WEAK-L^∞ INEQUALITY FOR NON-SYMMETRIC MARTINGALE TRANSFORMS AND HAAR SYSTEM

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ABSTRACT. Let $b < B$ be two real numbers. Suppose that $f = (f_n)_{n\geq 0}$ and $g = (g_n)_{n\geq 0}$ are two Hilbert-space-valued martingales satisfying

$$\left| dg_n - \frac{B+b}{2} df_n \right| \leq \left| \frac{B-b}{2} df_n \right|, \qquad n = 0, 1, 2, \ldots .$$

The paper contains the proof of the sharp weak-type inequality

$$\|g\|_{W(\Omega)} \leq 2 \max(-b, B) \|f\|_{L^\infty},$$

where W is the weak-L^∞ space introduced by Bennett, DeVore and Sharpley. As applications, we obtain related estimates for the Haar system and harmonic functions on Euclidean domains.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a non-atomic probability space, filtered by $(\mathcal{F}_n)_{n\geq 0}$, a non-decreasing family of sub-σ-fields of $\mathcal{F}$. Assume that $f = (f_n)_{n\geq 0}$, $g = (g_n)_{n\geq 0}$ are two martingales taking values in some separable Hilbert space $(\mathcal{H}, |\cdot|)$. Let $df = (df_n)_{n\geq 0}$, $dg = (dg_n)_{n\geq 0}$ denote the difference sequences of f and g, given by

$$df_0 = f_0 \quad \text{and} \quad df_n = f_n - f_{n-1}, \quad n = 1, 2, \ldots,$$

with a similar definition for dg. Then g is the transform of f by a predictable real-valued sequence $v = (v_n)_{n\geq 0}$, if for any n we have the identity $dg_n = v_n df_n$ almost surely. (Here by predictability of v we mean that for any n, the term v_n is measurable with respect to $\mathcal{F}_{(n-1)\vee 0}$.) A celebrated result of Burkholder [4, 5], asserts that if v is bounded in absolute value by 1, then we have the sharp strong-type inequality

$$\|g\|_p \leq \max\{p-1, (p-1)^{-1}\} \|f\|_p, \qquad 1 < p < \infty. \tag{1.1}$$

Here we have used the notation $\|f\|_p = \sup_n \|f_n\|_p$ for the p-th norm of f. In the boundary case $p = 1$ the above moment inequality does not hold with any finite constant, but we have the corresponding sharp weak-type bound

$$\|g\|_{1,\infty} \leq 2\|f\|_1, \tag{1.2}$$

where $\|g\|_{1,\infty} = \sup_{n\geq 0} \|g_n\|_{1,\infty} = \sup_{n\geq 0} \sup_{\lambda>0} \lambda \mathbb{P}(|g_n| \geq \lambda)$. These estimates were motivated by various questions concerning the properties of the Haar system $(h_n)_{n\geq 0}$, an important basis of $L^p(0,1)$, $1 \leq p < \infty$. Recall that this functional

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sequence is given by $h_0 = \chi_{[0,1)}$, $h_1 = \chi_{[0,1/2)} - \chi_{[1/2,1)}$, $h_2 = \chi_{[0,1/4)} - \chi_{[1/4,1/2)}$, $h_3 = \chi_{[1/2,3/4)} - \chi_{[3/4,1)}$, and so on. A celebrated result of Marcinkiewicz asserts that the Haar basis is unconditional in L^p, $1 < p < \infty$: there is a finite constant c_p such that for any $a_0, a_1, a_2, \ldots \in \mathbb{R}$ and any $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots \in \{-1, 1\}$ we have

$$\left\| \sum_{n=0}^{\infty} \varepsilon_n a_n h_n \right\|_{L^p(0,1)} \leq c_p \left\| \sum_{n=0}^{\infty} a_n h_n \right\|_{L^p(0,1)}.$$

Furthermore, in the boundary case $p = 1$ we have the weak-type bound

$$\left\| \sum_{n=0}^{\infty} \varepsilon_n a_n h_n \right\|_{L^{1,\infty}(0,1)} \leq c_1 \left\| \sum_{n=0}^{\infty} a_n h_n \right\|_{L^1(0,1)}.$$

Now, note that $(a_n h_n)_{n\geq 0}$, treated as a collection of random variables on the probability space $([0,1), \mathcal{B}(0,1), |\cdot|)$ with the dyadic filtration, becomes a martingale difference sequence and hence the above two estimates follow immediately from (1.1) and (1.2). Actually, as Burkholder proved in [5], the constants $\max\{p-1, (p-1)^{-1}\}$ and 2 remain optimal in the context of the Haar system.

The inequalities (1.1) and (1.2) have been extended in many directions and applied in various contexts of probability and harmonic analysis. The literature on this subject is very large, so we will only discuss a few results closely related to the contribution of this paper. First, the estimates were generalized to martingales satisfying the less restrictive condition of differential subordination, the definition of which we now recall. Following Burkholder [5], we say that g is differentially subordinate to f, if for any $n \geq 0$ we have the almost sure bound $|dg_n| \leq |df_n|$. Clearly, this condition is satisfied if g is the transform of f by some predictable sequence with values in $[-1, 1]$, but there are other important examples which, in turn, lead to deep results in the theory of Fourier multipliers (cf. [2]). Another extension of (1.1) and (1.2) concerns the case in which the transforming sequence $(v_n)_{n\geq 0}$ takes values in some fixed interval $[b, B]$ (cf. [2, 6, 9]). There is a corresponding less restrictive non-symmetric version of the differential subordination, which reads

$$\left| dg_n - \frac{B+b}{2} df_n \right| \leq \left| \frac{B-b}{2} df_n \right|, \qquad n = 0, 1, 2, \ldots. \tag{1.3}$$

(That is, $g - \frac{B+b}{2} f$ is differentially subordinate to $\frac{B-b}{2} f$). If $b = 0$ and $B = 1$, the above condition takes the more transparent form

$$|dg_n|^2 \leq df_n \cdot dg_n, \quad n = 0, 1, 2, \ldots.$$

The sharp versions of (1.1) and (1.2) under the domination (1.3) can be found in [2, 6, 9, 10].

Our purpose is to study another limiting case of the moment inequality (1.1), namely, an appropriate weak-type estimate for $p = \infty$, under the non-symmetric differential subordination. To describe the result precisely, we need to recall the notion of a weak-L^∞ space W, originally introduced by Bennett, DeVore and Sharpley in [3]. For a measurable function h on some measure space $(\mathcal{M}, \mathcal{G}, \mu)$, let $h^* : (0, \mu(\mathcal{M})) \to [0, \infty)$ stand for its decreasing rearrangement, defined by

$$h^*(t) = \inf\{\lambda \geq 0 : \ \mu(|h| > \lambda) \leq t\}, \qquad t > 0.$$

Then h^{**}, the maximal function of h^*, is given by

$$h^{**}(t) = \frac{1}{t}\int_0^t h^*(s)\,\mathrm{d}s\,, \qquad t \in (0, \mu(\mathcal{M})).$$

It is not difficult to show that h^{**} can be alternatively defined by

$$h^{**}(t) = \sup\left\{\frac{1}{\mu(E)}\int_E |h|\,\mathrm{d}\mu \ : \ E \in \mathcal{G},\ \mu(E) = t\right\}.$$

Now we introduce the weak-L^∞ space $W = W(\mathcal{M}, \mathcal{G}, \mu)$ by

$$W = \left\{h \ : \ \|h\|_{W(\mathcal{M},\mathcal{G},\mu)} = \sup_{t \in (0,\mu(\mathcal{M}))} (h^{**}(t) - h^*(t)) < \infty\right\}.$$

This space has several important properties which explain why it can be regarded as a weak version of L^∞. First, note that the classical definition of the Lorentz space $L^{p,\infty}$ does not extend to the case $p = \infty$ (in the function spaces theory, one sets $L^{\infty,\infty} = L^\infty$) and hence there seem to be no Marcinkiewicz interpolation theorem between L^1 and L^∞ for operators which are unbounded on L^∞. The introduction of the space W enables to fill this gap. Namely, we have $L^\infty \subset W$, and if an operator A is bounded from L^1 to $L^{1,\infty}$ and from L^∞ to W, then for any $1 < p < \infty$ it has an extension bounded on L^p. There are further close connections between W and the space BMO. For more detailed discussion on W and its interplay with the interpolation theory, see [3].

Equipped with the above definition, we return to the martingale setup and state the appropriate weak-L^∞ bound. It was proved in [11] that if g is differentially subordinate to f, then we have

$$\|g\|_W \leq 2\|f\|_\infty$$

and the constant 2 cannot be improved. Our contribution is the extension of this estimate to the context of the non-symmetric differential subordination.

Theorem 1.1. *Let $b < B$ be fixed real numbers. Then for any Hilbert-space-valued martingales f, g satisfying (1.3) we have the estimate*

$$\|g\|_{W(\Omega)} \leq 2\max(-b, B)\|f\|_{L^\infty}. \tag{1.4}$$

If $b < 0 < B$, then the constant is optimal, already in the context of the Haar system: for any $c < 2\max(-b, B)$ there is a sequence a_0, a_1, a_2, $\ldots$ of real numbers and a sequence ε_0, ε_1, ε_2, $\ldots$ with values in $\{b, B\}$ such that

$$\left\|\sum_{n=0}^\infty \varepsilon_n a_n h_n\right\|_{W(0,1)} > c\left\|\sum_{n=0}^\infty a_n h_n\right\|_{L^\infty(0,1)}.$$

This result will be established in the next section. The final part of the paper contains some applications to harmonic functions on Euclidean domains.

2. Proof of Theorem 1.1

For the sake of clarity, we split the contents of this section into two parts.

2.1. **Special functions and their properties.** Let $M = \max(-b, B)$ and distinguish the strip

$$S = \{(x, y) \in H \times H : |x| \le 1\}.$$

For any $\lambda \ge 0$, consider the functions V_λ and $U_\lambda : S \to \mathbb{R}$ given by

$$V_\lambda(x,y) = \left(\frac{|y|}{M} - \lambda - 2\right)\chi_{\left\{\frac{|y|}{M} > \lambda\right\}}$$

and

$$U_\lambda(x,y) \\ = \left[\frac{M}{B-b}\left(\left|\frac{y}{M} - \frac{B+b}{2M}x\right| - \lambda - 1\right)^2 - \frac{B-b}{4M}|x|^2\right]\chi_{\left\{\frac{B-b}{2M}|x| + \left|\frac{y}{M} - \frac{B+b}{2M}x\right| > \lambda+1\right\}}.$$

In the two lemmas below, we study certain crucial properties of U_λ and V_λ.

Lemma 2.1. *For any $\lambda \ge 0$ we have the majorization*

$$U_\lambda \ge V_\lambda. \tag{2.1}$$

Proof. We consider three cases. First, if $\frac{|y|}{M} \le \lambda$, then $\frac{B-b}{2M}|x| + \left|\frac{y}{M} - \frac{B+b}{2M}x\right| \le \lambda + 1$, so $U_\lambda(x,y) = V_\lambda(x,y) = 0$. The next case is described by the conditions $\frac{|y|}{M} > \lambda$ and $\frac{B-b}{2M}|x| + \left|\frac{y}{M} - \frac{B+b}{2M}x\right| \le \lambda + 1$: then we have $U_\lambda(x,y) = 0$ and $V_\lambda(x,y) = \frac{|y|}{M} - \lambda - 2 \le 0$, so the majorization is also satisfied. Finally, suppose that $\frac{|y|}{M} > \lambda$ and $\frac{B-b}{2M}|x| + \left|\frac{y}{M} - \frac{B+b}{2M}x\right| > \lambda + 1$. Then

$$U_\lambda(x,y) - V_\lambda(x,y) \ge \frac{M}{B-b}\left(\left|\frac{y}{M} - \frac{B+b}{2M}x\right| - \lambda - 1 - \frac{B-b}{2M}\right)^2 \ge 0,$$

and the proof is complete. □

To study further properties of U_λ, we need to introduce auxiliary functions $\Phi_\lambda : S \to \mathbb{R}$ and A_λ, $B_\lambda : S \to \mathcal{H}$ given by

$$\Phi_\lambda(x,y) = \left[\frac{1}{2}(|y| - \lambda - 1)^2 - \frac{1}{2}|x|^2\right]\chi_{\{|x|+|y|>\lambda+1\}},$$
$$A_\lambda(x,y) = -x\chi_{\{|x|+|y|>\lambda+1\}},$$
$$B_\lambda(x,y) = \left[y - (\lambda+1)\frac{y}{|y|}\right]\chi_{\{|x|+|y|>\lambda+1\}}.$$

Lemma 2.2. *For any $x, y, h, k \in S$ with $|k| \le |h|$, we have*

$$\Phi_\lambda(x+h, y+k) \le \Phi_\lambda(x,y) + A_\lambda(x,y)\cdot h + B_\lambda(x,y)\cdot k. \tag{2.2}$$

Proof. First, notice that $\Phi_\lambda(x,y)\leq \frac{1}{2}(|y|-\lambda-1)^2-\frac{1}{2}|x|^2$. Therefore, if $|x|+|y|>\lambda+1$, then

$$\begin{aligned}
&\Phi_\lambda(x+h,y+k)\\
&\leq \frac{1}{2}(|y+k|-\lambda-1)^2-\frac{1}{2}|x+h|^2\\
&=\frac{1}{2}(|y|-\lambda-1)^2-\frac{1}{2}|x|^2-\Big[-y\cdot k+(\lambda+1)|y+k|+x\cdot h-(\lambda+1)|y|\Big]\\
&\quad+\frac{1}{2}|k|^2-\frac{1}{2}|h|^2\\
&\leq \Phi_\lambda(x,y)+A_\lambda(x,y)\cdot h+B_\lambda(x,y)\cdot k.
\end{aligned}$$

If $|x|+|y|\leq\lambda+1$ and $|x+h|+|y+k|\leq\lambda+1$, then (2.2) is obvious since both sides are equal to 0. Finally, if $|x|+|y|\leq\lambda+1$ and $|x+h|+|y+k|>\lambda+1$, we distinguish two cases. If $|y+k|\leq\lambda+1$, then $0\geq|y+k|-\lambda-1>-|x+h|$, so $(|y+k|-\lambda-1)^2-|x+h|^2<0$ and (2.2) follows. On the other hand, if $|y+k|>\lambda+1$, then $|x|<\lambda+1-|y|<-|y|+|y+k|\leq|k|$, and

$$(|y+k|-\lambda-1)^2\leq(|y|+|k|-\lambda-1)^2\leq(|k|-|x|)^2\leq(|h|-|x|)^2\leq(|x+h|)^2.$$

This yields the desired claim. □

Lemma 2.3. *Suppose that martingales f, g satisfy the condition (1.3). Then for any $n\geq 0$ we have*

$$\mathbb{E}U_\lambda(f_n,g_n)\leq 0. \tag{2.3}$$

Proof. First, notice that

$$U_\lambda(x,y)=\frac{2M}{B-b}\Phi_\lambda\left(\frac{B-b}{2M}x,\frac{y}{M}-\frac{B+b}{2M}x\right).$$

Hence, in order to show (2.3), it suffices to prove that for any n we have

$$\mathbb{E}\Phi_\lambda\left(\frac{B-b}{2M}f_n,\frac{g_n}{M}-\frac{B+b}{2M}f_n\right)\leq 0. \tag{2.4}$$

To get this, we apply (2.2) with $x=\frac{B-b}{2M}f_{n-1}$, $y=\frac{g_{n-1}}{M}-\frac{B+b}{2M}f_{n-1}$, $h=\frac{B-b}{2M}df_n$ and $k=\frac{dg_n}{M}-\frac{B+b}{2M}df_n$ to obtain

$$\begin{aligned}
\Phi_\lambda\left(\frac{B-b}{2M}f_n,\frac{g_n}{M}-\frac{B+b}{2M}f_n\right)&\leq\Phi_\lambda\left(\frac{B-b}{2M}f_{n-1},\frac{g_{n-1}}{M}-\frac{B+b}{2M}f_{n-1}\right)\\
&\quad+A_\lambda\left(\frac{B-b}{2M}f_{n-1},\frac{g_{n-1}}{M}-\frac{B+b}{2M}f_{n-1}\right)\cdot h\\
&\quad+B_\lambda\left(\frac{B-b}{2M}f_{n-1},\frac{g_{n-1}}{M}-\frac{B+b}{2M}f_{n-1}\right)\cdot k
\end{aligned}$$

Both sides are integrable, and taking the conditional expectation with respect to $\mathcal{F}_{n-1}$ yields

$$\mathbb{E}\left(\Phi_\lambda\left(\frac{B-b}{2M}f_n,\frac{g_n}{M}-\frac{B+b}{2M}f_n\right)|\mathcal{F}_{n-1}\right)\leq\Phi_\lambda\left(\frac{B-b}{2M}f_{n-1},\frac{g_{n-1}}{M}-\frac{B+b}{2M}f_{n-1}\right).$$

This implies

$$\mathbb{E}\left(\Phi_\lambda\left(\frac{B-b}{2M}f_n, \frac{g_n}{M}-\frac{B+b}{2M}f_n\right)\right) \leq \mathbb{E}\Phi_\lambda\left(\frac{B-b}{2M}f_0, \frac{g_0}{M}-\frac{B+b}{2M}f_0\right) \leq 0,$$

where the latter estimate follows from (2.2) applied to $x=y=0$, $k=\frac{g_0}{M}-\frac{B+b}{2M}f_0$, and $h=\frac{B-b}{2M}f_0$. The proof is complete. □

Observe that the combination of (2.1) and (2.3) gives the estimate

$$\mathbb{E}\left(\frac{|g_n|}{M}-\lambda-2\right)\chi_{\left\{\frac{|g_n|}{M}>\lambda\right\}} \leq 0 \tag{2.5}$$

for all nonnegative integers n.

2.2. **Proof of the main result.** We turn to Theorem 1.1. We will first establish the weak type estimate, and then show its sharpness in the context of the Haar system.

Proof of (1.4). By homogeneity, we may assume that $\|f\|_{L^\infty} \leq 1$. Pick an arbitrary nonnegative integer n, a parameter $t \in (0,1]$ and recall the alternative definition of $\left(\frac{g_n}{M}\right)^{**}$:

$$\left(\frac{g_n}{M}\right)^{**}(t) = \sup\left\{\frac{1}{\mathbb{P}(E)}\int_E \frac{|g_n|}{M}\mathrm{d}\mathbb{P} \,:\, E\in\mathcal{F},\, \mathbb{P}(E)=t\right\}.$$

In particular, we see that

$$\left(\frac{g_n}{M}\right)^{**}(t)-\left(\frac{g_n}{M}\right)^{*}(t) = \sup\left\{\frac{1}{\mathbb{P}(E)}\mathbb{E}\left(\frac{|g_n|}{M}-\left(\frac{g_n}{M}\right)^*(t)\right)\chi_E \,:\, \mathbb{P}(E)=t\right\}.$$

By the definition of the decreasing rearrangement, we have $\mathbb{P}\left(|g_n|/M>\lambda\right)>t$ if $\lambda<(g_n/M)^*(t)$, and $\mathbb{P}\left(|g_n|/M>\lambda\right)\leq t$ if $\lambda>(g_n/M)^*(t)$. Therefore, we obtain the double estimate

$$\mathbb{P}\left(\frac{|g_n|}{M}\geq\left(\frac{g_n}{M}\right)^*(t)\right)\geq t\geq \mathbb{P}\left(\frac{|g_n|}{M}>\left(\frac{g_n}{M}\right)^*(t)\right).$$

Consequently, for any event E of probability t we get

$$\begin{aligned}\frac{1}{\mathbb{P}(E)}&\mathbb{E}\left(\frac{|g_n|}{M}-\left(\frac{g_n}{M}\right)^*(t)\right)\chi_E\\ &\leq \frac{1}{\mathbb{P}\left(\frac{|g_n|}{M}>\left(\frac{g_n}{M}\right)^*(t)\right)}\mathbb{E}\left(\frac{|g_n|}{M}-\left(\frac{g_n}{M}\right)^*(t)\right)\chi_{\left\{\frac{|g_n|}{M}>\left(\frac{g_n}{M}\right)^*(t)\right\}},\end{aligned}$$

which, by (2.5) applied to $\lambda=\left(\frac{g_n}{M}\right)^*(t)$, does not exceed 2. Taking the supremum over all E as above, we get $\|g/M\|_W \leq 2\|f\|_{L^\infty}$, which is the desired estimate. □

Sharpness for the Haar system. Fix $b<0<B$. We consider separately two possibilities.

Case 1: $\max(-b,B)=-b$. Introduce the function $f:[0,1)\to\{-1,1\}$ given by $f=-\chi_{[0,(1-a)/2)}+\chi_{[(1-a)/2,1)}$, where $a=-b/(B-b)$. Note that $\int_0^1 f=a$. Furthermore, f belongs to L^1, so it can be expanded into Haar series: there exist a_1, a_2, $\ldots\in\mathbb{R}$ such that $f=ah_0+a_1h_1+a_2h_2+\ldots$. Let $g=Bah_0+$

$ba_1h_1 + ba_2h_2 + \ldots = Ba + (f-a)b$: it is a transform of f by a sequence with values in $\{b, B\}$. It is easy to check that $g = -2b\chi_{[0,(1-a)/2)}$, by the above choice of the parameter a, and hence we have $g^*(t) = -2b\chi_{[0,(1-a)/2)}$ and $g^{**}(t) = -2b\chi_{[0,(1-a)/2]} - \frac{b(1-a)}{t}\chi_{((1-a)/2,1]}$. Consequently, we see that

$$\|g\|_{W(0,1)} \geq \lim_{t\downarrow(1-a)/2}(g^{**}(t) - g^*(t)) = -2b = -2b\|f\|_{L^\infty}.$$

Case 2: $\max(-b, B) = B$. Here the argumentation is similar to that above. We let $a = -B/(B-b)$ and consider the function $f = \chi_{[0,(1-a)/2)} - \chi_{[(1-a)/2,1)}$ of integral a and its transform $g = bah_0 + Ba_1h_1 + Ba_2h_2 + \ldots = ba + (f-a)B$ by a sequence with values in $\{b, B\}$. One easily verifies that $g = 2B\chi_{[0,(1-a)/2)}$, $g^*(t) = 2B\chi_{[0,(1-a)/2)}$ and $g^{**}(t) = 2B\chi_{[0,(1-a)/2]} + \frac{B(1-a)}{t}\chi_{((1-a)/2,1]}$. Thus we obtain

$$\|g\|_{W(0,1)} \geq \lim_{t\downarrow(1-a)/2}(g^{**}(t) - g^*(t)) = 2B = 2B\|f\|_{L^\infty}.$$

This establishes the desired sharpness. A different proof will be presented in the next section. □

3. Inequalities for harmonic functions

Now we will prove a version of Theorem 1.1 in the context of harmonic functions on Euclidean domains. Suppose that n is a positive integer and let D be an open connected subset of $\mathbb{R}^n$. Fix a base point ξ belonging to D and let $b < B$ be fixed real numbers. Assume further that two real-valued harmonic functions u, v on D satisfy the conditions

$$\left|v(\xi) - \frac{B+b}{2}u(\xi)\right| \leq \left|\frac{B-b}{2}u(\xi)\right| \tag{3.1}$$

and

$$\left|\nabla v(x) - \frac{B+b}{2}\nabla u(x)\right| \leq \left|\frac{B-b}{2}\nabla u(x)\right| \quad \text{for any } x \in D. \tag{3.2}$$

If $b = -1$ and $B = 1$, then (3.1) and (3.2) reduce to the differential subordination of harmonic functions introduced by Burkholder [8]. The general case $b < B$ can be considered as a non-symmetric version of this domination; it should also be compared with the condition (1.3) above.

Next, let D_0 be a bounded domain satisfying $\xi \in D_0 \subset D_0 \cup \partial D_0 \subset D$ and let $\mu^\xi_{D_0}$ stand for the harmonic measure on ∂D_0 corresponding to ξ. The weak-L^∞ norm of the function u is given by

$$||u||_{W(D)} = \sup_{D_0} \sup_{t\in(0,1]} (u^{**}_{D_0}(t) - u^*_{D_0}(t)),$$

where u_{D_0} is the restriction of u to D_0, and $u^*_{D_0}$, $u^{**}_{D_0}$ are the decreasing rearrangement and the associated maximal function of u_{D_0} with respect to the measure $\mu^\xi_{D_0}$. The harmonic analogue of Theorem 1.1 is the following.

Theorem 3.1. *Let $b < B$ be fixed numbers. If u, v satisfy (3.1) and (3.2), then*

$$||v||_{W(D)} \leq 2\max\{-b, B\}||u||_{L^\infty(D)}. \tag{3.3}$$

If $b < 0 < B$, then the constant is the best possible.

Proof. We may assume that $||u||_{L^\infty(D)} \leq 1$. Let D_0 be an arbitrary subdomain of D as above and let $\lambda > 0$ be a fixed parameter. It is easy to check by a direct differentiation that the function $U_\lambda(u, v)$ is superharmonic and hence, by (2.1), we obtain

$$\int_{\partial D_0} \left(\frac{|v(x)|}{M} - \lambda - 2 \right) \chi_{\{|v(x)|/M > \lambda\}} \mathrm{d}\mu^\xi_{D_0} \leq 0,$$

where, as above, $M = \max\{-b, B\}$. Now we repeat the argumentation from the proof of (1.4), replacing Ω , $\mathbb{P}$ by ∂D_0 and $\mu^\xi_{D_0}$, respectively. As the result, we obtain

$$\sup_{t\in(0,1]} (v^{**}_{D_0}(t) - v^*_{D_0}(t)) \leq 2M||u||_{L^\infty},$$

which is the desired estimate, since D_0 was chosen arbitrarily. It remains to show that the constant $2M$ is optimal. This will be accomplished by the construction of an appropriate example on $\mathbb{R}$. We will provide the details only in the case $\max\{-b, B\} = -b$, the other possibility is handled similarly. Consider the interval $D = (-b/(B - b) - 1, -b/(B - b) + 1)$, let $\xi = 0$ and define u, $v : D \to \mathbb{R}$ by

$$u(x) = -x - \frac{b}{B - b}, \qquad v(x) = -bx - \frac{Bb}{B - b}.$$

Then $u(0) = -b/(B - b)$ and $v(0) = -Bb/(B - b)$, so equality holds in (3.1). Furthermore, we have $\nabla u(x) = -1$ and $\nabla v(x) = -b$ for all $x \in D$, so both sides of (3.2) are equal for any x. Furthermore, it is easy to see that $||u||_{L^\infty} = 1$. To provide an appropriate lower bound for the weak norm of v, pick a subinterval $D_0 = aD$, where $a \in (0, 1)$ is a fixed parameter. Then the harmonic measure $\mu^\xi_{D_0}$ on $\partial D_0 = \left\{a\left(-\frac{b}{B-b} - 1\right), a\left(-\frac{b}{B-b} + 1\right)\right\} = \{a_-, a_+\}$ is given by

$$\mu^\xi_{D_0}(\{a_-\}) = \frac{1}{2}\left(1 - \frac{b}{B - b}\right), \qquad \mu^\xi_{D_0}(\{a_+\}) = \frac{1}{2}\left(1 + \frac{b}{B - b}\right).$$

Now, $v(a_-) = -ba_- - \frac{Bb}{B-b}$ and $v(a_+) = -ba_+ - \frac{Bb}{B-b} > v(a_-) > 0$, which implies

$$v^*_{D_0}(t) = \begin{cases} v(a_+) & \text{if } t \leq t_0, \\ v(a_-) & \text{if } t > t_0, \end{cases}$$

where $t_0 = \frac{1}{2}\left(1 + \frac{b}{B-b}\right)$, and

$$v^{**}_{D_0}(t) = \begin{cases} v(a_+) & \text{if } t \leq t_0, \\ \frac{1}{t}\left(t_0 v(a_+) + v(a_-)\,(t - t_0)\right) & \text{if } t > t_0. \end{cases}$$

Therefore, we obtain

$$\lim_{t\downarrow t_0} \left(v^{**}_{D_0}(t) - v^*_{D_0}(t)\right) = v(a_+) - v(a_-) = -b(a_+ - a_-) = -2ab,$$

so $||v||_{W(D)}/||u||_{L^\infty(D)} \geq -2ab$. Since $a \in (0, 1)$ was arbitrary, the constant $-2b$ in (3.3) is indeed the best possible. □

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