STRONG DIFFERENTIAL SUBORDINATION AND SHARP INEQUALITIES FOR ORTHOGONAL PROCESSES

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Abstract. We introduce a strong differential α-subordination for the continuous-time processes, which generalizes this notion from the discrete time setting, due to Burkholder and Choi. Then we determine the best constants in the L^p estimates for a nonnegative submartingale and its strong α-subordinate, under an additional assumption on the orthogonality of these two processes.

1. Introduction

The purpose of this paper is to study moment inequalities for a certain class of continuous-time processes. However, in order to introduce the basic concepts and to establish some connections with related results from the literature, let us start with the discrete-time setting. Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, equipped with a filtration $(\mathcal{F}_n)$, $n = 0, 1, 2, \ldots$. Let $f = (f_n)$, $g = (g_n)$ be adapted integrable processes taking values in a separable Hilbert space $\mathcal{H}$ which, as we can and will assume from now on, is equal to ℓ_2. The scalar product in $\mathcal{H}$ will be denoted by $(\cdot, \cdot)$ and $|y|$ will stand for the norm of $y \in \mathcal{H}$. The difference sequences of f and g are defined by

$$df_0 = f_0, \quad df_n = f_n - f_{n-1}, \qquad dg_0 = g_0, \quad dg_n = g_n - g_{n-1}, \quad n = 1, 2, \ldots.$$

Following Burkholder [4], we say that g is differentially subordinate to f, if

$$|dg_n| \leq |df_n|, \qquad n = 0, 1, 2, \ldots. \tag{1.1}$$

One of the main results of [4] can be stated as follows. We use the standard notation $||f||_p = \sup_{n \geq 0} ||f_n||_p$.

Theorem 1.1. *If f, g are martingales and g is differentially subordinate to f, then for any $1 < p < \infty$ we have a sharp estimate*

$$||g||_p \leq (p^* - 1)||f||_p.$$

Here $p^* = \max\{p, p/(p-1)\}$.

By sharpness we mean that for any $\varepsilon > 0$ there exist f, g satisfying the assumptions of the theorem with $||g||_p > (p^* - 1 - \varepsilon)||f||_p$. There is a number of related estimates comparing the sizes of a martingale and its differential subordinate. A good reference is a survey [5] by Burkholder, which also contains some applications of these results to harmonic analysis. See also papers [12] by Suh and [10] by the author for some recent results in this direction.

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The above moment inequality can be extended to a wider class of processes. Following [6], we say that g is strongly differentially subordinate to f, if g is differentially subordinate to f and, in addition,

$$|\mathbb{E}(dg_n|\mathcal{F}_{n-1})| \leq |\mathbb{E}(df_n|\mathcal{F}_{n-1})|, \qquad n = 1, 2, \dots. \tag{1.2}$$

Clearly, if a martingale g is differentially subordinate to a martingale f, then it is also strongly differentially subordinate to f. However, strong differential subordination implies some interesting sharp inequalities for sub- and supermartingales. For example, the paper [6] contains the following L^p-estimate.

Theorem 1.2. *Suppose f is a nonnegative submartingale and g is strongly differentially subordinate to f. Then for any $1 < p < \infty$ we have a sharp estimate*

$$||g||_p \leq (p^{**} - 1)||f||_p.$$

Here $p^{**} = \max\{2p, p/(p-1)\}$.

A further extensions of the strong differential subordination and the moment inequalities were provided by Choi [7]. Given a nonnegative number α, we say that g is α-strongly subordinate to f if g is differentially subordinate to f and for any positive integer n we have

$$|\mathbb{E}(dg_n|\mathcal{F}_{n-1})| \leq \alpha|\mathbb{E}(df_n|\mathcal{F}_{n-1})|.$$

Here is the main result of [7].

Theorem 1.3. *Let α be a fixed nonnegative integer. Suppose f is a nonnegative submartingale and g is α-strongly differentially subordinate to f and takes values in $\mathbb{R}^\nu$, where ν is a fixed integer. Then for any $1 < p < \infty$ we have*

$$||g||_p \leq (p^*_\alpha - 1)||f||_p,$$

where $p^*_\alpha = \max\{(\alpha+1)p, p/(p-1)\}$. The inequality is sharp provided $\alpha \leq 1$.

Comparing the constants in Theorem 1.1 and 1.3 we see, that they are equal for $p \leq (\alpha+2)/(\alpha+1)$ and different for the other vaules of p. We will see below a similar behavior of the constants in the case of continuous-time orthogonal processes.

Let us now turn to the continuous-time setting. Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, filtered by a right-continuous filtration $(\mathcal{F}_t)$, $t \geq 0$. Assume in addition, that $\mathcal{F}_0$ contains all the events of probability 0. Let $X = (X_t)$, $Y = (Y_t)$ be adapted $\mathcal{H}$-valued semimartingales with right-continuous paths with left limits. As shown by Bañuelos and Wang [1] and Wang [13], the notions of differential and strong differential subordination generalize to the continuous time setting (see Section 2 below). Furthermore, there are continuous-time versions of the Theorems 1.1 and 1.2, with the same constants $p^* - 1$ and $p^{**} - 1$, respectively. Using ideas from these two papers, we will provide a continuous time extension of α-strong differential subordination and show the version of Theorem 1.3. This is done in the Section 2 below.

However, the main interest of this paper is to study the processes under an additional *orthogonality* assumption. We say that $X = (X_1, X_2, \dots)$ and $Y = (Y_1, Y_2, \dots)$ are orthogonal, if for any $i, j \geq 1$ the process $[X_i, Y_j]$ is constant. Bañuelos and Wang [1] proved the following. Here $||X||_p = \sup_{t\geq 0} ||X_t||_p$.

Theorem 1.4. *Let X and Y be two $\mathbb{R}$-valued continuous-time continuous path orthogonal martingales starting from 0 such that Y is differentially subordinate to X. Then for $1 < p < \infty$, we have*

$$||Y||_p \leq \cot(\pi/2p^*)||X||_p.$$

This inequality is sharp. Moreover, if $1 < p < 2$, then X may be taken to be $\mathcal{H}$-valued, and if $2 < p < \infty$, then Y may be taken to be $\mathcal{H}$-valued.

The best constant for X, Y both taking values in $\mathcal{H}$ is not known.

Bañuelos and Wang use this result to provide a sharp L^p bound for Riesz transforms on $\mathbb{R}^n$ (see the paper by Iwaniec and Martin [9] for an alternative proof, using the method of rotations).

In this paper we study a related problem. Let $\alpha \in [0, 1]$ be fixed and let $\theta = \theta(\alpha) \in [0, \pi/4]$ be given by $\alpha = \tan\theta$. Set $p_0 = 2 - 2\theta/\pi$ and

$$C_p = \begin{cases} \tan\frac{\pi}{2p} & \text{if } p \leq p_0, \\ \cot\frac{\pi-2\theta}{2p} & \text{if } p > p_0. \end{cases} \tag{1.3}$$

Note that C_p is continuous as a function of p. Here is our main result.

Theorem 1.5. *Let X be a nonnegative submartingale and Y be an $\mathbb{R}$-valued process, which is α-strongly subordinate to X. If X and Y are orthogonal, then, for any $1 < p < \infty$,*

$$||Y||_p \leq C_p||X||_p \tag{1.4}$$

and the constant C_p is the best possible. Furthermore, if $p \geq 2$, then Y can be taken to be $\mathcal{H}$-valued.

We do not know what is the optimal constant in the above inequality for $1 < p < 2$ and Y taking values in $\mathcal{H}$. Comparing the constants appearing in the last two theorems, we see, that they are the same for $p \leq p_0$ and different for $p > p_0$.

Now we will present the application of this result to the study of the exit times of a cone. Let ψ be a positive number and let

$$K_\psi = \{(x, y) \in \mathbb{R}^2 : |y| \geq \cot\psi \cdot |x|\}$$

denote the cone of angle 2ψ, symmetric with respect to the y-axis. Let (X, Y) be two-dimensional Brownian motion starting from some $\xi \in K_\psi$ and let $\tau = \tau_\psi$ denote the exit-time of (X, Y) from the cone K_ψ. As shown by Burkholder [3] (see also page 196 in Revuz and Yor [11]), $||\tau||_p$ is finite if and only if $\psi < \pi/4p$.

We will extend this result to a wider class of processes. Let $\psi > 0$ be fixed and (V, W) be two-dimensional Brownian motion starting from $\xi \in C_\psi$. Let A denote the local time of V at 0. For a fixed $\alpha \in [0, 1]$, define the processes X, Y by

$$X_t = V_t, \qquad Y_t = W_t + \alpha A_t, \quad \text{for } t \geq 0$$

and let $\tau = \tau_\psi$ denote the exit time of (X, Y) from the cone C_ψ. We will establish the following fact.

Theorem 1.6. *The number $||\tau||_p$ is finite if and only if $\psi < (\pi - 2\theta)/4p$.*

Let us now describe the organization of the paper. As already mentioned, in the next section we study the continuous-time differential and α-strong differential subordination of semimartingales. Section 3 is devoted to the special functions which lead to the inequality (1.4). In the last section we complete the proof of

Theorem 1.5 by showing the sharpness of this inequality; here Theorem 1.6 comes into play.

2. Continuous-time differential and α-strong differential subordination. An extension of Choi's inequality.

Let X, Y be two right-continuous semimartingales with limits from the left. Let $[X,Y] = ([X,Y]_t)$ denote the quadratic covariance process (e.g. consult Delacherie and Meyer [8]). Then, following [1] and [13], Y is differentially subordinate to X, if the process $([X,X]_t - [Y,Y]_t)$ is nondecreasing and nonnegative as a function of t. Note that any two adapted sequences f, g of integrable functions can be thought of as continuous-time semimartingales and the above condition means that the process

$$[f,f]_n - [g,g]_n = \left(\sum_{k=0}^{n}(|df_k|^2 - |dg_k|^2)\right)$$

is nonnegative and nondecreasing (as a function of n). This is equivalent to (1.1) and hence the definition above is consistent with the discrete-time differential subordination.

For any semimartingale X there exists a unique continuous local martingale part X^c of X satisfying

$$[X,X]_t = |X_0|^2 + [X^c, X^c]_t + \sum_{0<s\leq t} |\triangle X_s|^2$$

for all $t \geq 0$. Furthermore, $[X^c, X^c] = [X,X]^c$, the pathwise continuous part of $[X,X]$. Here is Lemma 1 from [13].

Lemma 2.1. *If X and Y are semimartingales, then Y is differentially subordinate to X if and only if Y^c is differentially subordinate to X^c, the inequality $|\triangle Y_t| \leq |\triangle X_t|$ holds for all $t > 0$ and $|Y_0| \leq |X_0|$.*

Let us now turn to the strong differential subordination. Let us first consider the case $\mathcal{H} = \mathbb{R}$. If X is a sub- or supermartingale, the Doob-Meyer decomposition yields the existence of a unique local martingale M starting from 0 and a unique predictable finite variation process A starting from 0 such that

$$X_t = X_0 + M_t + A_t, \qquad \text{for } t \geq 0.$$

If X is a general real-valued semimartingale, then we have analogous decomposition, however, with A no longer predictable. Furthermore, the decomposition may not be unique.

Following [13], we say that a semimartingale Y is strongly differentially subordinate to a semimartingale X, if the two conditions below hold.

(i) The process Y is differentially subordinate to X,
(ii) There exist finite variation processes A and B, such that A is in the Doob-Meyer decomposition of X, B is in the Doob-Meyer decomposition of Y and $|A|_t - |B|_t$ is a nondecreasing function of t.

Here $|A|_t$ denotes the total variation of A on $[0,t]$ and $|B|_t$ is defined in a similar manner.

Now the generalization of α-strong differential subordination is clear: for a fixed nonnegative α, we say that Y is α-strongly differentially subordinate to X (or, shorter, α-subordinate to X), if the following conditions are satisfied.

(i) The process Y is differentially subordinate to X,
(ii) There exist finite variation processes A and B, such that A is in the Doob-Meyer decomposition of X, B is in the Doob-Meyer decomposition of Y and $\alpha|A|_t - |B|_t$ is a nondecreasing function of t.

Again, as any adapted sequence of integrable functions can be thought of as a continuous semimartingale, we see that the definition above generalizes the α-strong differential subordination: for any adapted sequences f, g of integrable functions, (ii) is equivalent to (1.2), since

$$\alpha|A|_n - |B|_n = \sum_{k=1}^{n} (\alpha|\mathbb{E}(df_k|\mathcal{F}_{k-1})| - |\mathbb{E}(dg_k|\mathcal{F}_{k-1})|).$$

All the results and definitions above can be transferred to the case of $\mathcal{H}$-valued semimartingales: the Doob-Meyer decomposition follows from applying coordinatewise its version for real processes and hence the notion α-strong differential subordination make sense if X and Y take values in a separable Hilbert space.

Let us extend Choi's result, Theorem 1.3, to this new setting.

Theorem 2.2. *Let X be a nonnegative submartingale and Y be α-strongly subordinate to X. Then for any $1 < p < \infty$ we have a sharp inequality*

$$||Y||_p \le (p^*_\alpha - 1)||X||_p. \tag{2.1}$$

Proof. Assume $X = X_0 + M + A$, $Y = Y_0 + N + B$ are the Doob-Meyer decompositions for X and Y such that the conditions (i), (ii) defining the α-subordination are satisfied. Let us start with some reductions. First, note that we may assume $||X||_p < \infty$. The second condition we may impose is that Y takes values in a finite dimensional subspace of $\mathcal{H}$. Indeed, suppose that Y is $\mathcal{H}$-valued and α-strongly subordinate to X. Denote, for $\nu \ge 2$,

$$Y^\nu = (Y_1, Y_2, \ldots, Y_{\nu-1}, 0, 0, \ldots), \quad \text{where} \quad Y = (Y_1, Y_2, \ldots).$$

Then Y^ν is α-strongly subordinate to X and $Y^\nu \to Y$ with probability 1. Now the finite-dimensional version of the theorem combined with Fatou's lemma yields the general case.

The further reduction is to bound the processes uniformly away from 0. Suppose Y takes values in $\mathbb{R}^{\nu-1}$. For any positive number a, consider the processes $\bar{X} = X + a$, $\bar{Y} = (Y, a) \in \mathbb{R}^\nu$. Clearly, they satisfy $\bar{X} \ge a$, $|\bar{Y}| \ge a$ and the subordination is preserved. If we can prove (2.1) for $\bar{X}$, $\bar{Y}$, then, letting $a \to 0$, we obtain (2.1) for X, Y.

The next step is to note that the stochastic integration preserves the local martingale property and, consequently, for the function U_p defined below, we can find a nondecreasing sequence (T_n) of bounded stopping times going to ∞ almost surely such that $(U_{px}(X_-, Y_-) \cdot M)^{T_n}$ and $(U_{py}(X_-, Y_-) \cdot N)^{T_n}$ are martingales, $n = 1, 2, \ldots$. Since for any bounded stoppping time T and continuous function f, $[X^T, Y^T] = [X, Y]^T$ and $(f(X_-, Y_-) \cdot M)^T = f(X^T_-, Y^T_-) \cdot M^T$, we may assume from the beginning that $U_{px}(X_-, Y_-) \cdot M$ and $U_{py}(X_-, Y_-) \cdot N$ are martingales.

The final observation is that it suffices to fix $t \ge 0$ and establish the inequality

$$\mathbb{E}|Y_t|^p \le (p^*_\alpha - 1)^p \mathbb{E}|X_t|^p.$$

We will use the following special function $U_p : \mathbb{R}_+ \times \mathbb{R}^\nu \to \mathbb{R}$ discovered by Choi.

$$U_p(x, y) = (|y| - (p^*_\alpha - 1)x)(x + |y|)^{p-1}.$$

Furthermore, let $V_p(x,y)=|y|^p-(p_\alpha^*-1)^p x^p$. As shown in [7], we have

$$V_p \le p(1-(p_\alpha^*)^{-1})^{p-1}U_p, \tag{2.2}$$

$$U_p(x,y)\le 0 \qquad \text{if } |y|\le x, \tag{2.3}$$

$$U_{px}+\alpha|U_{py}|\le 0 \qquad \text{if } x|y|\neq 0, \tag{2.4}$$

and the further property: for $(x,y)\in(0,\infty)\times\mathbb{R}^\nu$ and $(x+h,y+k)\in(0,\infty)\times\mathbb{R}^\nu$ such that $|h|\ge|k|$ and $y+tk\neq 0$ for all $t\in\mathbb{R}$, we have

$$U_p(x+h,y+k)\le U_p(x,y)+U_{px}(x,y)h+(U_{py}(x,y),k). \tag{2.5}$$

We will also need the following decomposition: for all $x>0$, $h>-x$ and y, $h\in\mathbb{R}^\nu$ with $y\neq 0$, we have

$$U_{pxx}(x,y)h^2+2hU_{pxy}(x,y)\cdot k+(kU_{pyy}(x,y),k)=C_1+C_2+C_3,$$

where

$$\begin{aligned}
C_1&=-\,p_\alpha^*(p-1)(x+|y|)^{p-2}(h^2-|k|^2),\\
C_2&=-\,(p_\alpha^*(p-1)-p)(x+|y|)^{p-1}|y|^{-1}(|k|^2-(y',k)^2),\\
C_3&=-\,(p-1)(x+|y|)^{p-3}[(p_\alpha^*-p)|y|+(p_\alpha^*(p-1)-p)x](h+(y',k))^2.
\end{aligned}$$

Here $y'=y/|y|$ for $y\neq 0$. As X, Y are bounded away from 0, we may use Itô's formula to U_p and obtain

$$\begin{aligned}
U_p(X_t,Y_t)=U_p(X_0,Y_0)&+\int_{0+}^t U_{px}(X_{s-},Y_{s-})dX_s\\
&+\int_{0+}^t (U_{py}(X_{s-},Y_{s-}),dY_s)+\frac{1}{2}I_1+I_2,
\end{aligned}\tag{2.6}$$

where

$$\begin{aligned}
I_1=\int_{0+}^t U_{pxx}(X_{s-},Y_{s-})d[X^c,X^c]_s&+2\sum_{i=1}^\nu U_{pxy_i}(X_{s-},Y_{s-})d[X^c,Y_i^c]_s\\
&+\sum_{i=1}^\nu\sum_{j=1}^\nu U_{py_iy_j}(X_{s-},Y_{s-})d[Y_i^c,Y_j^c]_s
\end{aligned}$$

and

$$I_2=\sum_{0<s\le t}\big[U_p(X_s,Y_s)-U_p(X_{s-},Y_{s-})-U_{px}(X_{s-},Y_{s-})\Delta X_s-(U_{py}(X_{s-},Y_{s-}),\Delta Y_s)\big].$$

Setting $x=X_{s-}$, $y=Y_{s-}$, $h=\Delta X_s$ and $k=\Delta Y_s$, we see that Lemma 2.1, combined with (2.5), implies $I_2\le 0$. Furthermore, taking into consideration the formulas for C_1, C_2 and C_3, we have

$$I_1=D+E+F,$$

where

$$D = -p_\alpha^*(p-1)\int_{0+}^t (X_{s-}+|Y_{s-}|)^{p-2}d([X^c,X^c]_s-[Y^c,Y^c]_s),$$
$$E = -(p_\alpha^*(p-1)-p)\int_{0+}^t (X_{s-}+|Y_{s-}|)^{p-1}|Y_{s-}|^{-1}d([Y^c,Y^c]_s-[G,G]_s),$$
$$F = -(p-1)\int_{0+}^t (X_{s-}+|Y_{s-}|)^{p-3}[(p_\alpha^*-p)|Y_{s-}|+(p_\alpha^*(p-1)-p)X_{s-}]d[H,H]_s$$

and

$$G_t = \int_0^t (Y_{s-}/|Y_{s-}|, dY_s^c), \qquad H_t = X_t + G_t.$$

By differential subordination, D is nonpositive. By Lemma 2 in [13], the process G is differentially subordinate to Y^c, which implies $E \leq 0$. Finally, $[H,H]_t$ is nondecreasing, which yields $F \leq 0$. This gives $I_1 \leq 0$. Furthermore, by (2.4) and α-strong subordination,

$$\begin{aligned}
&\int_{0+}^t U_{px}(X_{s-},Y_{s-})dA_s + \int_{0+}^t (U_{py}(X_{s-},Y_{s-}),dB_s)\\
&\leq \int_{0+}^t U_{px}(X_{s-},Y_{s-})dA_s + \int_{0+}^t |U_{py}(X_{s-},Y_{s-})|\, d|B_s|\\
&\leq \int_{0+}^t \big(U_{px}(X_{s-},Y_{s-}) + \alpha|U_{py}(X_{s-},Y_{s-})|\big)dA_s \leq 0.
\end{aligned}$$

Combining the above estimates with (2.3) and (2.6), we obtain

$$U_p(X_t,Y_t) \leq (U_{px}(X_-,Y_-)\cdot M)_t + (U_{py}(X_-,Y_-)\cdot N)_t.$$

Now we apply (2.2) and take expectation to complete the proof of (2.1). The sharpness of this estimate follows from the fact that the constant p_α^*-1 is already the best possible in the discrete-time setting. □

3. The proof of the inequality (1.4)

Let us start with the following auxiliary fact.

Lemma 3.1. *Let X be a nonnegative submartingale and Y be an $\mathcal{H}$-valued process, which is α-subordinate to X. If X and Y are orthogonal, then Y has continuous paths.*

Proof. For any j, the real-valued processes X and Y^j are orthogonal. Hence, by Lemma 1 in [2], we have $\Delta X_t \Delta Y_t^j = 0$ for every $t \geq 0$. This gives $\Delta X_t = 0$ or $\Delta Y_t^j = 0$ for every t, and since, in view of Lemma 2.1, we have $|\Delta Y_t| \leq |\Delta X_t|$, Y can not have any nonzero jumps. □

As in Theorem 2.2, the proof of (1.4) is based on the special functions and Itô's formula. We will keep the following notation: for $(x,y) \in \mathbb{R}_+\times\mathcal{H}$ or $(x,y)\in\mathbb{R}_+\times\mathbb{R}$, we define $r = r(x,y) \geq 0$, $\phi = \phi(x,y) \in [0,\pi/2]$ by the equations

$$x = r\sin\phi, \quad |y| = r\cos\phi.$$

For $1 < p < 2$ (respectively, $p \geq 2$), let $V_p : \mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ (respectively, $V_p : \mathbb{R}_+\times\mathcal{H}\to\mathbb{R}$) be given by

$$V_p(x,y) = |y|^p - C_p^p x^p = r^p h_p(\phi) = r^p[\cos^p\phi - C_p^p \sin^p\phi], \tag{3.1}$$

where the constant C_p is defined by (1.3).

We will consider three cases separately: $1 < p < p_0$, $p_0 \leq p < 2$ and $2 \leq p < \infty$. In all these cases, the argumentation is split into two parts: firstly, we introduce the special functions U_p and study their properties, secondly, we present the proof of the inequality (1.4).

3.1. **The case $1 < p < p_0$.** Let us consider the function $U_p : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, given by

$$U_p(x,y) = r^p g_p(\phi) = r^p \cdot \left[-\sin\frac{\pi}{2p}\cos^{p-1}\frac{\pi}{2p}\cos\Big(p(\frac{\pi}{2}-\phi)\Big)\right].$$

A similar function was used by Bañuelos and Wang [1] in the proof of Theorem 1.4 above.

Lemma 3.2. *The functions U_p, V_p have the following properties.*

(i) The function U_p is of class C^2 on the set $\{(x,y) : r > 0,\ 0 < \phi \leq \frac{\pi}{2}\}$.

(ii) We have $U_p(x,y) \leq 0$ if $|y| \leq x$.

(iii) We have $U_p \geq V_p$.

(iv) For any (x,y) with $\phi \neq 0$ and any h, $k \in \mathbb{R}$,

$$U_{pxx}(x,y)h^2 + U_{pyy}(x,y)k^2 = U_{pxx}(x,y)(h^2-k^2). \tag{3.2}$$

(v) The inequality $U_{pxx} \leq 0$ holds on the set $\{(x,y) : 0 < \phi \leq \frac{\pi}{2}\}$.

(vi) We have $U_{px} + \alpha|U_{py}| \leq 0$ on $\{(x,y) : 0 < \phi \leq \pi/2\}$.

Proof. (i) This follows from straightforward computations.

(ii) If $|y| \leq x$, then $\phi \geq \pi/4$ and $0 \leq p(\pi/2 - \phi)) \leq p\pi/4 \leq \pi/2$, which implies $\cos(p(\pi/2-\phi) \geq 0$ and $U_p(x,y) \leq 0$.

(iii) We must show that $g_p \geq h_p$. This is equivalent to the inequality (2.1) in [1].

(iv) It suffices to show that the function U_p is harmonic on the set $\{(x,y) : 0 < \phi < \frac{\pi}{2}\}$. We use the form of laplacian in the polar coordinates

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

and we get the claim after simple calculations.

(v) Taking into account the equalities $r_x = \sin\phi$, $\phi_x = \cos\phi/r$, we obtain

$$U_{pxx}(x,y) = p(p-1)\sin\frac{\pi}{2p}\cos^{p-1}\frac{\pi}{2p} r^{p-2}\cos\Big(\frac{\pi}{2}p - (p-2)\phi\Big) \leq 0,$$

as $\pi/2 \leq \frac{\pi}{2}p - (p-2)\phi \leq \pi$.

(vi) Since $r_y = \cos\phi$ and $\phi_y = -\sin\phi/r$, we compute that $U_{px}(x,y)+\alpha|U_{py}(x,y)|$ equals

$$pr^{p-1}\cdot\sin\frac{\pi}{2p}\cos^{p-1}\frac{\pi}{2p}\cdot\Big[-\sin(\frac{\pi}{2}p-(p-1)\phi) + \tan\theta\,|\cos(\frac{\pi}{2}p-(p-1)\phi)|\Big].$$

But $\pi/2 \leq \frac{\pi}{2}p - (p-1)\phi \leq \pi$, so we have

$$\tilde{U}_{px}(x,y) + \alpha|\tilde{U}_{py}(x,y)| = -pr^{p-1}\cdot\frac{\sin\frac{\pi}{2p}\cos^{p-1}\frac{\pi}{2p}}{\cos\theta}\sin\Big(\frac{\pi}{2}p-(p-1)\phi+\theta\Big).$$

It suffices to note that

$$0 \leq \frac{\pi}{2}p - (p-1)\phi + \theta \leq \pi,$$

the left inequality being trivial, the right one being equivalent to $p \leq p_0$. □

The proof of the inequality (1.4) *for* $1 < p < p_0$. In this case, we will present all the details. As in the remaining cases $p_0 \le p < 2$ and $2 \le p < \infty$ the proofs are similar, we will only indicate the necessary modifications of the arguments used here below.

First, we start from reductions analogous to those in the beginning of the proof of Theorem 2.2. That is, we may assume $||X||_p < \infty$ and, for a fixed $a > 0$, we set $\bar{X} = a + X$. Obviously, Y is α-subordinate to $\bar{X}$ and these two processes are orthogonal. Let M, N denote the local martingale parts of $\bar{X}$ and Y, respectively. Using a stopping time argument, we may assume that the stochastic integrals $(U_{px}(\bar{X}_-, Y_-) \cdot M)$ and $(U_{py}(\bar{X}_-, Y_-) \cdot N)$ are martingales. Finally, we fix $t \ge 0$ and observe it suffices to establish the inequality $\mathbb{E}V_p(X_t, Y_t) \le 0$.

By the property (i) in Lemma 3.2, we may apply Itô's formula to U_p to obtain

$$\begin{aligned} U_p(\bar{X}_t, Y_t) = U_p(\bar{X}_0, Y_0) &+ \int_{0+}^t U_{px}(\bar{X}_{s-}, Y_s) dX_s \\ &+ \int_{0+}^t U_{py}(\bar{X}_{s-}, Y_s) dY_s + \frac{1}{2} I_1 + \frac{1}{2} I_2 + I_3, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} I_1 &= \int_{0+}^t U_{pxx}(\bar{X}_{s-}, Y_s) d[\bar{X}^c, \bar{X}^c]_s + U_{pyy}(\bar{X}_{s-}, Y_s) d[Y, Y]_s, \\ I_2 &= 2 \int_{0+}^t U_{pxy}(\bar{X}_{s-}, Y_s) d[\bar{X}^c, Y]_s, \\ I_3 &= \sum_{0 < s \le t} \big[U_p(\bar{X}_s, Y_s) - U_p(\bar{X}_{s-}, Y_s) - U_{px}(\bar{X}_{s-}, Y_s) \Delta \bar{X}_s \big]. \end{aligned} \tag{3.4}$$

Note that above we have used the fact $Y_- = Y$, guaranteed by Lemma 3.1.

By orthogonality, the summand I_2 vanishes, as $[X^c, Y] = [X, Y]$ is constant. The part (v) of Lemma 3.2 implies that I_3 is nonpositive, while the part (iv) gives

$$\begin{aligned} &\int_{0+}^t U_{pxx}(\bar{X}_{s-}, Y_s) d[\bar{X}^c, \bar{X}^c]_s + U_{pyy}(\bar{X}_{s-}, Y_s) d[Y, Y]_s \\ &\qquad = \int_{0+}^t U_{pxx}(\bar{X}_{s-}, Y_s) d([\bar{X}_{s-}, \bar{X}_{s-}] - [Y_s, Y_s]), \end{aligned}$$

which is nonpositive due to α-subordination and the inequality $U_{pxx} \le 0$. Furthermore, using Lemma 3.2 (vi) and α-subordination, we get

$$\begin{aligned} &\int_{0+}^t \bar{U}_{px}(\bar{X}_{s-}, Y_s) dA_s + \int_{0+}^t \bar{U}_{py}(\bar{X}_{s-}, Y_s) dB_s \\ &\quad \le \int_{0+}^t \big(\bar{U}_{px}(\bar{X}_{s-}, Y_s) + \alpha |\bar{U}_{py}(\bar{X}_{s-}, Y_s)| \big) \, dA_s \le 0. \end{aligned}$$

Finally, by Lemma 3.2 (ii) and α-subordination, $U_p(X_0, Y_0) \le 0$. Combining all the above estimates with (3.3) and Lemma 3.2 (iii), we get

$$V_p(\bar{X}_t, Y_t) \le \int_{0+}^t U_{px}(\bar{X}_{s-}, Y_s) dM_s + \int_{0+}^t U_{py}(\bar{X}_{s-}, Y_s) dN_s$$

and take expectation to complete the proof. □

3.2. **The case $p_0 \le p < 2$.** Let $\psi_p = \frac{\pi}{2} - \frac{\pi}{2p}$ and $\phi_p = \frac{\pi - 2\theta}{2p}$. Note that $\psi_p \ge \phi_p$; this inequality is equivalent to $p \ge p_0$. Introduce the function $U_p : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ by

$$U_p(x, y) = r^p g_p(\phi), \tag{3.5}$$

where $g_p(\phi)$ equals

$$\begin{cases} \cos^{p-1}\phi_p\,(\sin\phi_p)^{-1}\cos(p\phi + \theta) & \text{if } 0 \le \phi \le \phi_p, \\ h_p(\phi) & \text{if } \phi_p < \phi \le \psi_p, \\ -\frac{\sin^{p-1}(\pi/2p)}{\cos(\pi/2p)}\cos\left(p\left(\frac{\pi}{2} - \phi\right)\right) + (\cot^p\psi_p - \cot^p\phi_p)\sin^p\phi & \text{if } \psi_p < \phi \le \frac{\pi}{2}. \end{cases}$$

Here is the analogue of Lemma 3.2.

Lemma 3.3. *The functions U_p, V_p have the following properties.*

(i) The function U_p is of class C^1 on $\{(x, y) : 0 < \phi \le \frac{\pi}{2}\}$.

(ii) We have $U_p(x, y) \le 0$ if $|y| \le x$.

(iii) We have $U_p \ge V_p$.

(iv) For any (x, y) with $\phi \notin \{0, \phi_p, \psi_p, \pi/2\}$ and any $h \in \mathbb{R}$, $k \in \mathbb{R}$,

$$U_{pxx}(x, y)h^2 + U_{pyy}(x, y)k^2 \le U_{pxx}(x, y)(h^2 - |k|^2). \tag{3.6}$$

(v) The inequality $U_{pxx} \le 0$ holds on $\{(x, y) : \phi \notin \{0, \phi_p, \psi_p, \pi/2\}\}$.

(vi) For any (x, y) with $\phi \notin \{0, \pi/2\}$, we have $U_{px} + \alpha|U_{py}| \le 0$. Furthermore, $U_{px}(0+, y) + \alpha|U_{py}(0+, y)| = 0$.

Proof. (i) We omit the standard calculations.

(ii) Let $|y| \le x$. Note that then we have $\psi_p \le \pi/4 \le \phi \le \pi/2$,

$$0 < p\left(\frac{\pi}{2} - \phi\right) \le \frac{\pi}{4}p < \frac{\pi}{2} \quad \text{and} \quad \cot^p\psi_p - \cot^p\phi_p \le 0. \tag{3.7}$$

This gives the desired estimate.

(iii) First we will establish the majorization on the set $\{(x, y) : 0 < \phi \le \phi_p\}$. We must show that $g_p \ge h_p$, or, in an equivalent form,

$$F_p(\phi) := \frac{\cos(p\phi + \theta)}{\sin^p\phi} - \frac{\sin\phi_p}{\cos^{p-1}\phi_p}\cot^p\phi + \frac{\cos\phi_p}{\sin^{p-1}\phi_p} \ge 0. \tag{3.8}$$

We have

$$F_p'(\phi) = -\frac{p\cos^{p-1}\phi}{\sin^{p-1}\phi}\left[\frac{\cos((p-1)\phi + \theta)}{\cos^{p-1}\phi} - \frac{\sin\phi_p}{\cos^{p-1}\phi_p}\right].$$

Denoting the expression in the square brackets by $G_p(\phi)$, we have

$$G_p'(\phi) = -\frac{p-1}{\cos^p\phi}\sin((p-2)\phi + \theta) \le 0.$$

The latter inequality holds since

$$\frac{\pi}{2} \ge (p-2)\phi + \theta \ge (p-2)\phi_p + \theta = \frac{\pi}{2p}\left(p - 2 + \frac{4\theta}{\pi}\right) \ge 0. \tag{3.9}$$

Therefore, as $G_p(\phi_p) = 0$, we have $G_p(\phi) \ge 0$ for $0 < \phi \le \phi_p$ and hence $F_p'(\phi) \le 0$ for those ϕ's. Now note that $F_p(\phi_p) = 0$ to obtain (3.8).

As $U_p = V_p$ for $\phi_p < \phi < \psi_p$, all that is left is to establish the inequality $U_p \ge V_p$ for $\psi_p \le \phi < \frac{\pi}{2}$. In an equivalent form, it reads

$$\cos^p\phi - \tan^p\frac{\pi}{2p}\sin^p\phi \le -\frac{\sin^{p-1}(\pi/2p)}{\cos(\pi/2p)}\cos\left(p\left(\frac{\pi}{2} - \phi\right)\right),$$

or

$$\hat{F}_p(\phi) := -\frac{\sin^{p-1}(\pi/2p)}{\cos(\pi/2p)} \cdot \frac{\cos\left(p\left(\frac{\pi}{2}-\phi\right)\right)}{\sin^p \phi} - \cot^p \phi + \tan^p \frac{\pi}{2p} \geq 0. \tag{3.10}$$

We have

$$\hat{F}_p'(\phi) = \frac{\sin^{p-1}(\pi/2p)}{\cos(\pi/2p)} \cdot \frac{p\cos^{p-1}\phi}{\sin^{p+1}\phi}\left[\frac{\cos((p-1)\phi - \frac{\pi}{2}p)}{\cos^{p-1}\phi} + \frac{\cos(\pi/2p)}{\sin^{p-1}(\pi/2p)}\right].$$

Denote the expression in the square brackets by $\hat{G}_p(\phi)$ and calculate its derivative. We obtain

$$\hat{G}_p'(\phi) = -\frac{p-1}{\cos^p \phi} \cdot \sin[(p-2)\phi - \frac{\pi}{2}p] \geq 0,$$

because of the inequality

$$-\frac{\pi}{2} > (p-2)\phi - \frac{\pi}{2}p \geq (p-2)\frac{\pi}{2} - \frac{\pi}{2}p = -\pi. \tag{3.11}$$

Now since $\hat{G}_p(\psi_p) = 0$, we obtain $\hat{G}_p(\phi) \geq 0$ for $\psi_p \leq \phi \leq \pi/2$ and $\hat{F}_p'(\phi) \geq 0$ for those ϕ's. It suffices to note that $\hat{F}_p(\psi_p) = 0$ to conclude that (3.10) is valid.

(iv) We will show that the function U_p is harmonic on the set $\{(x,y) : 0 < \phi < \phi_p\}$ and superharmonic on $\{(x,y) : \phi_p < \phi < \frac{\pi}{2},\ \phi \neq \psi_p\}$. This clearly gives (3.6). The first part of the statement can be verified easily using the form of the laplacian in the polar coordinates. For the second one, suppose first, that $\phi_p < \phi < \psi_p$. We have

$$\begin{aligned}\Delta U_p(x,y) &= p(p-1)r^{p-2}\sin^{p-2}\phi[\cot^{p-2}\phi - \cot^p \phi_p] \\ &\leq p(p-1)r^{p-2}\sin^{p-2}\phi[\cot^{p-2}\psi_p - \cot^p \phi_p] \leq 0.\end{aligned}$$

The latter inequality holds since

$$\cot^p \phi_p \geq \cot^p \psi_p \geq \cot^{p-2}\psi_p,$$

where in the last passage we have used the inequality $\psi_p \leq \pi/4$.

Therefore, all that is left is to check the superharmonicity of U_p on the set $\{(x,y) : \psi_p \leq \phi \leq \pi/2\}$. But

$$\Delta U_p(x,y) = p(p-1)r^{p-2}(\cot^p \psi_p - \cot^p \phi_p)\sin^{p-2}\phi \leq 0$$

and the claim follows.

(v) On the set $\{(x,y) : \phi \in (0,\phi_p)\}$, we have

$$U_{pxx}(x,y) = -p(p-1)r^{p-2}\cos^{p-1}\phi_p\,(\sin\phi_p)^{-1}\cos((p-2)\phi+\theta) \leq 0, \tag{3.12}$$

which is a consequence of the inequalities $(p-2)\phi + \theta \leq \theta$ and

$$(p-2)\phi + \theta \geq (p-2)\phi_p + \theta = \frac{\pi}{2} - \frac{\pi}{p} + \frac{2\theta}{p} \geq -\frac{\pi}{2}.$$

On the set $\{(x,y) : \phi \in (\phi_p, \psi_p)\}$ we check that

$$U_{pxx}(x,y) = -p(p-1)\cot^p \phi_p x^{p-2} \leq 0,$$

while for $\psi_p < \phi < \pi/2$,

$$\begin{aligned}U_{pxx}(x,y) = &-p(p-1)\frac{\sin^{p-1}(\pi/2p)}{\cos(\pi/2p)}r^{p-2}\cos(\frac{\pi}{2}p - (p-2)\phi) \\ &+ p(p-1)(\cot^p \psi_p - \cot^p \phi_p)x^{p-2} \leq 0.\end{aligned}$$

The latter inequality holds since both summands are nonnegative: this is due to the second inequality in (3.7) and (3.11).

(vi) Let us first consider the case $0 < \phi \leq \phi_p$. We have

$$\frac{\sin\phi_p}{\cos^{p-1}\phi_p}[U_{px}(x,y)+\alpha|U_{py}(x,y)|]$$
$$= pr^{p-1}[-\sin((p-1)\phi+\theta)+\tan\theta|\cos((p-1)\phi+\theta)|].$$

Since $0 < (p-1)\phi+\theta < (p-1)\phi_p+\theta = \pi/2+\theta/p-\pi/2p < \pi/2$, the cosinus inside the absolute value is positive and the expression in the square brackets equals $-\sin((p-1)\phi)/\cos\theta \leq 0$, with equality in the limit case $\phi = 0$.

If $\phi_p < \phi \leq \psi_p$, then

$$\begin{aligned} U_{px}(x,y)+\alpha|U_{py}(x,y)| &= -p\cot^p\phi_p x^{p-1}+\alpha p|y|^{p-1} \\ &= pr^{p-1}\sin^{p-1}\phi(-\cot^p\phi_p+\tan\theta\cot^{p-1}\phi) \\ &\leq pr^{p-1}\sin^{p-1}\phi(-\cot^p\phi_p+\tan\theta\cot^{p-1}\phi_p) \leq 0, \end{aligned}$$

as just proved above. Finally, if $\psi_p < \phi < \pi/2$, then

$$\begin{aligned} U_{px}(x,y)+\alpha|U_{py}(x,y)| =& \frac{pr^{p-1}}{\cos\theta}\Big[-\frac{\sin^{p-1}\frac{\pi}{2p}}{\cos\frac{\pi}{2p}}\sin\Big(\frac{\pi}{2}p+\phi(1-p)+\theta\Big) \\ &+(\cot^p\psi_p-\cot^p\phi_p)\sin^{p-1}\phi\cos\theta\Big] \leq 0. \end{aligned}$$

Here we have used the fact that both summands in the square brackets are nonpositive. This follows from the second inequality in (3.7) and

$$0 < \frac{\pi}{2}p+\phi(1-p)+\theta \leq \frac{\pi}{2}p+\psi_p(1-p)+\theta = \pi+(\theta-\frac{\pi}{2p}) \leq \pi. \quad \Box$$

The proof of the inequality (1.4) *for* $p_0 \leq p < 2$. As previously, assume $||X||_p < \infty$ and bound the process X from 0 by considering $\bar{X} = 2a+X$ for some fixed $a > 0$. However, we can not use Itô's formula to the function U_p, since it is no longer in C^2. We need a smoothing argument to overcome this problem. Let $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a C^∞ nonnegative function with support inside the unit ball and integral 1. Let

$$\bar{U}_p(x,y) = \int_{\mathbb{R}}\int_{\mathbb{R}} U_p(x-ua, y-va)g(u,v)dvdu, \qquad x \geq a,\ y \in \mathbb{R}.$$

The function $\bar{U}_p$ is of class C^∞ on $\{(x,y) \in \mathbb{R}\times\mathbb{R} : x > a\}$. Clearly, by Lemma 3.3, it has the following properties: for all x, y such that $|y| \leq x$,

$$\bar{U}_p(x,y) \leq 0.$$

Furthermore, for all $x \geq 2a$, y, h, $k \in \mathbb{R}$,

$$\bar{U}_{pxx}(x,y)h^2+\bar{U}_{pyy}(x,y)k^2 \leq \bar{U}_{pxx}(x,y)(h^2-|k|^2).$$

Finally, for all $x \geq 2a$ and $y \in \mathbb{R}$,

$$\bar{U}_{pxx}(x,y) \leq 0, \qquad \bar{U}_{px}(x,y)+\alpha|\bar{U}_{py}(x,y)| \leq 0$$

and

$$\bar{U}_p(x,y) \geq \int_{\mathbb{R}}\int_{\mathbb{R}} V_p(x-ua,y-va)g(u,v)dvdu \geq (|y|-a)_+^p-\cot^p\phi_p|x+a|^p. \tag{3.13}$$

Denoting by M, N the local martingale parts of $\bar{X}$ and Y, respectively, we may assume that the stochastic integrals $(\bar{U}_{px}(\bar{X}_-,Y_-)\cdot M)$ and $(\bar{U}_{py}(\bar{X}_-,Y_-)\cdot N)$

are martingales. Now we fix $t \geq 0$, repeat the arguments from the proof of the inequality (1.4) in the case $1 < p < p_0$ and arrive at

$$\bar{U}_p(\bar{X}_t, Y_t) \leq \int_{0+}^t \bar{U}_{px}(\bar{X}_{s-}, Y_s) dM_s + \int_{0+}^t \bar{U}_{py}(\bar{X}_{s-}, Y_s) dN_s$$

Finally, we apply (3.13), take expectation and let $a \to 0$ to complete the proof. □

3.3. **The case $2 \leq p < \infty$.** Introduce the function $U_p : \mathbb{R}_+ \times \mathcal{H} \to \mathbb{R}$ by

$$U_p(x, y) = r^p g_p(\phi), \tag{3.14}$$

where

$$g_p(\phi) = \begin{cases} \cos^{p-1}\phi_p\,(\sin\phi_p)^{-1}\cos(p\phi + \theta) & \text{if } 0 \leq \phi \leq \phi_p, \\ h_p(\phi) & \text{if } \phi_p < \phi \leq \frac{\pi}{2}. \end{cases}$$

Here, as in the previous case, $\phi_p = \frac{\pi - 2\theta}{2p}$. Let $\tilde{U}_p$ denote the function U_p in the special case $\mathcal{H} = \mathbb{R}$.

The analogue of the Lemmas 3.2 and 3.3 can be stated as follows.

Lemma 3.4. *The functions U_p, $\tilde{U}_p$ and V_p have the following properties.*

(i) The function U_p is of class C^1 on the set $\{(x, y) : r > 0,\ 0 < \phi \leq \pi/2\}$.

(ii) We have $U_p(x, y) \leq 0$ if $|y| \leq x$.

(iii) We have $U_p \geq V_p$.

(iv) The function $\tilde{U}_p$ is harmonic on the set $\{(x, y) : 0 < \phi < \phi_p\}$ and superharmonic on $\{(x, y) : \phi_p < \phi < \frac{\pi}{2}\}$.

(v) We have $U_{pxx} \leq 0$ on $\{(x, y) : \phi \notin \{0, \phi_p, \pi/2\}\}$.

(vi) For any (x, y) with $\phi \notin \{0, \pi/2\}$, we have $U_{px}(x, y) + \alpha|U_{py}(x, y)| \leq 0$ and $U_{px}(0+, y) + |U_{py}(0+, y)| = 0$.

(vii) For any (x, y) with $\phi \notin \{0, \phi_p, \pi/2\}$ and any $h \in \mathbb{R}$, $k \in \mathcal{H}$ we have

$$U_{pxx}(x, y)h^2 + (kU_{pyy}(x, y), k) \leq \tilde{U}_{pxx}(x, |y|)(h^2 - |k|^2).$$

Proof. The properties (i) - (vi) can be established using analogous arguments as in the proof of Lemma 3.3. We will only prove (vii). We have

$$U_{pxx}(x, y)h^2 = \tilde{U}_{pxx}(x, |y|)h^2$$

and

$$\begin{aligned} (kU_{pyy}, k) =& \tilde{U}_{pyy}(x, |y|)(y', k)^2 + \tilde{U}_{py}(x, |y|)\left(\frac{|k|^2 - (y', k)^2}{|y|}\right) \\ =& \tilde{U}_{pyy}(x, |y|)|k|^2 + \left(\frac{\tilde{U}_{py}(x, |y|)}{|y|} - \tilde{U}_{pyy}(x, |y|)\right)(|k|^2 - (y', k)^2), \end{aligned}$$

where $y' = y/|y|$. By (iv), $\tilde{U}_{pyy}(x, |y|)|k|^2 \leq -\tilde{U}_{pxx}(x, |y|)|k|^2$ and hence all we need is the estimate

$$\frac{\tilde{U}_{py}(x, y)}{|y|} - \tilde{U}_{pyy}(x, |y|) \leq 0.$$

On the set $\{(x, y) : \phi > \phi_p\}$ it takes form

$$p|y|^{p-2} - p(p-1)|y|^{p-2} \leq 0,$$

which is obvious. If $\phi < \phi_p$, the inequality is equivalent to

$$(p-1)\cos((p-2)\phi + \theta)\cos\phi \geq \cos((p-1)\phi + \theta),$$

or

$$(p-2)\cos((p-2)\phi+\theta)\cos\phi \geq -\sin((p-2)\phi+\theta)\sin\phi.$$

The estimate above holds since the left hand side is nonnegative, while the right hand side is nonpositive; this follows from

$$0 \leq (p-2)\phi+\theta \leq (p-2)\phi_p+\theta = \frac{\pi}{2}-\frac{\pi}{p}+\frac{2\theta}{p} < \frac{\pi}{2}. \quad \square$$

The proof of inequality (1.4) *in the case* $2 \leq p < \infty$. As in the proof of Theorem 2.2, we may assume that Y takes values in $\mathbb{R}^\nu$ for some integer ν. Now we repeat the arguments used in the proofs of the previous cases. The details are omitted. $\square$

4. Sharpness of the estimate

Throughout this section we assume $\mathcal{H}=\mathbb{R}$. We will show that the constant C_p in the inequality (1.4) is the best possible by constructing appropriate examples. It turns out, that the cases $p \leq p_0$ and $p > p_0$ are completely different in nature: in the case $p \leq p_0$ the constant C_p is the best possible even if one restricts oneself to the martingale setting.

Sharpness in the case $1 < p \leq p_0$*:* Let (X, Y) be two-dimensional Brownian motion starting from $(1, 0)$. Fix $q > p$ and let $\tau = \tau_q$ denote the exit time of (X, Y) from the cone $\{(x, y) : |y| \leq \tan \frac{\pi}{2q} x\}$. Let $\bar{X}_t = X_{\tau_q \wedge t}$ and $\bar{Y}_t = Y_{\tau_q \wedge t}$ for $t \geq 0$. Clearly, $\bar{Y}$ is α-strongly subordinate to $\bar{X}$ (for any α) and $\bar{X}$, $\bar{Y}$ are orthogonal.

It follows from the result of Burkholder [3], mentioned in the Introduction, that $\mathbb{E}\tau_q^{p/2} < \infty$ (so τ_q is finite almost surely) and hence, by Burkholder-Gundy inequality, $||\bar{X}||_p = ||X_{\tau_q}||_p < \infty$. Consequently,

$$||\bar{Y}||_p = ||Y_{\tau_q}||_p = \tan\frac{\pi}{2q}||X_{\tau_q}||_p = \tan\frac{\pi}{2q}||\bar{X}||_p.$$

It suffices to take $q \downarrow p$ to complete the proof. $\square$

The case $p > p_0$ is more involved. We will need Theorem 1.6. Recall the definition of the cone K_ψ from the Introduction. Let (V, W) be two-dimensional Brownian motion starting from $\xi \in K_\psi$ and let A denote the local time of V at 0. That is, we have, for all $t \geq 0$,

$$|V_t| = V_0 + \int_{0+}^t \operatorname{sgn}(V_s)dV_s + A_t.$$

Let $\alpha \in [0, 1]$ be fixed and, for $t \geq 0$, we define

$$X_t = |V_t|, \qquad Y_t = W_t + \alpha A_t.$$

Let $\tau = \tau_\psi$ be the exit time of (X, Y) from the cone K_ψ.

The proof of Theorem 1.6. Let $\psi < \psi_p$ and fix a (small) positive number a. Consider a function f defined on the cone $\{-a \leq \phi \leq \psi\}$ by

$$f(x, y) = r^p \cdot \cos^{p-1}\phi_p\,(\sin\phi_p)^{-1}\cos(p\phi+\theta).$$

The function f is harmonic in the interior of its domain and satisfies the condition

$$f_x(0, y) + \alpha f_y(0, y) = 0 \tag{4.1}$$

for all $y > 0$. Applying Itô's formula, we obtain, for any $t \geq 0$,

$$f(X_{\tau\wedge t}, Y_{\tau\wedge t}) = f(\xi) + \int_{0+}^{t} f_x(X_s, Y_s)dX_s + \int_{0+}^{t} f_y(X_s, Y_s)dY_s. \tag{4.2}$$

But the measure dA_s is concentrated on the set $\{s : X_s = 0\}$. Thus, by (4.1), we get

$$\int_{0+}^{t} f_x(X_s, Y_s)dA_s + \int_{0+}^{t} f_y(X_s, Y_s)d(\alpha A_s) = 0$$

and taking expectation in (4.2) yields

$$\mathbb{E}f(X_{\tau\wedge t}, Y_{\tau\wedge t}) = f(\xi). \tag{4.3}$$

Since on K_ψ we have $f(x, y) = U_p(x, y) \geq V_p(x, y) \geq (\cot^p \psi - \cot^p \phi_p)x^p$, the equation above yields

$$f(\xi) \geq (\cot^p \psi - \cot^p \phi_p)\mathbb{E}X^p_{\tau\wedge t} = (\cot^p \psi - \cot^p \phi_p)\mathbb{E}|V_{\tau\wedge t}|^p.$$

Thus, by Burkholder-Gundy inequality, we see that for some constant c_p depending only on p,

$$\mathbb{E}(\tau \wedge t)^{p/2} \leq c_p(\cot^p \psi - \cot^p \phi_p)^{-1} f(\xi)$$

and letting $t \to \infty$ yields $\mathbb{E}\tau^{p/2} < \infty$.

We will get the reverse implication by showing that $\mathbb{E}\tau^{p/2} \to \infty$ as $\psi \uparrow \phi_p$. Let $\psi < \phi_p$. As $\mathbb{E}\tau^{p/2} < \infty$, we have $\tau < \infty$ almost surely and $\mathbb{E}X^p_\tau < \infty$, which, combined with Theorem 1.5, yields $\mathbb{E}|Y_\tau|^p < \infty$. But $f(x, y) \leq Cr^p$ for some absolute constant C; hence if we let $t \to \infty$ in (4.3), we obtain

$$f(\xi) = \mathbb{E}f(X_\tau, Y_\tau) = \frac{\cos^{p-1} \phi_p(\sin \phi_p)^{-1}}{\sin^p \psi} \cos(p\psi + \theta)\mathbb{E}|V_\tau|^p,$$

by Lebesgue's dominated convergence theorem. Therefore, by Burkholder-Gundy inequality, for some constant $c(p)$ depending on p,

$$\mathbb{E}\tau^{p/2} \geq \frac{f(\xi) \sin^p \psi}{c(p) \cos(p\psi + \theta)}.$$

Hence, if $\psi \to \phi_p$, $\mathbb{E}\tau^{p/2} \to \infty$ and we are done. □

The proof of the sharpness of the estimate (1.4) *for* $p_0 < p < \infty$. Let $\psi < \phi_p$, $\xi = (\xi_1, \xi_2) \in K_\psi$, $\alpha \in [0, 1]$ and X, Y, $\tau = \tau_\psi$ be as above. Consider the processes $\bar{X}$, $\bar{Y}$ given by $\bar{X}_t = X_{\tau\wedge t}$, $\bar{Y}_t = Y_{\tau\wedge t} - \xi_2$ for all $t \geq 0$. Then $\bar{Y}$ is α-strongly subordinate to $\bar{X}$ and $\bar{X}$, $\bar{Y}$ are orthogonal. Moreover, by the theorem above, we can write

$$\cot \psi||\bar{X}_\infty||_p = \cot \psi||X_\tau||_p = ||Y_\tau||_p \leq ||\bar{Y}_\infty||_p + \xi_2,$$

all the terms being finite. Now if we let $\psi \uparrow \phi_p$, then $||\bar{X}||_p \to \infty$ and hence, by the inequality above, the number $\cot \phi_p$ can not be replaced in (1.4) by a smaller constant. □

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