Probab. Theory Relat. Fields (2008) 140:553–568
DOI 10.1007/s00440-007-0075-0

Weak type inequality for noncommutative differentially subordinated martingales

Adam Osękowski

Received: 25 August 2006 / Revised: 30 January 2007 / Published online: 4 May 2007

Abstract In the paper we focus on self-adjoint noncommutative martingales. We provide an extension of the notion of differential subordination, which is due to Burkholder in the commutative case. Then we show that there is a noncommutative analogue of the Burkholder method of proving martingale inequalities, which allows us to establish the weak type (1, 1) inequality for differentially subordinated martingales. Moreover, a related sharp maximal weak type (1, 1) inequality is proved.

Keywords Noncommutative probability space · Martingale · Weak type (1,1) inequality · Differentially subordinated martingales

Mathematics Subject Classification (2000) Primary: 46L53; Secondary: 60G42

1 Introduction

The theory of noncommutative martingales has been rapidly developed in recent years. Indeed, many of inequalities in the classical martingale theory have been successfully transferred into the noncommutative setting. Essentially, this direction of research started with the fundamental paper of Pisier and Xu [10], where the noncommutative martingale Hardy spaces were introduced and the right analogue of the Burkholder–Gundy inequalities was proved. Since then, several articles on this subject have appeared in the literature. The Burkholder–Gundy inequalities were further studied by Randrianantoanina in [12,13]. A noncommutative analogue of Doob's maximal inequality was

Research supported by MEN Grant 1 PO3A 012 29.

A. Osękowski (✉)
Department of Mathematics, Warsaw University, Banacha 2, 02-097 Warsaw, Poland
e-mail: ados@mimuw.edu.pl

proved by Junge in [4], noncommutative versions of Burkholder/Rosenthal inequalities and their extensions were established by Junge and Xu in [6] and Randrianantoanina in [14]. Noncommutative BMO-spaces were studied by Pisier and Xu in [10] and Musat in [7]. The classical John–Nirenberg inequalities were proved recently by Junge and Musat in [5]. The inequalities for martingale transforms appear in the papers by Pisier and Xu [10], Randrianantoanina [11] and Parcet and Randrianantoanina [8]. We also refer the reader to a survey by Xu [16] for more information on the subject.

In this paper we continue this line of research and present a new method of proving noncommutative martingale inequalities. We investigate the noncommutative analogue of differential subordination, introduced by Burkholder in the case of classical martingales. Let us briefly describe the problem in the commutative setting. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a classical probability space equipped with a discrete filtration $(\mathcal{F}_n)$. Let $x = (x_n)$, $y = (y_n)$ be two real-valued $(\mathcal{F}_n)$-martingales, with difference sequences (dx_n), (dy_n), respectively. In [1], Burkholder proved the weak type $(1, 1)$ inequality for commutative martingale transforms: there exists an absolute constant C such that if $dx_n = \xi_n dy_n$, $n = 0, 1, 2, \ldots$, for some predictable process (ξ_n) satisfying the condition $\sup_n |\xi_n| \leq 1$, then for any $n = 0, 1, 2, \ldots$, and any $\lambda > 0$,

$$\lambda\mathbb{P}(|x_n| \geq \lambda) \leq C\mathbb{E}|y_n|. \tag{1}$$

Then Burkholder [2] introduced the notion of differentially subordinated martingales, which generalized the martingale transforms: the martingale x is differentially subordinated to y, if, almost surely,

$$|dx_n| \leq |dy_n|, \quad n = 0, 1, 2, \ldots.$$

For such martingales Burkholder established the weak type $(1, 1)$ inequality (1) with an optimal constant $C = 2$.

The question about the weak type $(1, 1)$ inequality for noncommutative martingale transforms was raised by Pisier and Xu [10] and was answered positively by Randrianantoanina [11] under an additional assumption, that for any n, ξ_n commutes with $\mathcal{M}_n$. Later, Parcet and Randrianantoanina [8] gave another proof of this result, using the noncommutative version of Gundy's decomposition of a martingale.

A natural problem is whether there is any noncommutative analogue of differential subordination. And if it is the case, the next task is to determine, if the weak type $(1, 1)$ inequality holds in this setting. The aim of the paper is to answer positively these two questions.

The paper is organized as follows. In the next section we set some preliminary background on noncommutative spaces and martingale theory. In Sect. 3 we introduce the noncommutative differential subordination and prove it generalizes the martingale transforms. The main results of the paper are stated in Sect. 4. In Sect. 5 we explain our approach and its relation with Burkholder's original ideas, while Sect. 6 is completely devoted to the proofs of our main results.

2 Preliminary definitions

Let $\mathcal{M}$ be a finite von Neumann algebra equipped with a normal faithful normalized tracial state τ. Throughout, H will denote a Hilbert space, with $\mathcal{M} \subseteq B(H)$, $B(H)$ being the space of bounded linear operators on H. The identity element of $\mathcal{M}$ will be denoted by I. For any self-adjoint operator T, let $T = \int_{-\infty}^{\infty} \lambda \mathrm{d}e_\lambda^T$ be its spectral decomposition. Then for any Borel subset $B \subseteq \mathbb{R}$, let $I_B(T)$ stand for the spectral projection $\int_{-\infty}^{\infty} \chi_B(\lambda)\mathrm{d}e_\lambda^T$. The modulus of an operator $x \in \mathcal{M}$ is defined as $|x| = (x^*x)^{1/2}$.

Let $S,\ T$ be two projections belonging to $\mathcal{M}$. Then $S \vee T$ (resp., $S \wedge T$) will stand for the projection onto the sum $S(H) \cup T(H)$ (resp., onto the intersection $S(H) \cap T(H)$). We say that S and T are equivalent (and denote it by $S \sim T$), if there exists a partial isometry $u \in \mathcal{M}$ such that $u^*u = S$ and $uu^* = T$. We will need the following fact (cf. [15]).

Lemma 1 *Let S, T be two projections of $\mathcal{M}$. Then*

$$S - S \wedge T \sim S \vee T - T.$$

Furthermore, we say that S is subequivalent to T (and write $S \prec T$), if S is equivalent to a subprojection of T. Obviously, $S \prec T$ implies $\tau(S) \leq \tau(T)$.

Let us now introduce noncommutative L^p spaces associated with $(\mathcal{M}, \tau)$. For fixed $1 \leq p < \infty$ and $x \in \mathcal{M}$, we set

$$||x||_p = [\tau(|x|^p)]^{1/p}$$

and define the $L^p = L^p(\mathcal{M}, \tau)$ as completion of $(\mathcal{M}, || \cdot ||_p)$. For $p = \infty$, we set $L^\infty = L^\infty(\mathcal{M}, \tau) = \mathcal{M}$ with its usual operator norm. Then the trace τ extends to a positive linear functional on $L^p(\mathcal{M}, \tau)$, $1 \leq p \leq \infty$. Furthermore, for $1 \leq p, q \leq \infty$ such that $1/p + 1/q = 1/r \leq 1$, a product of $x \in L^p$ and $y \in L^q$ is in L^r and the tracial property $\tau(xy) = \tau(yx)$ holds. This will be used very frequently in the paper, usually with $p = 1$ and $q = \infty$.

We will recall the general setup for martingales. Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$. The restriction of τ to $\mathcal{N}$ is a normal faithful normalized trace on $\mathcal{N}$ and it is clear that the natural embedding $j : L^1(\mathcal{N}, \tau) \to L^1(\mathcal{M}, \tau)$ is isometric. The conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$ is defined as the dual map $\mathcal{E} = j^* : \mathcal{M} \to \mathcal{N}$. It satisfies

$$\mathcal{E}(axb) = a\mathcal{E}(x)b$$

for any $a,\ b \in \mathcal{N}$ and $x \in \mathcal{M}$. Furthermore, it preserves the trace, i.e. $\tau(\mathcal{E}(x)) = \tau(x)$ for any $x \in \mathcal{M}$ and extends to a contractive projection from $L^p(\mathcal{M}, \tau)$ onto $L^p(\mathcal{N}, \tau)$ for all $1 \leq p \leq \infty$.

Let $(\mathcal{M}_n)$, $n = 0, 1, 2, \ldots$ be a filtration, i.e., an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ whose union is weak* dense in the algebra $\mathcal{M}$. For each $n = 0, 1, 2, \ldots$, let $\mathcal{E}_n$ be the conditional expectation from $\mathcal{M}$ onto $\mathcal{M}_n$. A sequence

$x = (x_n)_{n\geq 0}$ in $L^1(\mathcal{M}, \tau)$ is called a martingale with respect to $(\mathcal{M}_n)$ (or $(\mathcal{M}_n)$-martingale), if

$$x_n = \mathcal{E}_n(x_{n+1}), \quad n = 0, 1, 2, \ldots.$$

For any $(\mathcal{M}_n)$-martingales $x = (x_n)$, $y = (y_n)$, the difference sequences of these martingales will be denoted by (dx_n), (dy_n), respectively, i.e.,

$$\begin{aligned} dx_0 &= x_0, \quad dx_n = x_n - x_{n-1}, \ \ n = 1, 2, \ldots. \\ dy_0 &= y_0, \quad dy_n = y_n - y_{n-1}, \ \ n = 1, 2, \ldots. \end{aligned}$$

3 Differential subordination

The following is the crucial notion of this paper.

Definition 1 Let $x = (x_n)_{n\geq 0}$, $y = (y_n)_{n\geq 0}$ be two self-adjoint martingales. We will say that x is differentially subordinated to y, if for any fixed $n = 0, 1, 2, \ldots$ the following two conditions hold.

(a) For any projection $S \in \mathcal{M}_n$,

$$\tau(Sdy_nSdy_nS - Sdx_nSdx_nS) \geq 0.$$

(b) For any two projections $S,\ T \in \mathcal{M}_n$, such that $ST = 0$ and $S + T \in \mathcal{M}_{n-1}$,

$$\tau(Sdy_nTdy_n - Sdx_nTdx_n) \geq 0.$$

Both traces may be infinite.

Remark 1 Note that in the commutative setting we obtain the usual differential subordination: indeed, the property (b) is always satisfied, while (a) reduces to $\mathbb{E}((|dy_n|^2 - |dx_n|^2)\chi_A) \geq 0$ for any $\chi_A \in \mathcal{M}_n$, $n = 0, 1, 2, \ldots$, which implies $|dx_n| \leq |dy_n|$ almost surely, for $n = 0, 1, 2, \ldots$.

Remark 2 The differential subordination is preserved if we multiply the dominated martingale x by a constant α, $|\alpha| \leq 1$ and/or multiply the dominating martingale y by a constant β, $|\beta| \geq 1$.

In the commutative setting the following property is valid. Suppose x is differentially subordinated to y. Then for any n, the inequality $dx_n + dy_n > 0$ implies $-dx_n + dy_n \geq 0$ (note that the first inequality is strict and the second one is not). The noncommutative analogue of this property is established in the following.

Lemma 2 *Suppose x, y are two self-adjoint $(\mathcal{M}_n)$-martingales such that x is differentially subordinated to y. Fix a nonnegative integer n and a projection $S \in \mathcal{M}_n$. If $S = I_{(0,\infty)}(S(dx_n + dy_n)S)$, then the operator $S(-dx_n + dy_n)S$ is nonnegative.*

Remark 3 In virtue of this Lemma and Remark 2, we have analogous statements for pairs $x, -y$; $-x, y$; $-x, -y$ of martingales.

Proof of the Lemma 2 Fix any negative number K and consider the projection

$$T = I_{(K,0)}(S(-dx_n + dy_n)S).$$

Obviously, the projection T belongs to $\mathcal{M}_n$ and $T \leq S$. By definition, the operator $T(-dx_n + dy_n)T$ is bounded and nonpositive and, since T is a subprojection of S, $T(dx_n + dy_n)T$ is nonnegative (and clearly belongs to L^1). However,

$$\begin{aligned}\tau(T(dx_n + dy_n)T(-dx_n + dy_n)T) &= \tau(T(-dx_n + dy_n)T(dx_n + dy_n)T)\\ &= \frac{1}{2}\Big\{\tau(T(dx_n + dy_n)T(-dx_n + dy_n)T)\\ &\qquad +\tau(T(-dx_n + dy_n)T(dx_n + dy_n)T)\Big\}\\ &= \tau(Tdy_nTdy_nT - Tdx_nTdx_nT)\end{aligned}$$

is nonnegative by differential subordination. This clearly gives $T = 0$, since in the definition of T we took the spectral projection corresponding to interval not containing 0. Now it suffices to take $K \to -\infty$. □

The following Lemma shows that differential subordination generalizes martingale transforms.

Lemma 3 *Suppose a sequence (ξ_n), predictable with respect to the filtration $(\mathcal{M}_n)$, satisfies* $\sup_n ||\xi_n|| \leq 1$ *and for any fixed n, ξ_n commutes with $\mathcal{M}_n$. Suppose y is any self-adjoint square integrable martingale and x, defined by $dx_n = \xi_n dy_n$, $n = 1, 2, \ldots$ is also self-adjoint. Then x is differentially subordinated to y.*

Proof Fix a nonnegative integer n. The operator ξ_n^* commutes with $\mathcal{M}_n$ as well and $\xi_n dy_n = (\xi_n dy_n)^* = dy_n\xi_n^*$.

Now for any $S \in \mathcal{M}_n$, using the tracial property, we obtain

$$\begin{aligned}\tau(Sdy_nSdy_nS - Sdx_nSdx_nS) &= \tau(Sdy_nSdy_nS - Sdy_n\xi_n^*S\xi_ndy_nS)\\ &= \tau((I - |\xi_n|^2)Sdy_nSdy_nS) \geq 0,\end{aligned}$$

as the operators $I - |\xi_n|^2$ and Sdy_nSdy_nS are nonnegative. The argument to establish the property (b) is exactly the same. □

4 The main results

For any two self-adjoint $(\mathcal{M}_n)$-martingales x, y, let

$$s_n = x_n + y_n, \quad d_n = -x_n + y_n, \quad n = 0, 1, 2, \ldots.$$

Obviously, (s_n), (d_n) are again self-adjoint $(\mathcal{M}_n)$-martingales. Now we may introduce the key families of projections: for a fixed $\lambda > 0$, let $R^\lambda_{-1} = I$,

$$Q^\lambda_n = I_{(-2\lambda,0)}(R^\lambda_{n-1}(s_n - \lambda I)R^\lambda_{n-1}),$$
$$R^\lambda_n = I_{(-2\lambda,0)}(Q^\lambda_n(d_n - \lambda I)Q^\lambda_n), \quad n = 0, 1, 2, \ldots.$$

Note that the projection R^λ_n, in the commutative setting, is the indicator function of the set $\{\max_{k\leq n}(|x_k| + |y_k|) < \lambda\}$. One could ask why we use such a strange expression for Q^λ_n, R^λ_n instead of simple

$$Q^\lambda_n = I_{(-\lambda,\lambda)}(R^\lambda_{n-1}s_n R^\lambda_{n-1}) \tag{2}$$

and similarly for the projection R^λ_n. The point is that we want to obtain a monotone (nonincreasing) chain of projections (see the Lemma 4 below) and this condition fails if we take the latter, "more natural" definition (2). However, note that an alternative way to define Q^λ_n, R^λ_n is

$$Q^\lambda_n = R^\lambda_{n-1}I_{(-\lambda,\lambda)}(R^\lambda_{n-1}s_n R^\lambda_{n-1}),$$
$$R^\lambda_n = Q^\lambda_n I_{(-\lambda,\lambda)}(Q^\lambda_n d_n Q^\lambda_n), \quad n = 0, 1, 2, \ldots.$$

We may now state the main results of the paper. The first theorem is a maximal weak type (1, 1) inequality.

Theorem 1 *Let x, y be two self-adjoint square integrable $(\mathcal{M}_n)$-martingales such that x is differentially subordinated to y. Then for any nonnegative integer n and any $\lambda > 0$ we have*

$$\lambda\tau(I - R^\lambda_n) \leq 2\tau(|y_n|). \tag{3}$$

The constant 2 is best possible.

The second theorem concerns the weak type (1, 1) inequality.

Theorem 2 *Let x, y be two self-adjoint square integrable $(\mathcal{M}_n)$-martingales such that x is differentially subordinated to y. Then for any nonnegative integer n and any $\lambda > 0$ we have*

$$\lambda\tau(I_{[\lambda,\infty)}(|x_n|)) \leq 4\tau(|y_n|).$$

Remark 4 Note that we impose an extra condition on the integrability of the martingales. We believe, however, that it is not really necessary. Most of the statements in the paper, as well as the methodology, are valid without this assumption.

Obviously, it suffices to prove the theorems above for $\lambda = 1$. Therefore, from now on, we assume this. For convenience, we will write R_n, Q_n instead of R^1_n, Q^1_n, respectively.

5 The noncommutative Burkholder operators for the weak type inequality

The plan is to study carefully Burkholder's proof and try to extend it to the noncommutative setting; in order to obtain the weak type (1, 1) inequality for differentially subordinated commutative martingales, Burkholder invented the function

$$u(x, y) = (|y|^2 - |x|^2)\chi_{\{|x|+|y|<1\}} + (2|y| - 1)\chi_{\{|x|+|y|\geq 1\}}, \tag{4}$$

enjoying the following properties: for any martingales x, y such that x is differentially subordinated to y,

1° $\mathbb{E}u(x_0, y_0) \geq 0$,
2° $\mathbb{E}u(x_n, y_n) \leq 2\mathbb{E}|y_n| - \mathbb{P}(|x_n| + |y_n| \geq 1)$,
3° the sequence $(\mathbb{E}u(x_n, y_n))$ is nondecreasing

(in fact he proved, that the process $(u(x_n, y_n))$ is a submartingale, satisfying $u(x_n, y_n) \leq 2|y_n| - \chi_{\{|x_n|+|y_n|\geq 1\}}$ almost surely). Obviously these conditions imply

$$2\mathbb{E}|y_n| - \mathbb{P}(|x_n| + |y_n| > 1) \geq \mathbb{E}u(x_0, y_0) \geq 0$$

and the weak type (1, 1) inequality. However, the process $(u(x_n, y_n))$ can not effectively be extended to the noncommutative setting and a certain modification is needed. As noted by Burkholder, much more can be extracted from the function u: consider a stopping time

$$\sigma = \inf\{n : |x_n| + |y_n| \geq 1\}$$

and martingales $x' = (x'_n) = (x_{\min(\sigma,n)})$, $y' = (y'_n) = (y_{\min(\sigma,n)})$. It can be easily checked that

$$dx'_n = dx_n\chi_{\{\sigma>n-1\}} \quad \text{and } dy'_n = dy_n\chi_{\{\sigma>n-1\}},$$

which immediately implies that x' is differentially subordinated to y'. Therefore we may apply the preceding procedure to this new pair of martingales. Before we do it, let us note, that by the definition of the stopping time σ, we have

$$\left\{\max_{k\leq n}(|x_k| + |y_k|) < 1\right\} = \{\sigma > n\} = \{|x'_n| + |y'_n| < 1\}. \tag{5}$$

Now we use Burkholder's argument: consider the process

$$\begin{aligned} u(x'_n, y'_n) &= (|y'_n|^2 - |x'_n|^2)\chi_{\{|x'_n|+|y'_n|<1\}} + (2|y'_n| - 1)\chi_{\{|x'_n|+|y'_n|\geq 1\}} \\ &= (|y_n|^2 - |x_n|^2)\chi_{\{|x'_n|+|y'_n|<1\}} + (2|y'_n| - 1)\chi_{\{|x'_n|+|y'_n|\geq 1\}}, \end{aligned} \tag{6}$$

where the last equality holds due to (5). This process leads to the inequality

$$\mathbb{P}(|x'_n| + |y'_n| \geq 1) \leq 2\mathbb{E}|y'_n| \leq 2\mathbb{E}|y_n|, \quad n = 0, 1, 2, \ldots,$$

or, in virtue of (5),

$$\mathbb{P}(\max_{k\le n}(|x_k|+|y_k|)\ge 1)\le 2\mathbb{E}|y_n|,\quad n=0,1,2,\ldots. \tag{7}$$

It turns out that the process (6) has a natural extension (u_n) to a noncommutative setting, which can be used to prove the noncommutative maximal weak type inequality (3). To define it, we will need certain additional projections. For convenience of the reader, let us recall the projections (Q_n), (R_n) (recall we assume $\lambda=1$): $R_{-1}=I$,

$$\begin{aligned} Q_n &= I_{(-2,0)}(R_{n-1}(s_n-I)R_{n-1}),\\ R_n &= I_{(-2,0)}(Q_n(d_n-I)Q_n),\quad n=0,1,2,\ldots \end{aligned}$$

and, for $n=0,1,2,\ldots$, define

$$\begin{aligned} U_n &= I_{(-\infty,-1]}(R_{n-1}s_nR_{n-1}),\\ D_n &= I_{[1,\infty)}(R_{n-1}s_nR_{n-1}),\\ N_n &= I_{(-\infty,-1]}(Q_nd_nQ_n),\\ P_n &= I_{[1,\infty)}(Q_nd_nQ_n). \end{aligned}$$

Now we may introduce the Burkholder operators. It can be done as follows: let

$$u_n=R_ns_nR_{n-1}d_n+v_n,\quad n=0,1,2,\ldots, \tag{8}$$

where

$$\begin{aligned} v_n &= \sum_{k=0}^{n}(U_k+N_k)(-s_k-d_k-I)+(D_k+P_k)(s_k+d_k-I)\\ &= \sum_{k=0}^{n}(U_k+N_k)(-2y_k-I)+(D_k+P_k)(2y_k-I). \end{aligned} \tag{9}$$

It can be checked, that in the commutative setting, the operator u_n is just the random variable (6).

Remark 5 For a better understanding of the term v_n in (9) we urge the reader to look at Lemma 4 below. Then it becomes clear that it provides a noncommutative generalization of the last term in (6). Indeed, the projections U_k, N_k, D_k, P_k are disjoint and the sum $U_k+N_k+D_k+P_k$ corresponds to $\{\sigma=k\}$ (or $\chi_{\{\sigma=k\}}$) in the commutative case. Thus we have to see that

$$2|y_k|=\begin{cases}-(s_k+d_k) & \text{in } U_k+N_k,\\ +(s_k+d_k) & \text{in } D_k+P_k.\end{cases}$$

Since N_k, P_k are subprojections of Q_k, in the commutative case we have

$$N_k\subset\{|s_k|<1,\ d_k\le -1\}\quad\text{and } P_k\subset\{|s_k|<1,\ d_k\ge 1\}.$$

Therefore, the assertion is clear in N_k and P_k. The situation for U_k and D_k is a bit more involved. Namely, to be in U_k means that $s_k \leq -1$ and that we are in R_{k-1}. In other words, we have the following characterization of U_k in the commutative case

$$U_k = \{s_k \leq -1, |s_j|, |d_j| < 1, 1 \leq j \leq k-1\}.$$

From this we can not deduce that $|d_k| < 1$ (as we did above), but we can prove that $s_k + d_k$ is negative. Indeed, we have $s_k + d_k = s_k + d_{k-1} + dy_k - dx_k < -1 + 1 + dy_k - dx_k$. Now we want to see that $dy_k \leq dx_k$, but we know that $s_k \leq -1$ and $|s_{k-1}| < 1$. This automatically gives that $dy_k + dx_k \leq 0$. Moreover, by differential subordination, $|dx_k| \leq |dy_k|$ so that $dy_k \leq 0$. The last two inequalities imply $dy_k \leq dx_k$. The argument for D_k is similar.

In order to prove the weak type inequality, we will show the Burkholder operators satisfy noncommutative analogues of the conditions 1°, 2° and 3°.

6 The proofs

The first statement describes the basic properties of the projections defined above.

Lemma 4 *Let x, y be two self-adjoint $(\mathcal{M}_n)$-martingales. For any $n = 0, 1, 2, \ldots$,*

(i) *the projections U_n, D_n, Q_n are subprojections of R_{n-1},*
(ii) *the projections N_n, P_n, R_n are subprojections of Q_n,*
(iii) *we have*

$$U_n + D_n + Q_n = R_{n-1}, \quad N_n + P_n + R_n = Q_n \tag{10}$$

(iv) *for any $\alpha \in \mathbb{R}$, the projections Q_n, U_n, D_n commute with $R_{n-1}(s_n + \alpha I)R_{n-1}$,*
(v) *for any $\alpha \in \mathbb{R}$, the projections R_n, N_n, P_n commute with $Q_n(d_n + \alpha I)Q_n$.*
(vi) *we have*

$$-R_n \leq R_n x_n R_n \leq R_n, \ -R_n \leq R_n y_n R_n \leq R_n.$$

Proof We will only show the property (vi). All the other properties are clear. By the definition of Q_n, we have

$$-Q_n \leq Q_n(x_n + y_n)Q_n \leq Q_n,$$

which implies

$$-R_n \leq R_n(x_n + y_n)R_n \leq R_n.$$

But by the definition of R_n, we have

$$-R_n \leq R_n(-x_n + y_n)R_n \leq R_n.$$

It suffices to combine the two inequalities above to obtain the claim. □

The next lemma is of great importance for the further analysis of the sequence $(\tau(u_n))$.

Lemma 5 *Let x, y be two self-adjoint $(\mathcal{M}_n)$-martingales. For $n = 0, 1, 2, \ldots,$ we have*

$$\tau(R_n s_n R_{n-1} d_n) = \tau(R_n s_n R_n d_n). \tag{11}$$

Proof Combining the properties (i), (ii), (iv) and (v) from Lemma 4 with the tracial property, we may write

$$\begin{aligned}
\tau(R_n s_n R_{n-1} d_n) &= \tau(R_n Q_n R_{n-1} s_n R_{n-1} d_n) \\
&= \tau(R_n R_{n-1} s_n R_{n-1} Q_n d_n) = \tau(R_n s_n Q_n d_n) = \tau(Q_n R_n s_n Q_n d_n) \\
&= \tau(R_n s_n Q_n d_n Q_n) = \tau(R_n s_n Q_n d_n Q_n R_n) = \tau(R_n s_n R_n Q_n d_n Q_n) \\
&= \tau(R_n s_n R_n d_n).
\end{aligned}$$

□

Now we are ready to establish the noncommutative conditions 1° and 2°.

Lemma 6 *Let x, y be two self-adjoint $(\mathcal{M}_n)$-martingales such that x is differentially subordinated to y. Then the trace $\tau(u_0)$ is nonnegative.*

Proof As $s_0 = dx_0 + dy_0$ and $d_0 = -dx_0 + dy_0$, the first term in (8) has positive trace due to the differential subordination. Let us now turn to $\tau(v_0)$. Since $U_0(dx_0 + dy_0)U_0 \leq -U_0$, we have $U_0 = I_{(-\infty,0)}(U_0(dx_0 + dy_0)U_0)$, so Lemma 2 yields $U_0(-dx_0 + dy_0)U_0 \leq 0$. Combining the last two inequalities we obtain

$$U_0(-2y_0 - I)U_0 \geq 0 \quad \text{and } \tau(U_0(-2y_0 - I)U_0) \geq 0.$$

In a similar manner we show

$$\tau(N_0(-2y_0 - I)N_0) \geq 0, \quad \tau(D_0(2y_0 - I)D_0) \geq 0, \quad \tau(P_0(2y_0 - I)P_0) \geq 0.$$

It suffices to add the last four inequalities to obtain $\tau(v_0) \geq 0$. The proof is complete. □

Lemma 7 *Let x, y be two self-adjoint $(\mathcal{M}_n)$-martingales. Then for any positive integer n,*

$$\tau(u_n) \leq 2\tau(|y_n|) - \tau(I - R_n). \tag{12}$$

Proof Let

$$T_n^+ = \sum_{k=1}^{n}(D_k + P_k), \quad T_n^- = \sum_{k=1}^{n}(U_k + N_k).$$

Note that by property (iii) from Lemma 4, we have $T_n^+ + T_n^- = I - R_n$. Hence, using the martingale property, we may write

$$\tau(v_n) = \tau(T_n^+(2y_n - I) + T_n^-(-2y_n - I)) = \tau(2(T_n^+ - T_n^-)y_n - (I - R_n)).$$

Furthermore, using Lemma 5, we have

$$\begin{aligned}\tau(R_n s_n R_{n-1} d_n) &= \tau(R_n(y_n + x_n)R_n(y_n - x_n))\\ &= \tau(R_n y_n R_n y_n - R_n x_n R_n x_n) + \tau(-R_n y_n R_n x_n + R_n x_n R_n y_n)\\ &= \tau(R_n y_n R_n y_n - R_n x_n R_n x_n).\end{aligned}$$

Indeed, here we used the tracial property: the operators $R_n x_n R_n$ and $R_n y_n R_n$ are bounded (Lemma 4 (vi)), so $R_n x_n R_n y_n,\ R_n y_n R_n x_n$ belong to L^1 and

$$\begin{aligned}\tau(R_n x_n R_n y_n) &= \tau(R_n R_n x_n R_n y_n) = \tau(R_n x_n R_n y_n R_n)\\ &= \tau(R_n x_n R_n R_n y_n R_n) = \tau(R_n y_n R_n R_n x_n R_n) = \tau(R_n y_n R_n x_n).\end{aligned}$$

Similarly, one shows that

$$\tau(R_n x_n R_n x_n) = \tau(R_n x_n R_n x_n R_n),\ \tau(R_n y_n R_n y_n) = \tau(R_n y_n R_n y_n R_n).$$

Therefore the inequality (12) is equivalent to

$$\begin{aligned}&\tau(R_n y_n R_n y_n R_n - R_n x_n R_n x_n R_n + 2(T_n^+ - T_n^-)y_n - (I - R_n))\\ &\leq 2\tau(|y_n|) - \tau(I - R_n).\end{aligned}$$

The operator $R_n x_n R_n x_n R_n$ is nonnegative, so it suffices to prove

$$\tau(R_n y_n R_n y_n + 2(T_n^+ - T_n^-)y_n) \leq 2\tau(|y_n|).$$

We have $y_n = y_n I_{(0,\infty)}(y_n) - y_n I_{(-\infty,0)}(y_n) = y_n^+ - y_n^-$ and

$$\tau((I - R_n)|y_n|) - \tau(T_n^+ y_n - T_n^- y_n) = 2\tau(T_n^- y_n^+ + T_n^+ y_n^-) \geq 0. \tag{13}$$

Again using the property (vi) in Lemma 4, along with the tracial property, we get

$$\tau(R_n y_n R_n y_n) = \tau(R_n y_n R_n(y_n^+ - y_n^-)) \leq 2\tau(R_n(y_n^+ + y_n^-)) = 2\tau(R_n|y_n|). \tag{14}$$

Combining (13) with (14) completes the proof. □

The proof of the condition 3° is more involved. We split it into several lemmas.

Lemma 8 *Let x, y be two self-adjoint $(\mathcal{M}_n)$-martingales. For any nonnegative integer n, the following inequalities hold.*

$$\tau(N_{n+1}(s_{n+1} + I)R_n(d_{n+1} + I)) \leq 0, \tag{15}$$
$$\tau(P_{n+1}(s_{n+1} - I)R_n(d_{n+1} - I)) \leq 0, \tag{16}$$
$$\tau[U_{n+1}(s_{n+1} + I)U_{n+1}(d_n + I)R_n] \leq 0, \tag{17}$$
$$\tau[D_{n+1}(s_{n+1} - I)D_{n+1}(d_n - I)R_n] \leq 0. \tag{18}$$

Proof The operator $Q_{n+1}R_n(s_{n+1}+I)R_n$ is nonnegative and bounded, while $Q_{n+1}(d_{n+1}+I)Q_{n+1}N_{n+1}$ is nonpositive. Hence, using Lemma 4 and the tracial property, we obtain

$$\begin{aligned}&\tau(N_{n+1}(s_{n+1}+I)R_n(d_{n+1}+I))\\&\quad=\tau(Q_{n+1}N_{n+1}Q_{n+1}R_n(s_{n+1}+I)R_n(d_{n+1}+I))\\&\quad=\tau(Q_{n+1}R_n(s_{n+1}+I)R_nQ_{n+1}(d_{n+1}+I)Q_{n+1}N_{n+1})\leq 0.\end{aligned}$$

The inequalities (16), (17) can be proved in the same manner; for instance, for (16), we repeat the above arguments, replacing N_{n+1} with P_{n+1}, $+I$ with $-I$ and thus obtain

$$\begin{aligned}&\tau(P_{n+1}(s_{n+1}-I)R_n(d_{n+1}-I))\\&\quad=\tau(Q_{n+1}R_n(s_{n+1}-I)R_nQ_{n+1}(d_{n+1}-I)Q_{n+1}P_{n+1})\leq 0,\end{aligned}$$

as under the trace we have a product of two operators: a nonpositive and bounded $Q_{n+1}R_n(s_{n+1}-I)R_n$ and a nonnegative $Q_{n+1}(d_{n+1}-I))Q_{n+1}P_{n+1}$. The proof is complete. □

Lemma 9 *Let x, y be two self-adjoint $(\mathcal{M}_n)$-martingales such that x is differentially subordinated to y. For any nonnegative integer n, we have*

$$\tau[U_{n+1}(s_n+I)U_{n+1}(-dx_{n+1}+dy_{n+1})R_n]\leq 0. \tag{19}$$

Moreover,

$$\begin{aligned}&\tau\big[U_{n+1}(s_{n+1}+I)R_n(d_{n+1}+I)R_n\\&\qquad-U_{n+1}(dx_{n+1}+dy_{n+1})U_{n+1}(-dx_{n+1}+dy_{n+1})R_n\big]\leq 0\end{aligned} \tag{20}$$

and

$$\begin{aligned}&\tau\big[D_{n+1}(s_{n+1}-I)R_n(d_{n+1}-I)R_n\\&\qquad-D_{n+1}(dx_{n+1}+dy_{n+1})D_{n+1}(-dx_{n+1}+dy_{n+1})R_n\big]\leq 0.\end{aligned} \tag{21}$$

Proof Note that

$$U_{n+1}(s_n+I)U_{n+1}=U_{n+1}R_n(s_n+I)R_nU_{n+1}$$

is nonnegative and bounded and write

$$\begin{aligned}&\tau[U_{n+1}(s_n+I)U_{n+1}(-dx_{n+1}+dy_{n+1})R_n]\\&\quad=\tau[U_{n+1}(s_n+I)U_{n+1}(-dx_{n+1}+dy_{n+1})U_{n+1}].\end{aligned} \tag{22}$$

We have

$$R_n=I_{(0,\infty)}(R_n(s_n+I)R_n),$$

which clearly gives

$$U_{n+1} = I_{(0,\infty)}(U_{n+1}(s_n + I)U_{n+1}).$$

Moreover, by the definition of U_{n+1}, we have

$$\begin{aligned} 0 &\geq U_{n+1}(s_{n+1} + I)U_{n+1} \\ &= U_{n+1}(s_n + I)U_{n+1} + U_{n+1}(dx_{n+1} + dy_{n+1})U_{n+1}, \end{aligned}$$

which combined with the preceding equality implies

$$U_{n+1} = I_{(-\infty,0)}(U_{n+1}(dx_{n+1} + dy_{n+1})U_{n+1}).$$

Now, by Lemma 2 and Remark following it, we obtain $U_{n+1}(-dx_{n+1}+dy_{n+1})U_{n+1} \leq 0$, which, together with (22), immediately yields (19). It suffices to add inequalities (17) and (19) to obtain (20). Exactly the same argumentation (along with (18)) leads to (21). □

Lemma 10 *Suppose self-adjoint martingales x, y are square integrable and x is differentially subordinated to y. Then*

$$\begin{aligned} \tau\big[&U_{n+1}(dx_{n+1} + dy_{n+1})U_{n+1}(-dx_{n+1} + dy_{n+1})R_n \\ &+D_{n+1}(dx_{n+1} + dy_{n+1})D_{n+1}(-dx_{n+1} + dy_{n+1})R_n \\ &-R_n(dx_{n+1} + dy_{n+1})R_n(-dx_{n+1} + dy_{n+1})R_n\big] \leq 0. \end{aligned}$$

Remark 6 This is the heart of the matter. The lemma concerns the only inequality, where the condition (b) from the differential subordination is used. Furthermore, in fact, the square integrability of the martingales is imposed only for the sake of this lemma. One could just *define* that x is differentially subordinated to y if the property (a) and the inequality above hold; then we would not need any integrability assumptions, all the arguments can be transferred to this more general setting.

Proof of the Lemma 10 We open the brackets and use the tracial property, thus obtaining

$$\begin{aligned} \tau\big[&- U_{n+1}dx_{n+1}U_{n+1}dx_{n+1} + U_{n+1}dy_{n+1}U_{n+1}dy_{n+1} \\ &-D_{n+1}dx_{n+1}D_{n+1}dx_{n+1} + D_{n+1}dy_{n+1}D_{n+1}dy_{n+1} \\ &-R_ndx_{n+1}R_ndx_{n+1} + R_ndy_{n+1}R_ndy_{n+1}\big] \leq 0. \end{aligned}$$

The property (b) from differential subordination gives the inequalities

$$\tau((R_n - U_{n+1})dy_{n+1}U_{n+1}dy_{n+1} - (R_n - U_{n+1})dx_{n+1}U_{n+1}dx_{n+1}) \geq 0,$$
$$\tau((R_n - D_{n+1})dy_{n+1}D_{n+1}dy_{n+1} - (R_n - D_{n+1})dx_{n+1}D_{n+1}dx_{n+1}) \geq 0,$$

as, clearly, $R_n - U_{n+1}, U_{n+1}$ and $R_n - D_{n+1}, D_{n+1}$ are pairs of orthogonal projections. Again by the property (b), since $U_{n+1} + D_{n+1}$ is a subprojection of R_n,

$$\tau((R_n - U_{n+1} - D_{n+1})dy_{n+1}(U_{n+1} + D_{n+1})dy_{n+1} \\ -(R_n - U_{n+1} - D_{n+1})dx_{n+1}(U_{n+1} + D_{n+1})dx_{n+1}) \geq 0.$$

Finally, noting that $R_n - U_{n+1} - D_{n+1} = Q_{n+1}$ (by (10)), the property (a) yields

$$\tau(Q_{n+1}dy_{n+1}Q_{n+1}dy_{n+1} - Q_{n+1}dx_{n+1}Q_{n+1}dx_{n+1}) \geq 0.$$

Now we combine the four inequalities above and that completes the proof. □

We are ready to deal with the condition 3°.

Lemma 11 *Suppose self-adjoint martingales x, y are square integrable and x is differentially subordinated to y. Then the sequence $(\tau(u_n))$ is nondecreasing.*

Proof Fix a nonnegative integer n. We have

$$\begin{aligned}\tau(u_n) - \tau(u_{n+1}) &= \tau(R_n s_n R_{n-1} d_n + v_n) - \tau(R_{n+1} s_{n+1} R_n d_{n+1} + v_{n+1}) \\ &= A + B,\end{aligned} \tag{23}$$

where

$$A = \tau(R_n s_n R_{n-1} d_n - R_{n+1} s_{n+1} R_n d_{n+1}), \quad B = \tau(v_n - v_{n+1}).$$

By the definition of v_n, the second summand is equal to

$$B = \tau\big[(U_{n+1} + N_{n+1})(s_{n+1} + d_{n+1} + I) + (D_{n+1} + P_{n+1})(-s_{n+1} - d_{n+1} + I)\big]. \tag{24}$$

Let us now deal with A. By Lemma 5 and the martingale property,

$$\begin{aligned}\tau(R_n s_n R_{n-1} d_n) &= \tau(R_n s_n R_n d_n) \\ &= \tau\big[R_n s_{n+1} R_n d_{n+1} - R_n(dx_{n+1} + dy_{n+1})R_n(dy_{n+1} - dx_{n+1})\big],\end{aligned} \tag{25}$$

so we may proceed as follows

$$\begin{aligned}A &= \tau\big[R_n s_{n+1} R_n d_{n+1} - R_n(dx_{n+1} + dy_{n+1})R_n(dy_{n+1} - dx_{n+1}) \\ &\qquad - R_{n+1} s_{n+1} R_n d_{n+1}\big] \\ &= \tau\big[(R_n - R_{n+1})s_{n+1} R_n d_{n+1} - R_n(dx_{n+1} + dy_{n+1})R_n(dy_{n+1} - dx_{n+1})\big] \\ &= \tau\big[(U_{n+1} + N_{n+1} + D_{n+1} + P_{n+1})s_{n+1} R_n d_{n+1} \\ &\qquad - R_n(dx_{n+1} + dy_{n+1})R_n(dy_{n+1} - dx_{n+1})\big].\end{aligned}$$

Now it can be easily checked, that

$$A + B = \tau\big[(U_{n+1} + N_{n+1})(s_{n+1} + I)R_n(d_{n+1} + I)$$
$$+(D_{n+1} + P_{n+1})(s_{n+1} - I)R_n(d_{n+1} - I)$$
$$-R_n(dx_{n+1} + dy_{n+1})R_n(-dx_{n+1} + dy_{n+1})\big].$$

We want to prove that $A + B$ is nonpositive. By Lemma 8 it suffices to show

$$\tau\big[U_{n+1}(s_{n+1} + I)R_n(d_{n+1} + I) + D_{n+1}(s_{n+1} - I)R_n(d_{n+1} - I)$$
$$-R_n(dx_{n+1} + dy_{n+1})R_n(-dx_{n+1} + dy_{n+1})\big] \le 0. \tag{26}$$

We may multiply the operator in the square bracket by R_n from the left and by the tracial property, (26) is equivalent to

$$\tau\big[U_{n+1}(s_{n+1} + I)R_n(d_{n+1} + I)R_n + D_{n+1}(s_{n+1} - I)R_n(d_{n+1} - I)R_n$$
$$-R_n(dx_{n+1} + dy_{n+1})R_n(-dx_{n+1} + dy_{n+1})R_n\big] \le 0. \tag{27}$$

By Lemma 9, we are left to show that

$$\tau\big[U_{n+1}(dx_{n+1} + dy_{n+1})U_{n+1}(-dx_{n+1} + dy_{n+1})R_n$$
$$+ D_{n+1}(dx_{n+1} + dy_{n+1})D_{n+1}(-dx_{n+1} + dy_{n+1})R_n$$
$$- R_n(dx_{n+1} + dy_{n+1})R_n(-dx_{n+1} + dy_{n+1})R_n\big] \le 0.$$

Now it suffices to use Lemma 10; this is the only place where we need the square integrability. The proof is complete. □

Finally we proceed to the proofs of the main results of the paper.

Proof of the Theorem 1 As noted above, the inequality (3) follows immediately from the conditions 1°, 2° and 3° (Lemmas 6, 7 and 11); indeed,

$$2\tau(|y_n|) - \tau(I - R_n) \ge \tau(u_n) \ge \tau(u_0) \ge 0.$$

The constant 2 is best possible even in the commutative case; see Burkholder [2]. □

Proof of the Theorem 2 We will use some properties of subequivalent projections. The reader is referred to [3] for similar arguments. We will prove that $I_{[1,\infty)}(x_n)$ and $I_{(-\infty,-1]}(x_n)$ are subequivalent to the projection $I - R_n$. This will immediately complete the proof.

To this end, denote $I_{[1,\infty)}(x_n)$ by f, and note that by Lemma 4, $R_n x_n R_n \le R_n$. Hence, it is clear (the spectral projections in Q_n and R_n are taken with respect to open intervals) that this implies $f \wedge R_n = 0$. By Lemma 1,

$$f = f - f \wedge R_n \sim f \vee R_n - R_n \le I - R_n.$$

The same arguments give $I_{(-\infty,-1]}(x_n) \prec I - R_n$. □

Acknowledgements I would like to express my gratitude to Quanhua Xu for introducing me into noncommutative probability theory. I am also indebted to Marek Bożejko and Stanisław Kwapień for discussions and encouragement. I would also like to thank the referees for careful reading of the first version of the paper.

References

1. Burkholder, D.L.: Martingale transforms. Ann. Math. Statist. **37**, 1494–1504 (1966)
2. Burkholder, D.L.: Sharp inequalities for martingales and stochastic integrals. Astérisque , 75–94 (1988)
3. Fack, T., Kosaki, H.: Generalized s-numbers of τ-measurable operators. Pacific J. Math. **123**(2), 269–300 (1986)
4. Junge, M.: Doob's inequality for noncommutative martingales. J. Reine Angew. Math **549**, 149–190 (2002)
5. Junge, M., Musat, M.: A noncommutative version of the John–Nirenberg Theorem. Trans. Am. Math. Soc. **359**(1), 115–142 (2007)
6. Junge, M., Xu, Q.: Noncommutative Burkholder/Rosenthal inequalities. Ann. Probab. **31**(2), 948–995 (2003)
7. Musat, M.: Interpolation between noncommutative BMO and noncommutative L^p spaces. J. Funct. Anal. **202**, 195–225 (2003)
8. Parcet, J., Randrianantoanina, N.: Gundy's decomposition for noncommutative martingales and applications. Proc. Lond. Math. Soc. (3) **93**(1), 227–252 (2006)
9. Pisier, G., Xu, Q.: Noncommutative martingale inequalities. Commun. Math. Phys. **189**, 667–698 (1997)
10. Pisier, G., Xu, Q.: Noncommutative L_p-spaces. In: Johnson, W.B., Lindenstrauss, J. (eds.) Handbook of the Geometry of Banach Spaces II. North Holland, Amsterdam, pp. 1459–1517 (2003)
11. Randrianantoanina, N.: Noncommutative martingale transforms. J. Funct. Anal. **194**, 181–212 (2002)
12. Randianantoanina, N.: Square function inequalities for noncommutative martingales. Israel J. Math. **140**, 333–365 (2004)
13. Randrianantoanina, N.: A weak-type inequality for noncommutative martingales and applications. Proc. Lond. Math. Soc. **91**(3), 509–544 (2005)
14. Randrianantoanina, N.: Conditioned square functions for noncommutative martingales, Preprint
15. Takesaki, M.: Theory of operator algebras I. Springer, New York (1979)
16. Xu, Q.: Recent development on noncommutative martingale inequalities. Functional Space Theory and its Applications. In: Proceedings of International Conference & 13th Academic Symposium in China, Wuhan, Research Information Ltd, UK, pp. 283–314 (2003)