A MAXIMAL INEQUALITY FOR NONNEGATIVE SUB- AND SUPERMARTINGALES

ADAM OSĘKOWSKI

Abstract. Let $X = (X_t)_{t\geq 0}$ be a nonnegative semimartingale and $H = (H_t)_{t\geq 0}$ be a predictable process taking values in $[-1, 1]$. Let Y denote the stochastic integral of H with respect to X. We show that

(i) If X is a supermartingale, then

$$||\sup_{t\geq 0} Y_t||_1 \leq 3||\sup_{t\geq 0} X_t||_1$$

and the constant 3 is the best possible.

(ii) If X is a submartingale satisfying $||X||_\infty \leq 1$, then

$$||\sup_{t\geq 0} Y_t||_p \leq 2\Gamma(p+1)^{1/p}, \qquad 1 \leq p < \infty.$$

The constant $2\Gamma(p+1)^{1/p}$ is the best possible.

Department of Mathematics, Informatics and Mechanics
University of Warsaw
Banacha 2, 02-097 Warsaw
Poland

email: ados@mimuw.edu.pl

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, equipped with nondecreasing right-continuous family $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-fields of $\mathcal{F}$. Assume that $\mathcal{F}_0$ contains all the events of probability 0. Suppose $X = (X_t)_{t\geq 0}$ is an adapted real-valued right-continuous semimartingale with left limits. Let Y denote the Itô integral of H with respect to X, given by

$$Y_t = H_0 X_0 + \int_{(0,t]} H_s dX_s, \quad t \geq 0.$$

Here H is a predictable process, taking values in the interval $[-1, 1]$. Let $||X||_p = \sup_{t\geq 0} ||X_t||_p$, $1 \leq p \leq \infty$, and $X^* = \sup_{t\geq 0} X_t$.

The problem of controlling the size of Y or Y^* by the size of $|X|^*$ has gained some interest in the literature. In [3], Burkholder introduced a method of proving maximal inequalities for martingales and obtained the following sharp estimate.

Theorem 1.1. *If X is a martingale and Y is as above, then we have*

$$||Y||_1 \leq \gamma|| \, |X|^*||_1, \tag{1.1}$$

2000 *Mathematics Subject Classification.* Primary: 60G42. Secondary: 60G44.
Key words and phrases. Supermartingale, stochastic integral, maximal function.
Partially supported by MEiN Grant 1 PO3A 012 29 and Foundation for Polish Science.

where $\gamma = 2,536\ldots$ is the unique solution of the equation

$$\gamma - 3 = -\exp\left(\frac{1-\gamma}{2}\right).$$

The constant is the best possible.

It was then proved by the author in [4], that if X is positive, then the optimal constant γ in (1.1) decreases to $2+(3e)^{-1} = 2,1226\ldots$. Furthermore, this inequality carries over to the case when X is assumed to be a positive supermartingale; the (best) constant remains the same. Then, in [5], the author studied a related problem, with Y replaced by its one-sided supremum; the main result of that paper can be stated as follows.

Theorem 1.2. *If X is a martingale and Y is as above, then the following inequality is sharp:*

$$||Y^*||_1 \leq \beta_0||\,|X|^*||_1, \tag{1.2}$$

where $\beta_0 = 2,0856\ldots$ is the positive solution to the equation

$$2\log\left(\frac{8}{3} - \beta_0\right) = 1 - \beta_0.$$

Furthermore, if X is nonnegative, then the best constant in (1.2) *equals $\frac{14}{9} = 1,555\ldots$.*

There is a natural question about the best constant in the inequality (1.2) in the case when X is a nonnegative supermartingale. It turns out that, in contrast with the estimate (1.1), the constant differs from the one in the setting of nonnegative martingales. Surprisingly, it differs quite much.

Theorem 1.3. *If X is a nonnegative supermartingale and Y is as above, then*

$$||Y^*||_1 \leq 3||X^*||_1 \tag{1.3}$$

and the inequality is sharp. It is already the best possible if the integrated process H takes values in $\{-1, 1\}$.

As usual, the inequalities for stochastic integrals are accompanied by its discrete-time versions. Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, filtered by $(\mathcal{F}_n)_{n\geq 0}$. Let $f = (f_n)_{n\geq 0}$ be an adapted nonnegative supermartingale and $g = (g_n)_{n\geq 0}$ be its transform by a predictable sequence $v = (v_n)_{n\geq 0}$ bounded in absolute value by 1. That is, we have

$$f_n = \sum_{k=0}^n df_k \quad \text{and} \quad g_n = \sum_{k=0}^n v_k df_k, \quad n = 0,\, 1,\, 2,\, \ldots.$$

By predictability of v we mean that v_0 is $\mathcal{F}_0$-measurable and for any $k \geq 1$, v_k is measurable with respect to $\mathcal{F}_{k-1}$. In the special case when each v_k is deterministic and takes values in $\{-1, 1\}$ we will say that g is a ± 1 transform of f. We will also use the notation $||f||_p = \sup_n ||f_n||_p$ for $1 \leq p \leq \infty$, $f_n^* = \max_{k\leq n} f_k$ and $f^* = \sup_k f_k$.

Here is the discrete-time version of Theorem 1.3.

Theorem 1.4. *Let f, g be as above. Then*

$$||g^*||_1 \leq 3||f^*||_1 \tag{1.4}$$

and the constant 3 *is the best possible. It is already the best possible for* ± 1 *transforms.*

The proofs of Theorem 1.3 and Theorem 1.4 are based on Burkholder's technique, which translates the problem of proving a given (sub-, super-)martingale inequality into the problem of finding a certain special function (for description of this method, see e.g. [3] or [4]). It turns out that the function we invent in order to deal with (1.3) and (1.4) can be used to obtain a related maximal inequality for bounded submartingales. Here is the precise statement, both in the continuous and discrete-time setting.

Theorem 1.5. *(i) If* X *is a nonnegative submartingale satisfying* $||X||_\infty \leq 1$ *and* Y *is as above, then*

$$||Y^*||_p \leq 2\Gamma(p+1)^{1/p}, \qquad 1 \leq p < \infty, \tag{1.5}$$

and the inequality is sharp. It is already the best possible if the integrated process H *takes values in* $\{-1, 1\}$.

(ii) If f *is a nonnegative submartingale satisfying* $||f||_\infty \leq 1$ *and* g *is as above, then*

$$||g^*||_p \leq 2\Gamma(p+1)^{1/p}, \qquad 1 \leq p < \infty. \tag{1.6}$$

The constant $2\Gamma(p+1)^{1/p}$ *is the best possible. It is already the best possible for* ± 1 *transforms.*

The paper is organized as follows. Section 2 contains some preliminary reductions in (1.3)–(1.6), which are needed to make Burkholder's technique applicable. In the next section we introduce the special function and study its properties. Then we provide the proofs of the announced inequalities and in the final section we show that the constants involved are the best possible.

2. Standard reductions

The first observation we make is that it suffices to deal with the discrete-time versions of the results above. This follows by approximation theorems of Bichteler [1]. Secondly, with no loss of generality we may restrict ourselves to processes f satisfying $\mathbb{P}(f_0 > 0) = 1$. Moreover, we may consider *simple* processes f only, that is, we may assume that for any integer n the random variable f_n takes only a finite number of values and there exists a number N such that $f_N = f_{N+1} = \ldots$ with probability 1. The next observation is that it suffices to prove Theorems 1.4 and 1.5 (ii) for ± 1 transforms. To see this, we present the following extension of the Lemma A.1 from [2]. The proof is identical as in the original setting and hence it is omitted.

Lemma 2.1. *Let* g *be the transform of a nonnegative supermartingale* f *(respectively, nonnegative submartingale bounded by* 1*) by a real-valued predictable sequence* v *uniformly bounded in absolute value by* 1*. Then there exist nonnegative supermartingales* $F^j = (F^j_n)_{n\geq 0}$ *(respectively, nonnegative submartingales bounded by* 1*) and Borel measurable functions* $\phi_j : [-1, 1] \to \{-1, 1\}$ *such that, for* $j \geq 1$ *and* $n \geq 0$,

$$f_n = F^j_{2n+1}, \quad f^* = (F^j)^*,$$

$$g_n = \sum_{j=1}^{\infty} 2^{-j}\phi_j(v_0)G^j_{2n+1},$$

where G^j *is the transform of* F^j *by* $\varepsilon = (\varepsilon_k)_{k\geq 0}$ *with* $\varepsilon_k = (-1)^k$.

Now assume we have established the estimate (1.4) for ± 1 transforms (the argumentation for (1.6) goes along the same lines). Lemma 2.1 gives us the processes F^j and the functions ϕ_j, $j \geq 1$. Conditionally on $\mathcal{F}_0$, the variable v_0 is deterministic and for any $j \geq 1$ the sequence $\phi_j(v_0)G^j$ is a ± 1 transform of F^j. Therefore we have the following chain of inequalities:

$$||g^*||_1 \leq \left|\left|\sum_{j=1}^{\infty} 2^{-j} \sup_n \left(\phi_j(v_0)G^j_{2n+1}\right)\right|\right|_1 \leq \sum_{j=1}^{\infty} 2^{-j}\left|\left|\left(\phi_j(v_0)G^j\right)^*\right|\right|_1$$

$$\leq 3\sum_{j=0}^{\infty} 2^{-j}||(F^j)^*||_1 = 3||f^*||_1.$$

As a result, we see that we may assume that f and g are simple. Such processes are in particular bounded; this will guarantee that the variables considered below are integrable.

The final reduction is that it suffices to prove that for any integer n we have

$$\mathbb{E}\left[g_n^* - 3f_n^*\right] \leq 0, \tag{2.1}$$

in the case of Theorem 1.4, and

$$\mathbb{E}(g_n^*)^p \leq 2^p\Gamma(p+1) \tag{2.2}$$

in the case of Theorem 1.5 (ii).

3. The special function

We start with an intermediate function $u : [0,1] \times (-\infty, 0] \to \mathbb{R}$ given by

$$u(x,y) = \begin{cases} 2x - x\log(x-y) - 3 & \text{if } x - y \leq 1, \\ 2x\exp\left[\frac{1}{2}(-x+y+1)\right] - 3 & \text{if } x - y > 1, \end{cases}$$

with the convention $0\log 0 = 0$. The main function in the paper is the function $U : [0,1] \times \mathbb{R} \times (0,\infty) \times \mathbb{R} \to \mathbb{R}$ defined by the formula

$$U(x,y,z,w) = y \vee w + (x \vee z)u\left(\frac{x}{x\vee z}, \frac{y - (y\vee w)}{x \vee z}\right), \tag{3.1}$$

where $a \vee b$ (respectively, $a \wedge b$) denotes the maximum (minimum) of a and b. It follows directly from the definition that U enjoys the following properties: for any (x,y,z,w) lying in the domain,

$$U(tx,ty,tz,tw) = tU(x,y,z,w) \qquad \text{for any } t > 0, \tag{3.2}$$

$$U(x,y,z,w) = U(x,y,x\vee z, y \vee w) \tag{3.3}$$

and, for any $a \in \mathbb{R}$,

$$U(x,y+a,z,w+a) = U(x,y,z,w) + a. \tag{3.4}$$

The lemma below is devoted to the main property of the function U. For fixed y, $w \in \mathbb{R}$ and $z > 0$, let $F_1 = F_{1,y,z,w}$, $F_{-1} = F_{-1,y,z,w}$ be functions from $\mathbb{R}_+$ to $\mathbb{R}$ given by

$$F_1(t) = U(t, y+t, z, w), \qquad F_{-1}(t) = U(t, y-t, z, w).$$

Lemma 3.1. *The functions $F_{\pm 1}$ are concave and nondecreasing on $[0,z]$. Furthermore, if $F = F_1$ or $F = F_{-1}$, then*

$$F(t) \le F(z) + F'(z-)(t-z), \qquad \text{for } t > z. \tag{3.5}$$

Remark 3.1. *In other words, the function F can be majorized by concave function Φ which coincides with F on $[0,z]$. As a consequence, we have the following property: suppose $x \in [0,z]$, $z > 0$ and $y, w \in \mathbb{R}$. Let $X \ge -x$ be a random variable with nonpositive mean. Then, by Jensen's inequality, if $F = F_1$ or $F = F_{-1}$,*

$$\mathbb{E}F(x+X) \le F(x + \mathbb{E}X) \le F(x). \tag{3.6}$$

The proof of Lemma 3.1. First, observe that we may assume $z = 1$ and $w = 0$, in view of (3.2) and (3.4). We will consider the functions F_1 and F_{-1} separately.

The function F_1. Let us start with the concavity. If $y \le -1$, then this is obvious: $F_1(t) = 2t\exp(\frac{1}{2}(y+1)) - 3$ for $t \in [0,1]$. If $y > -1$, then

$$F(t) = \begin{cases} 2t - t\log(-y) - 3 & \text{if } t \le -y, \\ y + 3t - t\log t & \text{if } t \in (-y, 1], \end{cases}$$

which is concave. Moreover, observe that $F_1'(1-) = 2[\exp(\frac{1}{2}(y+1)) \wedge 1] > 0$ which, together with the concavity just established, yields the monotonicity. To check (3.5), note that for $t > 1$,

$$F_1(t) = \begin{cases} 2t\exp(\frac{y+t}{2t}) - 3t & \text{if } t < -y, \\ y & \text{if } t \ge -y. \end{cases}$$

It is straightforward to check that F_1 is convex and $\lim_{t\to\infty} F_1'(t) = 0 < F_1'(1-)$. This gives (3.5).

The function F_{-1}. As previously, first we deal with the concavity on $[0,1]$. We have, for t lying in this interval,

$$F_{-1}(t) = \begin{cases} y + t - t\log t - 3 & \text{if } y \ge t, \\ 2t - t\log(2t - y) - 3 & \text{if } t > y \ge -1 + 2t, \\ 2t\exp(\frac{y-2t+1}{2}) - 3 & \text{if } y < -1 + 2t, \end{cases}$$

a concave function on $[0,1]$. The monotonicity is a consequence of $F_{-1}'(1-) = 0$, which can be derived directly from the formula above.

It remains to show (3.5). If $t > 1$, then

$$F_{-1}(t) = \begin{cases} y - 2t & \text{if } y > t, \\ 2t\exp(\frac{y-t}{2t}) - 3t & \text{if } y \le t, \end{cases}$$

which is a convex function satisfying $\lim_{t\to\infty} F_{-1}'(t) = 2e^{-1/2} - 3 < 0 = F_{-1}(1-)$. This proves the claim. □

We will also need the following majorization property.

Lemma 3.2. *For any (x,y,z,w) from the domain of U we have*

$$U(x,y,z,w) \ge y \vee w - 3x \vee z. \tag{3.7}$$

Proof. The properties (3.2), (3.3) and (3.4) imply that we may assume $x \leq z = 1$ and $y \leq w = 0$. Observe that we have

$$\begin{aligned} U(0,y,1,0) &= 0, \\ U(x,0,1,0) &= 2x - x\log x - 3 \geq -3, \\ U(1,y,1,0) &= 2\exp(y/2) - 3 \geq -3. \end{aligned}$$

Now it suffices to use the concavity of the functions $F_{\pm 1}|_{[0,1]}$ established in Lemma (3.1) to obtain the desired estimate. □

4. The proofs of the inequalities (1.3)

4.1. **The proof of** (2.1)**.** First we will show that the process $(U(f_n, g_n, f_n^*, g_n^*))_{n\geq 0}$ is a supermartingale. To this end, fix $n \geq 1$ and observe that, in view of (3.3),

$$\begin{aligned} \mathbb{E}(U(f_n, g_n, f_n^*, g_n^*)|\mathcal{F}_{n-1}) &= \mathbb{E}(U(f_n, g_n, f_{n-1}^* \vee f_n, g_{n-1}^* \vee g_n)|\mathcal{F}_{n-1}) \\ &= \mathbb{E}(U(f_n, g_n, f_{n-1}^*, g_{n-1}^*)|\mathcal{F}_{n-1}) \\ &= \mathbb{E}(U(f_{n-1} + df_n, g_{n-1} + \varepsilon df_n, f_{n-1}^*, g_{n-1}^*)|\mathcal{F}_{n-1}), \end{aligned}$$

where $\varepsilon \in \{-1, 1\}$. Using the functions $F_{\pm 1}$ from the previous section, we see that

$$\begin{aligned} &\mathbb{E}(U(f_{n-1} + df_n, g_{n-1} + \varepsilon df_n, f_{n-1}^*, g_{n-1}^*)|\mathcal{F}_{n-1}) \\ &= \mathbb{E}(F_{\varepsilon, g_{n-1} - \varepsilon f_{n-1}, f_{n-1}^*, g_{n-1}^*}(f_{n-1} + df_n)|\mathcal{F}_{n-1}). \end{aligned}$$

Now apply (3.6) conditionally on $\mathcal{F}_{n-1}$, with $x = f_{n-1}$ and $X = df_n$, to bound the expression from above by

$$F_{\varepsilon, g_{n-1} - \varepsilon f_{n-1}, f_{n-1}^*, g_{n-1}^*}(f_{n-1}) = U(f_{n-1}, g_{n-1}, f_{n-1}^*, g_{n-1}^*).$$

This gives the supermartingale property. Hence, applying (3.7), we may write, for any $n \geq 0$,

$$\mathbb{E}g_n^* - 3\mathbb{E}f_n^* \leq \mathbb{E}U(f_n, g_n, f_n^*, g_n^*) \leq \mathbb{E}U(f_0, g_0, f_0^*, g_0^*),$$

which is nonpositive: this is due to

$$U(f_0, g_0, f_0^*, g_0^*) = U(f_0, g_0, f_0, g_0) = g_0 + f_0 u(1,0) = g_0 - f_0 \leq 0.$$

The proof is complete.

4.2. **The proof of** (2.2) **in the case** $p = 1$**.** Let us introduce the special function $W_1 : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$W_1(x, y, w) = U(1 - x, y, 1, w) + 3.$$

It follows immediately from Lemma 3.1 and Lemma 3.2, that W_1 has the following properties:

(4.1) if $\varepsilon \in \{-1, 1\}$ and $y, w \in \mathbb{R}$, then the function $t \mapsto W_1(t, y + \varepsilon t, w)$, $t \in [0,1]$, is concave and nonincreasing.

$$W_1(x, y, w) \geq y \vee w. \tag{4.2}$$

The condition (4.1) implies that the process $(W_1(f_n, g_n, g_n^*))_{n\geq 0}$ is a supermartingale. To see this, we proceed as previously: for $n \geq 1$, there is $\varepsilon \in \{-1, 1\}$ such that we have

$$\begin{aligned} \mathbb{E}(W_1(f_n, g_n, g_n^*)|\mathcal{F}_{n-1}) &= \mathbb{E}(W_1(f_{n-1} + df_n, g_{n-1} + \varepsilon df_{n-1}, g_{n-1}^*|\mathcal{F}_{n-1}) \\ &\leq W_1(f_{n-1}, g_{n-1}, g_{n-1}^*), \end{aligned}$$

where in the last passage we have used (4.1) and Jensen's inequality. Together with (4.2) and (4.1) again, this gives

$$\mathbb{E}g_n^* \leq \mathbb{E}W_1(f_n, g_n, g_n^*) \leq \mathbb{E}W_1(f_0, g_0, g_0^*) \leq \mathbb{E}W_1(0,0,0) = 2, \tag{4.3}$$

which is the desired bound.

4.3. **The proof of** (2.2) **in the case** $p > 1$**.** As previously, we start with defining the special function. Let $W_p : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by

$$W_p(x,y,w) = p(p-1)\int_0^\infty a^{p-2} W_1(x, y-a, (w-a)\vee 0)da.$$

By (4.1), for any $\varepsilon \in \{-1,1\}$ and $y, w \in \mathbb{R}$, $a > 0$, the function

$$t \mapsto W_1(t, y-a+\varepsilon t, (w-a)\vee 0)$$

is concave and nonincreasing. This implies that $t \mapsto W_p(t, y+t, w)$ also has analogous property, which, in turn, yields that the process $(W_p(f_n, g_n, g_n^*))_{n\geq 0}$ is a supermartingale. On the other hand, by (4.2), we have

$$\begin{aligned} W_p(x,y,w) &\geq p(p-1)\int_0^\infty a^{p-2}\left[(y-a)\vee(w-a)\vee 0\right] da \\ &= p(p-1)\int_0^{y\vee w} a^{p-2}((y\vee w)-a)da = (y\vee w)^p. \end{aligned} \tag{4.4}$$

Therefore,

$$\begin{aligned} \mathbb{E}(g_n^*)^p &\leq \mathbb{E}W_p(f_n, g_n, g_n^*) \leq \mathbb{E}W_p(f_0, g_0, g_0^*) \leq W_p(0,0,0) \\ &= p(p-1)\int_0^\infty a^{p-2} W_1(0,-a,0)da = 2^p\Gamma(p+1), \end{aligned}$$

as claimed.

5. Sharpness

We will show the optimality of the constants appearing in Theorems 1.3, 1.4 and 1.5 by providing appropriate examples.

5.1. **The constant** 3 **is the best in** (1.3) **and** (1.4)**.** Clearly, it suffices to show the sharpness of (1.4). For a fixed $\delta \in (0,1)$, let $(X_n)_{n\geq 0}$ be a sequence of independent random variables such that $\mathbb{P}(X_0 = 1) = 1$ and, for $k \geq 1$,

$$\mathbb{P}(X_{2k-1} = -\delta) = 1,$$
$$\mathbb{P}(X_{2k} = \delta - 1) = \delta = 1 - \mathbb{P}(X_{2k} = \delta).$$

As $\mathbb{E}X_n \leq 0$ for $n \geq 1$, the process $S_n = X_0 + X_1 + X_2 + \ldots + X_n$, $n = 0, 1, 2, \ldots$, is a supermartingale. Let $\tau = \inf\{n : S_n = 0\}$ and observe that $\mathbb{P}(\tau < \infty) = 1$ and τ takes only even values, with

$$\mathbb{P}(\tau = 2k) = (1-\delta)^{k-1}\delta, \qquad k = 1, 2, \ldots.$$

Now define $f_n = S_{\tau\wedge n}$ and $dg_n = (-1)^n df_n$. Then f is a nonnegative supermartingale satisfying $\mathbb{P}(f^* = 1) = 1$ and g is its ± 1 transform. Note that $\mathbb{P}(dg_0 = 1) = 1$, $dg_n = \delta$ for $1 \leq n < \tau$ and $dg_\tau = \delta - 1$, which implies $g^* = g_\tau^* = g_{\tau-1} = 1 + (\tau-1)\delta$. Hence

$$\mathbb{E}g^* = \sum_{k=1}^\infty (1 + (2k-1)\delta)(1-\delta)^{k-1}\delta = 3 - \delta.$$

As δ was arbitrary, the constant 3 can not be replaced by a smaller number in (1.3) and (1.4).

5.2. **The constant** $2\Gamma(p+1)^{1/p}$ **is the best in** (1.5) **and** (1.6)**.** Again, we may restrict ourselves to the discrete-time case. Fix $\delta \in (0,1)$ and let (S_n), τ be the random variables considered above. Define

$$f_n = 1 - S_{\tau\wedge n}, \qquad dg_n = (-1)^{n+1} df_n.$$

Then f is a nonnegative submartingale bounded from above by 1 and g is its ± 1 transform. We see that $\mathbb{P}(dg_0 = 0) = 1$, $dg_n = \delta$ for $1 \leq n < \tau$ and $dg_\tau = \delta - 1$, from which it follows that $g^* = g^*_\tau = g_{\tau-1} = (\tau - 1)\delta$. Therefore

$$\begin{aligned} \mathbb{E}(g^*)^p &= \sum_{k=1}^{\infty} [(2k-1)\delta]^p (1-\delta)^{k-1}\delta \\ &= \frac{(1-\delta)^{1/2}}{2} \cdot 2\delta \sum_{k=1}^{\infty} [(2k-1)\delta]^p \left[(1-\delta)^{1/(2\delta)}\right]^{(2k-1)\delta}. \end{aligned}$$

The above expression can be easily shown to converge to

$$\frac{1}{2}\int_0^\infty s^p \exp(-s/2) ds = 2^p \Gamma(p+1),$$

as $\delta \to 0$. This proves the claim.

References

[1] K. Bichteler, *Stochastic integration and L^p-theory of semimartingales*, Ann. Probab. 9 (1980), pp. 49–89.

[2] ———, *Explorations in martingale theory and its applications*, Ecole d'Ete de Probabilits de Saint-Flour XIX—1989, pp. 1–66, Lecture Notes in Math., 1464, Springer, Berlin, 1991.

[3] ———, *Sharp norm comparison of martingale maximal functions and stochastic integrals,* Proceedings of the Norbert Wiener Centenary Congress, 1994 (East Lansing, MI, 1994), pp. 343–358, Proc. Sympos. Appl. Math., 52, Amer. Math. Soc., Providence, RI, 1997.

[4] A. Osękowski, *Sharp maximal inequality for stochastic integrals*, Proc. Amer. Math. Soc. **136** (2008), 2951–2958.

[5] ———, *Sharp maximal inequality for martingales and stochastic integrals*, Elect. Comm. in Probab., to appear.