[w,v]_{A_\infty} \geq e^{Kq/p}$. Now we will prove that $[w,v]_{A_\infty} \leq e^{Kq/p}(p-q)(1+2\varepsilon)/p$, which will yield the desired claim. We need to show that for any $R \in \mathcal{T}$ we have

$$\left(\frac{1}{\mu(R)}\int_R w \mathrm{d}\mu\right)\exp\left(\frac{1}{\mu(R)}\int_R \log(v^{-1})\mathrm{d}\mu\right) \leq e^{Kq/p}\frac{(p-q)(1+2\varepsilon)}{p}. \tag{2.15}$$

There is a positive integer k such that $R \subseteq A_{k-1}$ and $R \not\subseteq A_k$, and the set R splits into $R \cap A_k$ and $R \setminus A_k$. If Q is an atom of A_k contained in R, then by (2.14) applied to $r = -q$,

$$\frac{1}{\mu(Q)}\int_Q w \mathrm{d}\mu = \frac{1}{\mu(Q)}\int_Q f^{-q}\mathrm{d}\mu = \frac{\delta}{K+\delta}e^{q(K-k\delta)/p}\left(1 - \frac{Ke^{-q\delta/p}}{K+\delta}\right)^{-1}. \tag{2.16}$$

Furthermore, by (2.12),

$$\exp\left(\frac{1}{\mu(Q)}\int_Q \log(v^{-1})\mathrm{d}\mu\right)=\exp\left(\frac{q}{\mu(Q)}\int_Q \log f\mathrm{d}\mu\right)=e^{qk\delta/p}. \tag{2.17}$$

Multiply (2.16) throughout by $\mu(Q)$ and sum over all Q as above to obtain

$$x_{R\cap A_k}:=\frac{1}{\mu(R\cap A_k)}\int_{R\cap A_k} w\mathrm{d}\mu=\frac{\delta}{K+\delta}e^{q(K-k\delta)/p}\left(1-\frac{Ke^{-q\delta/p}}{K+\delta}\right)^{-1}.$$

Similarly, if we rise (2.17) to power $\mu(Q)$ and multiply over all Q as above, we get

$$e^{y_{R\cap A_k}}:=\exp\left(\frac{1}{\mu(R\cap A_k)}\int_{R\cap A_k}\log(v^{-1})\mathrm{d}\mu\right)=e^{qk\delta/p}.$$

An application of (2.9) yields

$$x_{R\cap A_k}e^{y_{R\cap A_k}}=\frac{\delta e^{Kq/p}}{K+\delta}\left(1-\frac{Ke^{-q\delta/p}}{K+\delta}\right)^{-1}\leq e^{Kq/p}\frac{(p-q)}{p}(1+\varepsilon). \tag{2.18}$$

Next, observe that since f is constant on $R\setminus A_k$, so are w and $\log(v^{-1})$, and hence

$$x_{R\setminus A_k}:=\frac{1}{\mu(R\setminus A_k)}\int_{R\setminus A_k} w\mathrm{d}\mu=w|_{R\setminus A_k}=e^{q(K-(k-1)\delta)/p}$$

and

$$e^{y_{R\setminus A_k}}:=\exp\left(\frac{1}{\mu(R\setminus A_k)}\int_{R\setminus A_k}\log(v^{-1})\mathrm{d}\mu\right)=v^{-1}|_{R\setminus A_k}=e^{-Kq/p}\cdot e^{q(k-1)\delta/p}.$$

Step 5. Application of Lemma 2.4. Observe that the point

$$(x_R,y_R):=\left(\frac{1}{\mu(R)}\int_R w\mathrm{d}\mu,\frac{1}{\mu(R)}\int_R\log(v^{-1})\mathrm{d}\mu\right)$$

lies on the line segment I joining the points $(x_{R\cap A_k},y_{R\cap A_k})$, $(x_{R\setminus A_k},y_{R\setminus A_k})$: indeed,

$$(x_R,y_R)=\frac{\mu(R\cap A_k)}{\mu(R)}(x_{R\cap A_k},y_{R\cap A_k})+\frac{\mu(R\setminus A_k)}{\mu(R)}(x_{R\setminus A_k},y_{R\setminus A_k}).$$

We will show that this line segment lies entirely under the curve γ_b, where, as above, $b=e^{Kq/p}(p-q)(1+2\varepsilon)/p$: this will immediately yield (2.15).

First, note that by (2.18), the point $(x_{R\cap A_k},y_{R\cap A_k})$ lies below the curve γ_a (where $a=e^{Kq/p}(p-q)(1+\varepsilon)/p$). Furthermore, we have $y_{R\setminus A_k}\leq y_{R\cap A_k}$. Therefore, if we take $(x_0,y_0)\in\gamma_a$ with $x_0=x_{R\cap A_k}$, then it suffices to show that the slope of the segment I is not bigger than the slope of the line ℓ_{x_0,y_0}. By Lemma 2.4, this is equivalent to saying that

$$\frac{y_{R\cap A_k}-y_{R\setminus A_k}}{x_{R\cap A_k}-x_{R\setminus A_k}}\leq-\frac{\kappa_{a,b}}{x_{R\cap A_k}},$$

which, after some lengthy but straightforward computations, reduces to (2.10). □

Remark 2.5. The proof shows that the weight $w=v$ constructed above satisfies

$$[w]_{A_\infty}\leq\exp\left(\frac{q}{p-q}\right)\frac{p-q}{p}(1+2\varepsilon),$$

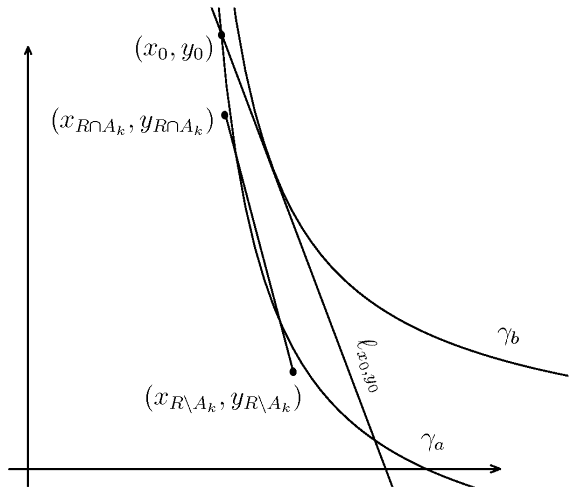

FIGURE 1. The line segment with endpoints $(x_{R\cap A_k}, y_{R\cap A_k})$, $(x_{R\setminus A_k}, y_{R\setminus A_k})$ lies below the curve γ_b

which follows from (2.15) and the assumption $K < p/(q-p)$. Increasing K if necessary, we may assume that

$$[w]_{A_\infty} \geq \exp\left(\frac{q}{p-q}\right)\frac{p-q}{p}(1-2\varepsilon).$$

This follows from (2.16), (2.17) and the left bound in (2.9): there is $Q \in \mathcal{T}$ with

$$\begin{aligned}\left(\frac{1}{\mu(Q)}\int_Q w\mathrm{d}\mu\right)\exp\left(\frac{1}{\mu(Q)}\int_Q \log(w^{-1})\mathrm{d}\mu\right) &= \frac{\delta e^{Kq/p}}{K+\delta}\left(1-\frac{Ke^{-q\delta/p}}{K+\delta}\right)^{-1}\\ &\geq \exp\left(\frac{q}{p-q}\right)\frac{p-q}{p}(1-2\varepsilon).\end{aligned}$$

The final part of this section is devoted to the proof of Theorem 1.3.

Proof of (1.1). Pick $q \in (0,p)$ and apply (1.4) to $v = w$ to obtain

$$||\mathcal{G}_{\mathcal{T}}f||_{L^p(w)} \leq \left([w]_{A_\infty}\frac{p}{p-q}\right)^{1/q}||f||_{L^p(w)}. \tag{2.19}$$

Now we optimize over q. Consider two cases. If $[w]_{A_\infty} = 1$, then we let $q \downarrow 0$ to get $||\mathcal{G}_{\mathcal{T}}f||_{L^p(w)} \leq e^{1/p}||f||_{L^p(w)} = C_{p,[w]_{A_\infty}}||f||_{L^p(w)}$, as desired. If $[w]_{A_\infty} > 1$, then $C_{p,[w]_{A_\infty}} > e^{1/p}$: indeed, if we plug $C = e$ into (1.2), then the left-hand side is smaller that the right-hand side; on the other hand, this inequality is reversed for sufficiently large C. Therefore, the number $q = p - (\log C_{p,[w]_{A_\infty}})^{-1}$ belongs to $(0,p)$, and plugging it into (2.19) yields (1.1), because

$$\begin{aligned}[w]_{A_\infty}\frac{p}{p-q} = [w]_{A_\infty}\log C^p_{p,[w]_{A_\infty}} &= C^p_{p,[w]_{A_\infty}}e^{-1}\\ &= C^q_{p,[w]_{A_\infty}}\exp\left((p-q)\log C_{p,[w]_{A_\infty}}-1\right)\\ &= C^q_{p,[w]_{A_\infty}}.\end{aligned}$$

□

Sharpness of (1.1)*, the case* $r = 1$. The required condition $[w]_{A_\infty} = 1$ forces that w is constant (by Jensen's inequality). Fix an arbitrary constant $K \in (0,1)$. We will be done if we show construct a nonnegative, μ-integrable function f for which $\int_X \mathcal{G}_\mathcal{T} f \mathrm{d}\mu \geq e^K \int_X f \mathrm{d}\mu$; this will yield the sharpness for $p = 1$, for other values of the exponent p, the function $f^{1/p}$ will do the job, since $\left[\mathcal{G}_\mathcal{T}(f^{1/p})\right]^p = \mathcal{G}_\mathcal{T} f$.

Actually, the appropriate construction has been carried out above. Fix $\delta > 0$ and let f be the function given by (2.11), with the use of parameters K, δ and $p = 1$. By (2.13), we have $\mathcal{G}_\mathcal{T} f \geq e^K f$ and hence all we need is the μ-integrability of f. To this end, we proceed as in the proof of (2.14) and compute that

$$\int_X f \mathrm{d}\mu = \frac{\delta}{K+\delta} e^{-K} \sum_{m=0}^\infty \left(\frac{Ke^\delta}{1+\delta}\right)^m,$$

which is finite for sufficiently small δ, since $K < 1$. □

Sharpness of (1.1)*, the case* $r > 1$. Fix $\varepsilon \in (0, (r-1)/2)$ and $\eta < 1$. It is easy to check that the function $\xi(x) = e^{x-1}/x$, $x \in (1,\infty)$, is strictly increasing from 1 to infinity. Therefore, there is a unique number $q = q(\varepsilon) \in (0,p)$ such that

$$\frac{r}{1+2\varepsilon} = \xi\left(\frac{p}{p-q}\right) = \exp\left(\frac{q}{p-q}\right)\frac{p-q}{p},$$

which is equivalent to $C_{p,r/(1+2\varepsilon)} = \exp(1/(p-q))$. Note that $q(\varepsilon)$ is bounded away from 0 as $\varepsilon \to 0$; furthermore, observe that for a fixed p, the function $r \mapsto C_{p,r}$ is continuous (being the inverse to the continuous and strictly increasing function $C \mapsto C^p/(ep\log C)$, $C \geq e^{1/p}$). Putting the above facts together, we see that if ε is sufficiently close to 0, then

$$\left(\frac{1-2\varepsilon}{1+2\varepsilon}\right)^{1/q} C_{p,r/(1+2\varepsilon)} > \eta C_{p,r}. \tag{2.20}$$

Fix such an ε. By Remark 2.5, decreasing ε if necessary, we can construct a weight w and a function f such that $r(1-2\varepsilon)/(1+2\varepsilon) \leq [w]_{A_\infty} \leq r$ and

$$||\mathcal{G}_\mathcal{T} f||_{L^p(w)} > \left([w]_{A_\infty}\frac{p}{(p-q)(1+2\varepsilon)}\right)^{1/q}||f||_{L^p(w)}.$$

Consequently,

$$\begin{aligned} ||\mathcal{G}_\mathcal{T} f||_{L^p(w)} &> \left(\frac{r(1-2\varepsilon)}{(1+2\varepsilon)^2}\frac{p}{p-q}\right)^{1/q}||f||_{L^p(w)} \\ &= \left(\frac{1-2\varepsilon}{1+2\varepsilon}\right)^{1/q}\exp\left(\frac{1}{p-q}\right)||f||_{L^q(w)} \\ &= \left(\frac{1-2\varepsilon}{1+2\varepsilon}\right)^{1/q} C_{p,r/(1+2\varepsilon)}||f||_{L^p(w)} \\ &> \eta C_{p,r}||f||_{L^p(w)}. \end{aligned}$$

This is precisely the desired sharpness. □

3. Proof of Theorem 1.5

The proof of the strong-type result rests on the following exponential estimate for Carleson sequences, which is of independent interest.

Theorem 3.1. *Let K be a positive constant and let α_Q, $Q \in \mathcal{T}$, be nonnegative numbers satisfying*

$$\frac{1}{\mu(R)} \sum_{Q \subseteq R} \alpha_Q \leq K \tag{3.1}$$

for all $R \in \mathcal{T}$. Then for any integrable function f on X we have

$$\sum_{Q \in \mathcal{T}} \alpha_Q \exp\left(\frac{1}{\mu(Q)} \int_Q f \mathrm{d}\mu\right) \leq Ke \int_X e^f \mathrm{d}\mu. \tag{3.2}$$

Proof. By homogeneity, we may and do assume that $K = 1$. Consider the functional sequences $(f_n)_{n\geq 0}$, $(g_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ given by

$$f_n(x) = \frac{1}{\mu(Q_n(x))} \int_{Q_n(x)} e^f \mathrm{d}\mu, \qquad g_n(x) = \frac{1}{\mu(Q_n(x))} \int_{Q_n(x)} f \mathrm{d}\mu$$

and

$$z_n(x) = \frac{1}{\mu(Q_n(x))} \sum_{Q \subseteq Q_n(x)} \alpha_Q$$

(recall that $Q_n(x)$ is the unique element of $\mathcal{T}^n$ which contains x). Introduce the function $B : \mathbb{R}^3 \to \mathbb{R}$ by the formula $B(x, y, z) = ex - e^{y-z+1}$. Clearly, this function is concave on $\mathbb{R}^3$, so for any x, y, z, h, k and ℓ we have

$$\begin{aligned} &B(x + h, y + k, z + \ell) \\ &\leq B(x, y, z) + \frac{\partial B}{\partial x}(x, y, z)h + \frac{\partial B}{\partial y}(x, y, z)k + \frac{\partial B}{\partial z}(x, y, z)\ell. \end{aligned} \tag{3.3}$$

Now we will show that the sequence $(\int_X B(f_n, g_n, z_n)\mathrm{d}\mu)_{n\geq 0}$ enjoys a certain monotonicity-type property. To this end, fix $n \geq 0$, $Q \in \mathcal{T}^n$ and pairwise disjoint elements Q_1, Q_2, …, Q_m of $\mathcal{T}^{n+1}$ whose union is Q. Put $x = f_n|_Q$, $y = g_n|_Q$ and $z = z_n|_Q$; furthermore, for any $j = 1, 2, \ldots, m$, let $x + h_j = f_{n+1}|_{Q_j}$, $y + k_j = g_{n+1}|Q_j$ and $z + \ell_j = h_{n+1}|Q_j$. Observe that z and $z + \ell_j$ belong to the interval $[0, 1]$, by the assumption of the lemma (and the equality $K = 1$ we imposed at the beginning). Furthermore, arguing as in the proof of the weak-type estimate, we see that

$$\sum_{j=1}^{m} \frac{\mu(Q_j)}{\mu(Q)} h_j = \sum_{j=1}^{m} \frac{\mu(Q_j)}{\mu(Q)} k_j = 0. \tag{3.4}$$

Finally, concerning the dynamics of the sequence $(z_n)_{n\geq 0}$, we easily check that

$$\begin{aligned} z = \frac{1}{\mu(Q)} \sum_{R \subseteq Q} \alpha_R &= \frac{\alpha_Q}{\mu(Q)} + \sum_{j=1}^{m} \frac{\mu(Q_j)}{\mu(Q)} \cdot \frac{1}{\mu(Q_j)} \sum_{R \subset Q_j} \alpha_R \\ &= \frac{\alpha_Q}{\mu(Q)} + \sum_{j=1}^{m} \frac{\mu(Q_j)}{\mu(Q)} (z + \ell_j), \end{aligned}$$

which is equivalent to

$$\sum_{j=1}^{m} \frac{\mu(Q_j)}{\mu(Q)}\ell_j = -\frac{\alpha_Q}{\mu(Q)}. \tag{3.5}$$

Let us apply (3.3), with $h = h_j$, $k = k_j$, $\ell = \ell_j$, multiply throughout by $\mu(Q_j)/\mu(Q)$ and sum the obtained estimates over j. By (3.4) and (3.5), we get

$$\sum_{j=1}^{n} \frac{\mu(Q_j)}{\mu(Q)} B(x+h_j, y+k_j, z+\ell_j) \le B(x,y,z) - \frac{\partial B}{\partial z}(x,y,z)\cdot\frac{\alpha_Q}{\mu(Q)}.$$

Since $z \le 1$, we see that $\frac{\partial B}{\partial z}(x,y,z) = e^{y-z+1} \ge e^y$ and hence the above estimate implies

$$\int_Q B(f_{n+1}, g_{n+1}, z_{n+1})\mathrm{d}\mu \le \int_Q B(f_n, g_n, z_n)\mathrm{d}\mu - \alpha_Q \exp\left(\frac{1}{\mu(Q)}\int_Q f\mathrm{d}\mu\right).$$

Summing over all $Q \in \mathcal{T}^n$ we get

$$\int_X B(f_{n+1}, g_{n+1}, z_{n+1})\mathrm{d}\mu \le \int_X B(f_n, g_n, z_n)\mathrm{d}\mu - \sum_{Q\in\mathcal{T}^n} \alpha_Q \exp\left(\frac{1}{\mu(Q)}\int_Q f\mathrm{d}\mu\right)$$

and hence for each n we have

$$\int_X B(f_{n+1}, g_{n+1}, z_{n+1})\mathrm{d}\mu \le \int_X B(f_0, g_0, z_0)\mathrm{d}\mu - \sum_{Q\in\mathcal{T}^k,\, k\le n} \alpha_Q \exp\left(\frac{1}{\mu(Q)}\int_Q f\mathrm{d}\mu\right).$$

Observe that the assumption $z_{n+1} \le 1$ and Jensen's inequality imply

$$B(f_{n+1}, g_{n+1}, z_{n+1}) \ge e f_{n+1} - e^{g_{n+1}+1} \ge 0.$$

Consequently, the preceding estimate implies

$$\sum_{Q\in\mathcal{T}^k,\, k\le n} \alpha_Q \exp\left(\frac{1}{\mu(Q)}\int_Q f\mathrm{d}\mu\right) \le B(f_0, g_0, z_0) \le e f_0 = e\int_X e^f \mathrm{d}\mu.$$

It remains to let $n \to \infty$ to complete the proof. □

Proof of Theorem 1.5. Fix f, w and v as in the statement. As previously, we may assume that $f \ge 0$ and $p = 1$, replacing f with $|f|^{1/p}$ if necessary. Consider the "truncated" geometric maximal operator $\mathcal{G}_{\mathcal{T}}^n$, given by

$$\mathcal{G}_{\mathcal{T}}^n(f)(x) = \max_{0\le k\le n} \exp\left(\frac{1}{\mu(Q_k(x))}\int_{Q_k(x)} \log f\mathrm{d}\mu\right).$$

Let us apply Theorem 3.1 to the sequence $(\alpha_Q)_{Q\in\mathcal{T}}$ defined as follows. For any $x \in X$ there is a set Q^x belonging to $\mathcal{T}^0 \cup \mathcal{T}^1 \cup \ldots \cup \mathcal{T}^n$ and containing x such that $\mathcal{G}_{\mathcal{T}}^n(f)(x) = \exp\left(\frac{1}{\mu(Q^x)}\int_{Q^x}\log|f|\mathrm{d}\mu\right)$. There might be several sets with this property; if this is the case, we choose Q^x to be the set of the largest measure. For $Q \in \mathcal{T}^0 \cup \mathcal{T}^1 \cup \ldots \cup \mathcal{T}^n$, define

$$\alpha_Q = \int_{\{x\in X:Q^x=Q\}} \exp\left(\frac{1}{\mu(Q)}\int_Q \log(v^{-1})\mathrm{d}\mu\right) w\mathrm{d}\mu$$

and for other Q's, put $\alpha_Q = 0$. This sequence satisfies the condition (3.1) with $K = S_{w,v}$ (the number $S_{w,v}$ was defined in (1.5)). Indeed, this is clear if $R \in \mathcal{T}^k$ for some $k > n$, since then the sum on the left is zero. If $R \in \mathcal{T}^k$ for some $k \le n$, then we observe that the sets $(\{x \in X : Q^x = Q\})_{Q\subseteq R}$ are pairwise disjoint and contained in R. Furthermore, we have

$$\begin{aligned}\alpha_Q &= \int_{\{x\in X:Q^x=Q\}} \exp\left(\frac{1}{\mu(Q)}\int_Q \log(v^{-1})\mathrm{d}\mu\right) w\mathrm{d}\mu \\ &\leq \int_{\{x\in X:Q^x=Q\}} \mathcal{G}_{\mathcal{T}}(v^{-1}\chi_R)w\mathrm{d}\mu\end{aligned}$$

and therefore

$$\frac{1}{\mu(R)}\sum_{Q\subseteq R}\alpha_Q \leq \frac{1}{\mu(R)}\int_R \mathcal{G}_{\mathcal{T}}(v^{-1}\chi_R)w\mathrm{d}\mu \leq S_{w,v}.$$

Consequently, the inequality (3.2) applied to the function $\log(fv)$ gives

$$\begin{aligned}\int_X \mathcal{G}^n_{\mathcal{T}}(f)w\mathrm{d}\mu &= \sum_{Q\in\mathcal{T}}\int_{\{x\in X:Q^x=Q\}} \mathcal{G}^n_{\mathcal{T}}(f)w\mathrm{d}\mu \\ &= \sum_{Q\in\mathcal{T}}\int_{\{x\in X:Q^x=Q\}} \exp\left(\frac{1}{\mu(Q)}\int_Q \log f\mathrm{d}\mu\right)w\mathrm{d}\mu \\ &= \sum_{Q\in\mathcal{T}}\alpha_Q\exp\left(\frac{1}{\mu(Q)}\int_Q \log(fv)\mathrm{d}\mu\right) \\ &\leq eS_{w,v}\int_X e^{\log(fv)}\mathrm{d}\mu = eS_{w,v}\int_X fv\mathrm{d}\mu.\end{aligned}$$

It remains to note that if we let $n\to\infty$, then $\mathcal{G}^n_{\mathcal{T}}(f)$ increases to $\mathcal{G}_{\mathcal{T}}f$. Therefore, the claim follows from Lebesgue's monotone convergence theorem. □

Sharpness of the factor $e^{1/p}$. This follows at once from the fact that the constant $e^{1/p}$ is optimal even in the unweighted context, as we have already shown above. □

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