SHARP INEQUALITIES FOR THE DYADIC SQUARE FUNCTION IN THE BMO SETTING

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Abstract. We introduce a method for the simultaneous study of a BMO function φ and its dyadic square function $S(\varphi)$ that can yield sharp norm inequalities between the two. One of the applications is the sharp bound for the p-th moment of $S(\varphi)$, $0 < p < \infty$, which in turn implies the square-exponential integrability of the square function. We also present sharp refinements of these inequalities in the more restrictive case when φ is assumed to be bounded.

1. Introduction

Square function inequalities play an important role in both classical and noncommutative probability theory, harmonic analysis, potential theory and many other areas of mathematics. The purpose of this paper is to establish sharp bounds in the dyadic case, which are closely related to the works of Bollobás [2], Davis [3], John and Nirenberg [6], Littlewood [7], Marcinkiewicz [8], Paley [10], Slavin and Vasyunin [15], Wang [16] and many others.

Let us start with introducing some background and notation. In what follows, $\mathcal{I} = [0, 1]$ and $\mathcal{D}$ stands for the collection of all dyadic subintervals of $\mathcal{I}$. Let $(h_n)_{n\geq 0}$ be the Haar system on $[0, 1]$:

$$h_0 = \chi_{[0,1]}, \qquad h_1 = \chi_{[0,1/2)} - \chi_{[1/2,1)},$$
$$h_2 = \chi_{[0,1/4)} - \chi_{[1/4,1/2)}, \qquad h_3 = \chi_{[1/2,3/4)} - \chi_{[3/4,1)},$$
$$h_4 = \chi_{[0,1/8)} - \chi_{[1/8,1/4)}, \qquad h_5 = \chi_{[1/4,3/8)} - \chi_{[3/8,1/2)},$$

and so on. Let $\mathcal{H}$ be a separable Hilbert space with scalar product $\cdot$ and norm $|\cdot|$. For any $I \in \mathcal{D}$ and an integrable function $\varphi : \mathcal{I} \to \mathcal{H}$, we will write $\langle\varphi\rangle_I$ for the average of φ over I: that is, $\langle\varphi\rangle_I = \frac{1}{|I|}\int_I \varphi$ (throughout, unless stated otherwise, the integration is with respect to Lebesgue's measure). Furthermore, for any such φ and any nonnegative integer n, we use the notation

$$\varphi_n = \sum_{k=0}^{n} \frac{1}{|I_k|}\int_{\mathcal{I}} \varphi(s)h_k(s)\mathrm{d}s\, h_k$$

for the projection of φ on the subspace generated by the first $n + 1$ Haar functions (I_k is the support of h_k). We define the dyadic square function of φ by the formula

$$S(\varphi)(x) = \left(\sum \left|\frac{1}{|I_n|}\int_{\mathcal{I}} \varphi(s)h_n(s)\mathrm{d}s\right|^2\right)^{1/2},$$

2000 *Mathematics Subject Classification.* Primary: 42A05, 42B35. Secondary: 49K20.
Key words and phrases. Square function, dyadic, BMO, best constants.
Partially supported by Polish Ministry of Science and Higher Education (MNiSW) grant N N201 364436.

where the summation runs over all nonnegative integers n such that $x \in I_n$.

The inequalities comparing the sizes of φ and its square function $S(\varphi)$ are of importance in analysis and have interested many mathematicians. A classical result of Paley [10] and Marcinkiewicz [8] states that there are finite absolute constants c_p $(0 < p < \infty)$ and C_p $(1 < p < \infty)$, such that for any $\varphi : \mathcal{I} \to \mathbb{R}$,

$$||\varphi||_p \leq c_p ||S(\varphi)||_p \tag{1.1}$$

and

$$||S(\varphi)||_p \leq C_p ||\varphi||_p. \tag{1.2}$$

The question about the optimal values of c_p and C_p (still in the real-valued setting) was studied by Davis [3]. For $0 < p < \infty$, let ν_p denote the smallest positive zero of a confluent hypergeometric function M_p and let μ_p be the largest positive zero of the parabolic cylinder function of order p (see Abramovitz and Stegun [1] for details). Using a related estimate for continuous-time martingales and Skorokhod embedding theorems, Davis [3] showed that if $0 < p \leq 2$, then the best choice for c_p is ν_p, while for $p \geq 2$, the optimal value of C_p is ν_p^{-1}. This result was taken up by Wang [16], who studied (1.1) and (1.2) for Hilbert-space-valued functions: he proved that Davis' constants are the best in this setting and also showed that when $p \geq 3$, then the optimal choice for c_p is μ_p. In the remaining cases the best constants are not known. For $p = 1$ the inequality (1.2) fails to hold, but there is a related weak-type estimate

$$\lambda \big| \{ S(\varphi) \geq \lambda \} \big| \leq K ||\varphi||_1, \qquad \lambda > 0,$$

due to Bollobás [2], where $K = 1,463\ldots$. This constant is optimal, see [9].

We will study related sharp bounds for the dyadic square function, but with the particular emphasis put on φ with bounded mean oscillation. Let

$$\mathrm{BMO} = \big\{ \varphi : \mathcal{I} \to \mathcal{H} : \langle |\varphi - \langle \varphi \rangle_I|^2 \rangle_I < \infty \text{ for every } I \in \mathcal{D} \big\},$$

and for any $\varphi \in BMO$, define the corresponding norm

$$||\varphi||_{BMO} = \inf_{I \in \mathcal{D}} \big(\langle |\varphi - \langle \varphi \rangle_I|^2 \rangle_I \big)^{1/2}.$$

It is well-known that any function $\varphi \in BMO$ has very strong integrability properties, see the classical paper of John and Nirenberg [6], consult also the recent work of Slavin and Vasyunin [15]. In particular, p-th norms $\langle |\varphi|^p \rangle_{\mathcal{I}}$, as well as exponential expressions $\langle e^{c\varphi} \rangle_{\mathcal{I}}$ (for c sufficiently small), are comparable to $\langle \varphi \rangle_{\mathcal{I}}$ (in the sense given below). We will introduce a Bellman-type method which enables the study of sharp bounds for various norms of $S(\varphi)$ under the assumption that φ belongs to BMO. In particular, this method will lead us to the following statements. For any $0 < p < \infty$ and $x \geq 0$, let

$$C_p(x) = \begin{cases} (x^2+1)^{1/2} & \text{if } p < 2, \\ \left[e^{x^2} \int_{x^2}^\infty e^{-s} s^{p/2} \mathrm{d}s \right]^{1/p} & \text{if } p \geq 2. \end{cases}$$

Theorem 1.1. *Suppose that $\varphi : \mathcal{I} \to \mathcal{H}$ satisfies $||\varphi||_{BMO} \leq 1$. Then for any $0 < p < \infty$,*

$$||S(\varphi)||_p \leq C_p(|\langle \varphi \rangle_{\mathcal{I}}|) \tag{1.3}$$

and the inequality is sharp, even if $\mathcal{H} = \mathbb{R}$.

Here by sharpness we mean that for any $x \in \mathbb{R}$ and any $c < C_p(|x|)$, there is a function $\varphi : \mathcal{I} \to \mathbb{R}$ such that $\langle\varphi\rangle_{\mathcal{I}} = x$ and $||S(\varphi)||_p > c$. The above theorem yields the following square-exponential integrability (for an alternative proof in the real-valued setting, see e.g. Garsia [5]).

Theorem 1.2. *Suppose that* $\varphi : \mathcal{I} \to \mathcal{H}$ *satisfies* $||\varphi||_{BMO} \leq 1$. *Then for any* $0 < c < 1$,

$$\mathbb{E}e^{cS^2(\varphi)} \leq (1-c)^{-1}e^{c\langle\varphi\rangle_{\mathcal{I}}^2} \tag{1.4}$$

and the inequality is sharp, even if $\mathcal{H} = \mathbb{R}$.

Observe that in the both results above, the upper bounds depend on the function φ. This dependence cannot be removed, since the average $\langle\varphi\rangle_{\mathcal{I}}$ can take arbitrarily large values. By standard scaling, Theorems 1.1 and 1.2 can be formulated for arbitrary BMO functions, not necessarily satisfying $||\varphi||_{BMO} \leq 1$.

In the second part of the paper we study the special case $\varphi \in L^\infty$. Clearly, if $||\varphi||_\infty \leq 1$, then $||\varphi||_{BMO} \leq 2$ and the above theorems (applied to $\varphi/2$) yield L^p and square-exponential inequalities for $S(\varphi)$. Since $|\langle\varphi\rangle_{\mathcal{I}}| \leq 1$, one can choose the upper bounds which depend only on p or c. However, this approach does not produce the optimal inequalities and our next contribution will be to provide sharp versions of these. Define $K_p = 1$ for $0 < p < 2$ and

$$K_p^p = \frac{\Gamma(p+1)2^{p/2-1}}{\Gamma\left(\frac{p+1}{2}\right)\pi^{p-3/2}} \cdot \frac{1 - \frac{1}{3^{p+1}} + \frac{1}{5^{p+1}} - \frac{1}{7^{p+1}} + \dots}{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots}$$

for $p \geq 2$. We will establish the following statements.

Theorem 1.3. *Suppose that* $\varphi : \mathcal{I} \to \mathcal{H}$ *satisfies* $||\varphi||_\infty \leq 1$. *Then for* $0 < p < \infty$,

$$||S(\varphi)||_p \leq K_p, \tag{1.5}$$

and the inequality is sharp, even if $\mathcal{H} = \mathbb{R}$.

Theorem 1.4. *Suppose that* $\varphi : \mathcal{I} \to \mathcal{H}$ *satisfies* $||\varphi||_\infty \leq 1$. *Then for any number* $0 < c < \pi^2/8$ *we have*

$$\mathbb{E}e^{cS^2(\varphi)} \leq (\cos\sqrt{2c})^{-1}, \tag{1.6}$$

and the inequality is sharp, even if $\mathcal{H} = \mathbb{R}$.

The description of the method and the proofs of Theorems 1.1 and 1.2 can be found in the next section. The final part of the paper concerns the bounded case.

2. BMO inequalities

2.1. **On the method.** Throughout this section, we distinguish the set

$$\mathcal{C} = \{(x, y, z) \in \mathcal{H} \times [0, \infty) \times (0, \infty) : |x|^2 \leq y \leq |x|^2 + 1\} \cup \{(0, 0, 0)\}.$$

Let $V : \mathcal{H} \times [0, \infty) \to \mathbb{R}$, $\alpha : \mathcal{H} \to \mathbb{R}$ be given functions and suppose that we are interested in showing the estimate

$$\int_{\mathcal{I}} V(\varphi, S(\varphi)) \leq \alpha(\langle\varphi\rangle_{\mathcal{I}}) \tag{2.1}$$

for all simple functions $\varphi : \mathcal{I} \to \mathcal{H}$ satisfying $||\varphi||_{BMO} \leq 1$. Here by simplicity we mean that φ can be written as a finite sum $\sum_{k=0}^m a_k h_k$ for some integer m and some coefficients $a_0, a_1, \ldots, a_m \in \mathcal{H}$. The key to study such an estimate is a class $\mathcal{U}(V)$, which consists of all $U : \mathcal{C} \to \mathbb{R}$ satisfying

$$U(x, |x|^2, z) \geq V(x, z) \qquad \text{for all } x,\ z \text{ such that } (x, |x|^2, z) \in \mathcal{C}, \tag{2.2}$$

$$U(x, y, |x|) \leq \alpha(x) \qquad \text{for all } x,\ y \text{ such that } (x, y, |x|) \in \mathcal{C} \tag{2.3}$$

and the further condition that for any $(x, y, z) \in \mathcal{C}$ and any $d \in \mathcal{H}$, $e \in \mathbb{R}$ satisfying $(x_\pm, y_\pm, \sqrt{z^2 + |d|^2}) := (x \pm d, y \pm e, \sqrt{z^2 + |d|^2}) \in \mathcal{C}$,

$$U(x, y, z) \geq \frac{1}{2}\left[U\big(x_-, y_-, \sqrt{z^2 + |d|^2}\big) + U\big(x_+, y_+, \sqrt{z^2 + |d|^2}\big)\right]. \tag{2.4}$$

The interplay between the class $\mathcal{U}(V)$ and the inequality (2.1) is described in the following statement.

Theorem 2.1. *If the class $\mathcal{U}(V)$ is nonempty, then* (2.1) *is valid for all simple φ satisfying $||\varphi||_{BMO} \leq 1$.*

Proof. Fix φ as in the statement. First we prove that the sequence

$$\left(\int_{\mathcal{I}} U\big(\varphi_n, (|\varphi|^2)_n, S(\varphi_n)\big)\right)_{n \geq 0} \tag{2.5}$$

is nonincreasing (note that $(|\varphi|^2)_n$ denotes the projection of the real-valued function $|\varphi|^2$ on the subspace spanned by the first $n+1$ Haar functions). To show this monotonicity, fix $n \geq 1$, denote by I the support of h_n and let I^ℓ, I^r be the left and the right halves of I. We have

$$\begin{aligned}
&\int_I U\big(\varphi_n, (|\varphi|^2)_n, S(\varphi_n)\big)\\
&\qquad = \int_{I^\ell} U\big(\varphi_n, (|\varphi|^2)_n, S(\varphi_n)\big) + \int_{I^r} U\big(\varphi_n, (|\varphi|^2)_n, S(\varphi_n)\big)\\
&\qquad = \frac{|I|}{2}\left[U(x - d, y - e, \sqrt{z^2 + |d|^2}) + U(x + d, y + e, \sqrt{z^2 + |d|^2})\right],
\end{aligned}$$

where x, y, z are the (constant) values of φ_{n-1}, $(|\varphi|^2)_{n-1}$ and $S(\varphi_{n-1})$ on I, respectively, and $d \in \mathcal{H}$, $e \in \mathbb{R}$ are determined by the conditions $\varphi_n - \varphi_{n-1} \in \{-d, d\}$, $(|\varphi|^2)_n - (|\varphi|^2)_{n-1} \in \{-e, e\}$. Using (2.4), we obtain the bound

$$\int_I U\big(\varphi_n, (|\varphi|^2)_n, S(\varphi_n)\big) \leq |I| U(x, y, z) = \int_I U\big(\varphi_{n-1}, (|\varphi|^2)_{n-1}, S(\varphi_{n-1})\big),$$

and since $(\varphi_n, (|\varphi|^2)_n, S(\varphi_n))$ and $(\varphi_{n-1}, (|\varphi|^2)_{n-1}, S(\varphi_{n-1}))$ coincide on $\mathcal{I} \setminus I$, we get the announced monotonicity property of the sequence (2.5). Next, since φ is simple, there is m such that $\varphi_m = \varphi$, $(|\varphi|^2)_m = |\varphi|^2$ and $S(\varphi_m) = S(\varphi)$. Combining this with (2.2) and (2.3), we obtain

$$\begin{aligned}
\int_{\mathcal{I}} V(\varphi, S(\varphi)) &\leq \int_{\mathcal{I}} U(\varphi_m, (|\varphi|^2)_m, S(\varphi_m))\\
&\leq \int_{\mathcal{I}} U(\varphi_0, (|\varphi|^2)_0, S(\varphi_0))\\
&= \int_{\mathcal{I}} U(\langle\varphi\rangle_{\mathcal{I}}, \langle|\varphi|^2\rangle_{\mathcal{I}}, |\langle\varphi\rangle_{\mathcal{I}}|)\\
&\leq \alpha(\langle\varphi\rangle_{\mathcal{I}}).
\end{aligned} \tag{2.6}$$

This completes the proof. □

It turns out that the implication of the above theorem can be reversed. For $(x,y)\in\mathcal{H}\times[0,\infty)$ such that $|x|^2\le y\le |x|^2+1$, let $\mathcal{M}(x,y)$ denote the class of all simple functions $\varphi:\mathcal{I}\to\mathcal{H}$ from the unit ball of BMO, satisfying $\langle\varphi\rangle_{\mathcal{I}}=x$ and $\langle|\varphi|^2\rangle_{\mathcal{I}}=y$. The class $\mathcal{M}(x,y)$ is nonempty: for example, it contains the function $\varphi=(x-d)\chi_{[0,1/2)}+(x+d)\chi_{[1/2,1)}$, where $d\in\mathcal{H}$ is a vector satisfying $|d|^2=y-|x|^2$. Define $U^0:\mathcal{C}\to(-\infty,\infty]$ by the formula

$$U^0(x,y,z)=\sup\left\{\int_{\mathcal{I}}V\left(\varphi,\sqrt{z^2-|x|^2+S^2(\varphi)}\right):\varphi\in\mathcal{M}(x,y)\right\}. \tag{2.7}$$

Theorem 2.2. *Suppose that the inequality* (2.1) *holds for all simple* $\varphi:\mathcal{I}\to\mathcal{H}$ *satisfying* $||\varphi||_{BMO}\le 1$. *Then the class* $\mathcal{U}(V)$ *is nonempty and* U^0 *is its least element.*

Proof. To see that U^0 is the least pick an element U of $\mathcal{U}(V)$ and $\varphi\in\mathcal{M}(x,y)$. Repeating the argumentation from (2.6), we obtain

$$\begin{aligned}\int_{\mathcal{I}}V\left(\varphi,\sqrt{z^2-|x|^2+S^2(\varphi)}\right)&\le\int_{\mathcal{I}}U\left(\varphi_0,(|\varphi|^2)_0,\sqrt{z^2-|x|^2+S^2(\varphi_0)}\right)\\&=U(x,y,z).\end{aligned}$$

Thus, taking supremum over $\varphi\in\mathcal{M}(x,y)$, we get $U^0(x,y,z)\le U(x,y,z)$, i.e. the minimality of U^0.

Now we check that U^0 belongs to the class $\mathcal{U}(V)$. The majorization (2.2) is obvious: if $y=|x|^2$, then $\varphi\equiv x$ belongs to $\mathcal{M}(x,y)$, and for this choice of φ,

$$\int_{\mathcal{I}}V\left(\varphi,\sqrt{z^2-|x|^2+S^2(\varphi)}\right)=V(x,z).$$

The condition (2.3) is also straightforward: by (2.1), for any $\varphi\in\mathcal{M}(x,y)$ we have

$$\int_{\mathcal{I}}V(\varphi,S(\varphi))\le\alpha(x)$$

and hence, taking supremum over φ, we get $U^0(x,y,|x|)\le\alpha(x)$. To prove (2.4), fix any x, y, z, d, e as in the statement of this condition and pick $\varphi_\pm\in\mathcal{M}(x_\pm,y_\pm)$. Next, splice these two functions together, using the formula

$$\varphi(t)=\begin{cases}\varphi_-(2t) & \text{if } t\in[0,1/2),\\ \varphi_+(2t-1) & \text{if } t\in[1/2,1].\end{cases}$$

The new function belongs to $\mathcal{M}(x,y)$. Indeed, it is simple,

$$\langle\varphi\rangle_{\mathcal{I}}=\int_0^{1/2}\varphi_-(2t)\mathrm{d}t+\int_{1/2}^1\varphi_+(2t-1)\mathrm{d}t=\frac{\langle\varphi_-\rangle_{\mathcal{I}}+\langle\varphi_+\rangle_{\mathcal{I}}}{2}=x$$

and similarly,

$$\langle|\varphi|^2\rangle_{\mathcal{I}}=\frac{\langle|\varphi_-|^2\rangle_{\mathcal{I}}+\langle|\varphi_+|^2\rangle_{\mathcal{I}}}{2}=y.$$

Of course, $||\varphi||_{BMO}\le 1$. Finally, it is a matter of a simple verification that

$$-\langle\varphi\rangle_{\mathcal{I}}^2+S^2(\varphi)(t)=\begin{cases}|d|^2-\langle\varphi_-\rangle_{\mathcal{I}}+S^2(\varphi_-)(2t) & \text{for } t\in[0,1/2),\\ |d|^2-\langle\varphi_+\rangle_{\mathcal{I}}+S^2(\varphi_+)(2t-1) & \text{for } t\in[1/2,1].\end{cases}$$

In consequence, by the definition of U^0,

$$\begin{aligned} U^0(x,y,z) &\geq \int_{\mathcal{I}} V\left(\varphi, \sqrt{z^2-|x|^2+S^2(\varphi)}\right) \\ &= \int_0^{1/2} V\left(\varphi, \sqrt{z^2-\langle\varphi\rangle_{\mathcal{I}}^2+S^2(\varphi)}\right) \\ &\quad + \int_{1/2}^1 V\left(\varphi, \sqrt{z^2-\langle\varphi\rangle_{\mathcal{I}}^2+S^2(\varphi)}\right) \\ &= \frac{1}{2}\Bigg[\int_{\mathcal{I}} V\left(\varphi_-, \sqrt{z^2+|d|^2-\langle\varphi_-\rangle_{\mathcal{I}}^2+S^2(\varphi_-)}\right) \\ &\qquad + \int_{\mathcal{I}} V\left(\varphi_+, \sqrt{z^2+|d|^2-\langle\varphi_+\rangle_{\mathcal{I}}^2+S^2(\varphi_+)}\right)\Bigg]. \end{aligned}$$

Taking supremum over all $\varphi_\pm \in \mathcal{M}(x_\pm, y_\pm)$ yields (2.4). The final step is to show that $-\infty < U^0(x,y,z) < \infty$ for any $(x,y,z) \in \mathcal{C}$. The lower bound follows from the definition of U^0 and the already mentioned fact that the function $\varphi = (x-d)\chi_{[0,1/2)} + (x+d)\chi_{[1/2,1)}$ (where $|d|^2 = y - |x|^2$) belongs to $\mathcal{M}(x,y)$. We turn to the upper bound. The case $(x,y,z) = (0,0,0)$ follows immediately from (2.3): $U^0(0,0,0) \leq \alpha(0) < \infty$. Thus we may assume that $z > 0$. Using induction, we shall prove that for any $\overline{x}$, d_1, d_2, …, $d_n \in \mathcal{H}$ we have

$$U^0(\overline{x}+d_1+d_2+\ldots+d_n, y, \sqrt{|\overline{x}|^2+|d_1|^2+|d_2|^2+\ldots+|d_n|^2}) < \infty.$$

The case $n=0$ follows at once from (2.3). To deal with the induction step, put $x = \overline{x}+d_1+\ldots+d_n$, $z = \sqrt{|\overline{x}|^2+|d_1|^2+\ldots+|d_n|^2}$ and $d = d_{n+1}$. An application of (2.4) gives

$$U^0(x+d, y, \sqrt{z^2+|d|^2}) \leq 2U^0(x,y,z) - U^0(x-d, y, \sqrt{z^2+|d|^2}).$$

which is finite by the induction assumption and the lower bound

$$-\infty < U^0(x-d, y, \sqrt{z^2+|d|^2}),$$

which we have already established. It suffices to note that any point $(x,y,z) \in \mathcal{C}$ with $z > 0$ can be written in the form

$$(\overline{x}+d_1+d_2+\ldots+d_n, y, \sqrt{|\overline{x}|^2+|d_1|^2+|d_2|^2+\ldots+|d_n|^2})$$

for appropriate choice of vectors $\overline{x}$, d_1, d_2, … and d_n. A straightforward proof of this fact is left to the reader. □

The method, after straightforward modifications, can be applied in many other settings. For example, if we want to establish (2.1) for nonnegative BMO functions φ only, we need to take $\mathcal{H} = \mathbb{R}$ and change the definition of $\mathcal{C}$ to

$$\{(x,y,z) \in [0,\infty)\times[0,\infty)\times(0,\infty)\} \cup \{(0,0,0)\}.$$

Another modification, which will be used below, concerns the case in which φ is assumed to be bounded and there are no additional assumptions on the mean oscillation of φ. Then the method rests on the existence of appropriate special functions of *two* variables; we may remove the variable y, which controls the BMO norm of φ. To be more precise, suppose that $\mathcal{B}$ is the unit ball of $\mathcal{H}$ and let $V : \mathcal{B}\times[0,\infty) \to \mathbb{R}$, $\alpha : \mathcal{B} \to \mathbb{R}$ be given functions. Suppose we are interested in showing (2.1) for all simple $\varphi : \mathcal{I} \to \mathcal{B}$. Put $\mathcal{C} = \big(\mathcal{B}\times(0,\infty)\big) \cup \{(0,0)\}$. Then the validity of the estimate is equivalent to the existence of a function $U : \mathcal{C} \to \mathbb{R}$, which satisfies

$$U(x,z) \geq V(x,z) \qquad \text{for all } (x,z) \in \mathcal{C}, \tag{2.8}$$

$$U(x,|x|) \leq \alpha(x) \qquad \text{for all } x \in \mathcal{B} \tag{2.9}$$

and the further condition that for any $(x,z) \in \mathcal{C}$ and any $d \in \mathcal{H}$ such that $x \pm d \in \mathcal{B}$ we have

$$U(x,z) \geq \frac{1}{2}\left[U(x+d, \sqrt{x^2+|d|^2}) + U(x-d, \sqrt{z^2+|d|^2})\right]. \tag{2.10}$$

Essentially, the proof of the equivalence is a word-by-word repetition of the reasoning presented above. We leave the details to the interested reader; consult also Slavin and Vasyunin [15].

2.2. **Applications: proofs of Theorem 1.1 and Theorem 1.2.**

Proof of (1.3) *for* $p \geq 2$. By Lebesgue's monotone convergence theorem, it suffices to prove the claim for simple functions φ only. The estimate (1.3) is of the form (2.1), with the following choice of parameters:

$$V(x,z) = z^p \quad \text{ and } \quad \alpha(x) = e^{|x|^2}\int_{|x|^2}^\infty e^{-s}s^{p/2}\mathrm{d}s.$$

The corresponding special function $U : \mathcal{C} \to \mathbb{R}$ is given by

$$U(x,y,z) = (y-|x|^2)e^{z^2}\int_{z^2}^\infty e^{-s}s^{p/2}\,\mathrm{d}s + (1+|x|^2-y)z^p.$$

Let us check the conditions (2.2)–(2.4). The majorization is trivial; in fact, both sides are equal. The condition (2.3) is also straightforward: if $s \geq |x|^2$, then $s^{p/2} \geq |x|^p$ and hence

$$\begin{aligned} U(x,y,|x|) &= (y-|x|^2)e^{|x|^2}\int_{|x|^2}^\infty e^{-s}s^{p/2}\,\mathrm{d}s + (1+|x|^2-y)|x|^p \\ &\leq (y-|x|^2)e^{|x|^2}\int_{|x|^2}^\infty e^{-s}s^{p/2}\,\mathrm{d}s + (1+|x|^2-y)e^{|x|^2}\int_{|x|^2}^\infty e^{-s}s^{p/2}\,\mathrm{d}s \\ &= \alpha(x,y). \end{aligned}$$

Finally, (2.4) is equivalent to

$$\begin{aligned} (y-|x|^2-|d|^2)e^{z^2+|d|^2}&\int_{z^2+|d|^2}^\infty e^{-s}s^{p/2}\mathrm{d}s + (1+|x|^2+|d|^2-y)(z^2+|d|^2)^{p/2} \\ &\leq (y-|x|^2)e^{z^2}\int_{z^2}^\infty e^{-s}s^{p/2}\mathrm{d}s + (1+|x|^2-y)z^p. \end{aligned}$$

If we substitute $A = y-|x|^2 \in [0,1]$, $Z = z^2 \geq 0$, we see that all we need is to show that the function

$$F(D) := (A-D)e^{Z+D}\int_{Z+D}^\infty e^{-s}s^{p/2}\mathrm{d}s + (1-A+D)(Z+D)^{p/2}$$

is nonincreasing on $[0,\infty)$. A little calculation yields

$$\begin{aligned} F'(D) &= (1-A+D)\left[(Z+D)^{p/2}+\frac{p}{2}(Z+D)^{p/2-1}-e^{Z+D}\int_{Z+D}^\infty e^{-s}s^{p/2}\mathrm{d}s\right] \\ &= -\frac{p}{2}\left(\frac{p}{2}-1\right)(1-A+D)e^{Z+D}\int_{Z+D}^\infty e^{-s}s^{p/2-2}\mathrm{d}s \leq 0. \end{aligned}$$

This completes the proof of (2.4) and the inequality (1.3) is established. □

Sharpness of (1.3) *for* $p\geq 2$. Here we exploit Theorem 2.2. Fix a function $\beta:\mathbb{R}\to\mathbb{R}$ and suppose that the inequality

$$\int_{\mathcal{I}} S^p(\varphi)\leq \beta(\langle\varphi\rangle_{\mathcal{I}})$$

holds true for all simple real-valued φ on $\mathcal{I}$ satisfying $||\varphi||_{BMO}\leq 1$. By Theorem 2.2, the function U^0, given by (2.7), belongs to the class $\mathcal{U}(V)$ (here, as previously, $V(x,z)=z^p$). For any simple $\varphi:I\to\mathbb{R}$ and any $d\in\mathbb{R}$, we easily check that $||\varphi||_{BMO}=||\varphi+d||_{BMO}$ and, in addition, $\varphi\in\mathcal{M}(x,x^2+1)$ if and only if $\varphi+d\in\mathcal{M}(x+d,(x+d)^2+1)$. In consequence, since $S^2(\varphi+d)-\langle\varphi+d\rangle_{\mathcal{I}}^2=S^2(\varphi)-\langle\varphi\rangle_{\mathcal{I}}^2$, we obtain the following additional property of U^0: for any $x\in\mathbb{R}$ and $z>0$,

$$\begin{aligned} U^0(x,x^2+1,z) &= \sup_{\varphi\in\mathcal{M}(x,x^2+1)}\left\{\int_{\mathcal{I}}(z^2-x^2+S^2(\varphi))^p\right\} \\ &= \sup_{\varphi\in\mathcal{M}(x,x^2+1)}\left\{\int_{\mathcal{I}}(z^2-(x+d)^2+S^2(\varphi+d))^p\right\} \\ &= \sup_{\varphi\in\mathcal{M}(x+d,(x+d)^2+1)}\left\{\int_{\mathcal{I}}(z^2-(x+d)^2+S^2(\varphi))^p\right\} \\ &= U^0(x+d,(x+d)^2+1,z). \end{aligned} \tag{2.11}$$

Now we are ready to prove the sharpness. By (2.4), we have, for any $x>0$, $z>0$ and any $d\in(0,1)$,

$$\begin{aligned} U^0(x,x^2+1,z)\geq \frac{1}{2}\Big[&U^0(x-d,x^2+1-2xd,\sqrt{x^2+d^2}) \\ &+U^0(x+d,x^2+1+2xd,\sqrt{x^2+d^2})\Big]. \end{aligned} \tag{2.12}$$

Next, note that for any x, z, the function $U(x,\cdot,z)$, given on $[x^2,x^2+1]$, is midconcave and hence concave, since U^0 is locally bounded from below (see the previous subsection). Thus,

$$\begin{aligned} U^0(x-d,x^2+1-2xd,\sqrt{z^2+d^2}) &= U^0(x-d,(x-d)^2+1-d^2,\sqrt{z^2+d^2}) \\ &\geq d^2\,U^0(x-d,(x-d)^2,\sqrt{z^2+d^2}) \\ &\quad +(1-d^2)U^0(x-d,(x-d)^2+1,\sqrt{z^2+d^2}) \\ &\geq d^2(z^2+d^2)^{p/2} \\ &\quad +(1-d^2)U^0(x,x^2+1,\sqrt{x^2+d^2}), \end{aligned}$$

where in the last passage we have used (2.2) and (2.11). The analogous reasoning leads to the same lower bound for $U^0(x+d,x^2+1+2xd,\sqrt{z^2+d^2})$. Plugging these two estimates into (2.12) yields

$$U^0(x, x^2+1, z) \geq d^2(z^2+d^2)^{p/2} + (1-d^2)U^0(x, x^2+1, \sqrt{x^2+d^2}).$$

Therefore, by induction,

$$\begin{aligned} &U^0(x, x^2+1, x) \\ &\geq d^2 \sum_{k=1}^{n} (1-d^2)^{k-1}(x^2+kd^2)^{p/2} + (1-d^2)^n U^0(x, x^2+1, \sqrt{x^2+nd^2}) \\ &\geq d^2 \sum_{k=1}^{n} (1-d^2)^{k-1}(x^2+kd^2)^{p/2}. \end{aligned}$$

Here, in the last line, we have used the fact that the function V is nonnegative and hence so is U^0, by the very definition. Let $n \to \infty$ and pick $c > 1$. If d is sufficiently close to 0, then $1 - d^2 \geq e^{-cd^2}$ and hence we obtain that

$$U^0(x, x^2+1, x) \geq d^2 \sum_{k=1}^{\infty} e^{-c(k-1)d^2}(x^2+kd^2)^{p/2} \xrightarrow{d \to 0} e^{x^2} \int_{x^2}^{\infty} e^{-cs} s^{p/2} \mathrm{d}s.$$

Since $c > 1$ was arbitrary, (2.3) implies

$$\beta(x) \geq U^0(x, x^2+1, x) \geq e^{x^2} \int_{x^2}^{\infty} e^{-s} s^{p/2} \mathrm{d}s,$$

which is the desired sharpness. □

Proof of (1.3), $0 < p < 2$. Though the bound follows easily from Jensen inequality:

$$\int_{\mathcal{I}} S^p(\varphi) \leq \left[\int_{\mathcal{I}} S^2(\varphi) \right]^{p/2} = \langle |\varphi|^2 \rangle_{\mathcal{I}}^{p/2} \leq \big(|\langle \varphi \rangle_{\mathcal{I}}|^2 + 1 \big)^{p/2},$$

it is instructive to see how it can be established using the above methodology. We see that the estimate is of the form (2.1), with

$$V(x, z) = z^p \qquad \text{and} \qquad \alpha(x) = (|x|^2+1)^{p/2}.$$

Consider the function $U : \mathcal{C} \to \mathbb{R}$ given by $U(x, y, z) = (y - |x|^2 + z^2)^{p/2}$. Then $U \in \mathcal{U}(V)$: indeed, the conditions (2.2) and (2.3) are trivial, while (2.4) follows from the concavity of the function $t \mapsto t^{p/2}$, $t \geq 0$:

$$\begin{aligned} &U(x+d, y+e, \sqrt{z^2+d^2}) + U(x-d, y-e, \sqrt{z^2+d^2}) \\ &= \big(y - |x|^2 + z^2 + (e - 2x \cdot d)\big)^{p/2} + \big(y - |x|^2 + z^2 - (e - 2x \cdot d)\big)^{p/2} \\ &\leq 2(y - |x|^2 + z^2)^{p/2} = 2U(x, y, z). \end{aligned}$$

This finishes the proof. □

Sharpness of (1.3), $0 < p < 2$. This time the simplest approach is to provide an appropriate example. For any $x \in \mathbb{R}$, put $\varphi = x + h_1$ and observe that $\langle \varphi \rangle_{\mathcal{I}} = x$ and $S(\varphi) = \sqrt{\langle \varphi \rangle_{\mathcal{I}}^2 + 1}$. Thus, both sides of (1.3) are equal and hence the bound is sharp. □

Proof of (1.4). This follows immediately from the above considerations. Indeed, for any $c<1$ and any $\varphi : I \to \mathcal{H}$ with $||\varphi||_{BMO} \leq 1$,

$$\begin{aligned}\int_{\mathcal{I}} e^{cS^2(\varphi)} &= \sum_{k=0}^{\infty} \frac{c^k}{k!} \int_{\mathcal{I}} S^{2k}(\varphi)\\ &\leq \sum_{k=0}^{\infty} \frac{c^k}{k!} e^{\langle\varphi\rangle_{\mathcal{I}}^2} \int_{\langle\varphi\rangle_{\mathcal{I}}^2}^{\infty} e^{-s} s^k \mathrm{d}s\\ &= e^{\langle\varphi\rangle_{\mathcal{I}}^2} \int_{\langle\varphi\rangle_{\mathcal{I}}^2}^{\infty} e^{-s} e^{cs} \mathrm{d}s = \frac{e^{c\langle\varphi\rangle_{\mathcal{I}}^2}}{1-c},\end{aligned}$$

which is the claim. □

Sharpness of (1.4) *for* $\mathcal{H}=\mathbb{R}$. Again, we make use of Theorem 2.2. Fix $c\in(0,1)$, a function $\beta:\mathbb{R}\to\mathbb{R}$ and suppose that the inequality

$$\int_{\mathcal{I}} e^{cS^2(\varphi)} \leq \beta(\varphi\rangle_{\mathcal{I}})$$

holds true for all simple $\varphi:\mathcal{I}\to\mathbb{R}$ satisfying $||\varphi||_{BMO}\leq 1$. Let $V:\mathbb{R}\times[0,\infty)\to\mathbb{R}$ be given by $V(x,z)=e^{cz^2}$ and let U^0 be defined by (2.7). Repeating the reasoning leading to (2.11), we see that here we also have

$$U^0(x,x^2+1,z)=U^0(x+d,(x+d)^2+1,z) \tag{2.13}$$

for all $x\in\mathbb{R}$, $z>0$ and $d\in\mathbb{R}$. Furthermore, for all $(x,y,z)\in\mathcal{C}$ and $d>0$,

$$\begin{aligned}U^0(x,y,\sqrt{z^2+d^2}) &= \sup_{\varphi\in\mathcal{M}(x,y)} \left\{\int_{\mathcal{I}} e^{c(z^2+d^2-x^2+S^2(\varphi))}\right\}\\ &= e^{cd^2} \sup_{\varphi\in\mathcal{M}(x,y)} \left\{\int_{\mathcal{I}} e^{c(z^2-x^2+S^2(\varphi))}\right\} = e^{cd^2} U^0(x,y,z).\end{aligned} \tag{2.14}$$

Finally, a reasoning similar to that presented above yields that for any $d\in(-1,1)$,

$$\begin{aligned}&U^0(x-d,x^2+1-2xd,\sqrt{x^2+d^2})\\ &\quad= U^0(x-d,(x-d)^2+1-d^2,\sqrt{x^2+d^2})\\ &\quad\geq d^2\, U^0(x-d,(x-d)^2,\sqrt{x^2+d^2}) + (1-d^2) U^0(x-d,(x-d)^2+1,\sqrt{x^2+d^2})\\ &\quad\geq d^2 e^{c(x^2+d^2)} + (1-d^2) U^0(x,x^2+1,\sqrt{x^2+d^2})\\ &\quad= d^2 e^{c(x^2+d^2)} + (1-d^2) e^{cd^2} U^0(x,x^2+1,|x|),\end{aligned}$$

where in the last passage we have used (2.14). Now, write down (2.12), combine it with the above lower bound and calculate a little bit to obtain

$$U^0(x,x^2+1,|x|) \geq \frac{e^{c(x^2+d^2)}}{1-(1-d^2)e^{cd^2}}.$$

Letting $d\to 0$ and applying (2.3) gives

$$\beta(x) \geq U^0(x,x^2+1,|x|) \geq \frac{e^{cx^2}}{1-c}.$$

Thus the bound (1.4) is optimal. □

3. Inequality for bounded functions

Now we will study the case in which the functions φ are bounded by 1. It will be convenient to make use of some classical probabilistic arguments. Throughout this section, $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, equipped with a continuous-time filtration $(\mathcal{F}_t)_{t\geq 0}$, and $X = (X_t)_{t\geq 0}$ is an adapted, one-dimensional Brownian motion starting from the origin. Then $[X, X]$, the quadratic covariance process (square bracket) of X is given by $[X, X]_t = t$; for the necessary definitions, we refer the reader to Dellacherie and Meyer [4].

As in the BMO setting, the main difficulty lies in proving the L^p estimates (1.5) for $p \geq 2$. Fix such a number p and let S denote the strip $[-1, 1] \times [0, \infty)$. Consider a function $v : S \to \mathbb{R}$, satisfying the heat equation

$$v_y + \frac{1}{2} v_{xx} = 0 \tag{3.1}$$

with the boundary condition $v(\pm 1, y) = y^p$, $y \geq 0$. It is well-known (and follows immediately from some very basic facts from semigroup theory; see e.g. Revuz and Yor [12]) that v admits the following stochastic representation: if $\tau_x = \inf\{t : |x + X_t| \geq 1\}$ is the first exit time of the process $((t, x + X_t))_{t\geq 0}$ from the strip S, then

$$v(x, y) = \mathbb{E}(y + \tau_x)^{p/2}. \tag{3.2}$$

Clearly, v is of class C^∞ in the interior of the strip S and continuous up to its boundary. In fact, it is not difficult to write down the explicit integral formula for v. However, we will not do this; we will only need the explicit value of v at the point $(0, 0)$, which can be directly derived from (3.2).

Let us gather some further information on v.

Lemma 3.1. *The function v has the following properties.*

(i) We have $v_y \geq 0$ and $v_{yy} \geq 0$ in the interior of S.

(ii) For any $y \geq 0$, the function $v(\cdot, y)$ is even and concave.

(iii) We have

$$v(0, 0) = \frac{\Gamma(p + 1)2^{p/2-1}}{\Gamma\left(\frac{p+1}{2}\right)\pi^{p-3/2}} \cdot \frac{1 - \frac{1}{3^{p+1}} + \frac{1}{5^{p+1}} - \frac{1}{7^{p+1}} + \dots}{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots}.$$

Proof. (i) The bound $v_y \geq 0$ is obvious. The inequality $v_{yy} \geq 0$ follows from (3.2) and the convexity of the function $t \mapsto t^{p/2}$ on $[0, \infty)$. Indeed, if y_1, $y_2 \geq 0$ and $\lambda \in (0, 1)$, then

$$\lambda\mathbb{E}(y_1 + \tau_x)^{p/2} + (1 - \lambda)\mathbb{E}(y_2 + \tau_x)^{p/2} \geq \mathbb{E}(\lambda y_1 + (1 - \lambda)y_2 + \tau_x)^{p/2}. \tag{3.3}$$

(ii) The first assertion follows at once from the stochastic representation of v and the fact that $(-X_t)_{t\geq 0}$ is also a Brownian motion. The second property is an immediate consequence of (3.1) and part (i).

(iii) Let $Y = (Y_t)_{t\geq 0}$ be a Brownian motion independent of X. It is a classical result that Y_{τ_0} has the density $g(u) = e^{-\pi u/2}/(1 + e^{-\pi u})$. The process Y is independent from τ_0, so Y_{τ_0} has the same distribution as $\tau_0^{1/2}Y_1$ and thus

$$\begin{aligned} v(0,0) &= \mathbb{E}\tau_0^{p/2} = \frac{\mathbb{E}|Y_{\tau_0}|^p}{\mathbb{E}|Y_1|^p} \\ &= \frac{\sqrt{\pi}}{2^{p/2}\Gamma\left(\frac{p+1}{2}\right)}\int_{\mathbb{R}} |x|^p \frac{e^{-\pi x/2}}{1+e^{-\pi x}}\,\mathrm{d}x \\ &= \frac{2\sqrt{\pi}}{2^{p/2}\Gamma\left(\frac{p+1}{2}\right)}\int_0^\infty x^p \sum_{k=0}^\infty (-1)^k e^{-(2k+1)\pi x/2}\,\mathrm{d}x \qquad (3.4)\\ &= \frac{\sqrt{\pi}}{2^{p/2-1}\Gamma\left(\frac{p+1}{2}\right)}\sum_{k=0}^\infty \left(\frac{(2k+1)\pi}{2}\right)^{-p-1}\int_0^\infty x^p e^{-x}\mathrm{d}x \\ &= \frac{\Gamma(p+1)2^{p/2-1}}{\Gamma\left(\frac{p+1}{2}\right)\pi^{p-3/2}}\cdot \frac{1-\frac{1}{3^{p+1}}+\frac{1}{5^{p+1}}-\frac{1}{7^{p+1}}+\dots}{1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots}. \end{aligned}$$

Here, in the last passage, we have used the identity

$$\frac{\pi^2}{8} = 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots. \qquad \square$$

Proof of (1.5). Suppose first that $p \geq 2$. For any $(x, y) \in \mathcal{B}\times[0,\infty)$, let $V(x,y) = y^p$, $\alpha(x) = v(0,0)$ and $U(x,y) = v(|x|, y^2)$. We will prove that the conditions (2.8)–(2.10) are satisfied. The majorization follows immediately from the concavity of $v(\cdot, y)$ and the fact that $U(x,y) = V(x,y)$ for x lying on the unit sphere. To prove (2.9), note that for $t \in (0,1)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t,t^2) = v_x(t,t^2) + 2tv_y(t,t^2) = v_x(t,t^2) - tv_{xx}(t,t^2).$$

By parts (i) and (ii) of Lemma 3.1, we have $v_x(0,t^2) = 0$, $v_{xxx}(0,t^2) = 0$ and $v_{xxxx} = 4v_{yy} \geq 0$, which implies $v_{xxx}(\cdot, t^2) \geq 0$ on $[0,1]$. Thus, by the mean-value theorem, there is $t' \in (0,t)$ such that $v_x(t,t^2) = tv_{xx}(t',t^2)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t,t^2) = t\big(v_{xx}(t',t^2) - v_{xx}(t,t^2)\big) \leq 0.$$

In consequence, $U(x,|x|) = v(x,|x|^2) \leq v(0,0)$. Finally, we turn to (2.4). Fix x, $d \in \mathcal{H}$ and $y \geq 0$ such that $|x| \leq 1$, $|x+d| \leq 1$. We must prove that

$$v(|x+d|, y^2+|d|^2) + v(|x-d|, y^2+|d|^2) \leq 2v(|x|, y^2). \qquad (3.5)$$

The left-hand side can be written in the form $F(2x\cdot d)$, where

$$F(s) = v(\sqrt{A-s}, y^2+|d|^2) + v(\sqrt{A+s}, y^2+|d|^2)$$

and $A = |x|^2 + |d|^2$. The function F is nondecreasing on $[0, 2|x||d|]$: we have

$$F'(s) = -\frac{v_x(\sqrt{A-s}, y^2+|d|^2)}{2\sqrt{A-s}} + \frac{v_x(\sqrt{A+s}, y^2+|d|^2)}{2\sqrt{A+s}} \geq 0,$$

where the latter bound is due to $\frac{d}{dx}(v_x(x,y)/x) = (xv_{xx}(x,y) - v_x(x,y))x^{-2} \geq 0$ for $x > 0$ (see the above verification of (2.9)). Furthermore, F is even; thus, it suffices to prove (3.5) for $x\cdot d = |x||d|$, i.e. in the case when x and d are linearly dependent. Introduce an auxiliary continuous function $G : [0,1] \to \mathbb{R}$, given by

$$G(t) = v\big(|x+td|, y^2+t^2|d|^2\big) + v\big(|x-td|, y^2+t^2|d|^2\big).$$

We will prove that G is nonincreasing, which will immediately yield (3.5). Fix $t \in (0,1)$ satisfying $x \pm td \neq 0$ and let $x_\pm = |x \pm td|$, $y_+ = \sqrt{y^2 + t^2|d|^2}$. Since $x \cdot d = |x||d|$, $G'(t)$ equals

$$v_x(x_+, y_+^2)\frac{(x+td)\cdot d}{|x+td|} - v_x(x_-, y_+^2)\frac{(x-td)\cdot d}{|x-td|} + 2\big(v_y(x_-, y_+^2) + v_y(x_+, y_+^2)\big)t|d|^2$$
$$= v_x(x_+, y_+^2)|d| - v_x(x_-, y_+^2)\frac{(|x|-t|d|)|d|}{|x-td|} - (v_{xx}(x_-, y_+^2) + v_{xx}(x_+, y_+^2))t|d|^2.$$

Now, if $|x| \geq t|d|$, the above expression is equal to

$$|d|\left[v_x(x_+, y_+^2) - v_x(x_-, y_+^2) - (v_{xx}(x_-, y_+^2) + v_{xx}(x_+, y_+^2))t|d|\right]$$
$$= |d|\left[\int_{x_-}^{x_+} v_{xx}(s, y_+^2)\,\mathrm{d}s - (v_{xx}(x_-, y_+^2) + v_{xx}(x_+, y_+^2))\frac{|x_+ - x_-|}{2}\right],$$

which is nonpositive: this follows at once from the estimates $v_{xx} \leq 0$ and $v_{xxxx} \geq 0$, which we have already established above. Similarly, if $t|d| > |x|$, then

$$G'(t) = |d|\left[v_x(x_+, y_+^2) + v_x(x_-, y_+^2) - (v_{xx}(x_-, y_+^2) + v_{xx}(x_+, y_+^2))t|d|\right]$$
$$= |d|\left[\int_{-x_-}^{x_+} v_{xx}(s, y_+^2)\,\mathrm{d}s - (v_{xx}(x_-, y_+^2) + v_{xx}(x_+, y_+^2))\frac{|x_+ + x_-|}{2}\right] \leq 0,$$

because of the same reasons as above. This completes the proof of (2.10) and hence (1.5) is established for $p \geq 2$. If $0 < p < 2$, then the estimate follows immediately from Jensen's inequality combined with the case $p = 2$, or by the use of the function $U(x,z) = (1 - |x|^2 + z^2)^{p/2}$. Since the calculations are the same as in the BMO case, we omit them. □

Sharpness. For $0 < p \leq 2$, the optimality of the constant is trivial, so assume that $p > 2$. Suppose that the best constant in (1.5) (for real-valued functions) equals c_p. Apply the version of Theorem 2.2: the function $U^0 : ([-1,1]\times(0,\infty)) \cup \{(0,0)\} \to \mathbb{R}$, given by

$$U^0(x,z) = \sup\left\{\mathbb{E}(z^2 - x^2 + S^2(\varphi))^{p/2} \mid \varphi : \mathcal{I} \to [-1,1],\ \langle\varphi\rangle_{\mathcal{I}} = x\right\},$$

satisfies (2.8), (2.9) and (2.10) with $V(x,z) = z^{p/2}$ and $\alpha(x) = c_p^p$. Next, let N be a positive integer. Introduce the family $(\tau_n^N)_{n\geq 0}$ of stopping times, given by $\tau_0 \equiv 0$ and, for $n \geq 1$,

$$\tau_n^N = \inf\{t > \tau_{n-1}^N : |X_t - X_{\tau_{n-1}^N}| \geq 1/N\}.$$

Then $(R_n)_{n\geq 0} = (X_{\tau_n^N})_{n\geq 0}$ is a symmetric random walk of step of size $1/N$. Consider the stopping time $\sigma^N = \inf\{n : |R_n| = 1\}$. If $\sigma^N > n$ and we apply (2.10) to $x = R_n$, $z = \sqrt{n}/N$ and $d = R_{n+1} - R_n$, then we get

$$U^0(R_n, \sqrt{n}/N) \geq \frac{1}{2}\Big[U^0(R_{n+1}, \sqrt{n+1}/N) + U^0(R_n - (R_{n+1} - R_n), \sqrt{n+1}/N)\Big].$$

Since $R_{n+1} - R_n$ is symmetric and independent from the event $\{\sigma^N > n\}$, the above estimate yields

$$\mathbb{E}U^0(R_n, \sqrt{n}/N)\chi_{\{\sigma^N > n\}} \geq \mathbb{E}U^0(R_{n+1}, \sqrt{n+1}/N)\chi_{\{\sigma^N > n\}},$$

which can be rewritten in the form

$$\mathbb{E}U^0\left(R_{\sigma^N\wedge n}, \sqrt{\sigma^N \wedge n}/N\right) \geq \mathbb{E}U^0\left(R_{\sigma^N\wedge(n+1)}, \sqrt{\sigma^N \wedge (n+1)}/N\right).$$

Consequently, applying (2.9) and then (2.8), we get

$$c_p^p \geq U^0(0,0) \geq \mathbb{E}U^0\left(R_{\sigma^N\wedge n}, \sqrt{\sigma^N \wedge n}/N\right) \geq \mathbb{E}\left(\sqrt{\sigma^N \wedge n}/N\right)^p$$

and hence, by Lebesgue's monotone convergence theorem,

$$\mathbb{E}\left(\sqrt{\sigma_N}/N\right)^p \leq c_p^p. \tag{3.6}$$

However, by the very definition of (τ_n^N) and σ^N, we have that $\tau := \tau^N_{\sigma^N}$ is the first exit time of X from the interval $[-1,1]$. Since $\sup_{0\leq k<\sigma^N}|\tau^N_{k+1}-\tau^N_k| \to 0$ almost surely as $N\to\infty$, we have

$$\frac{\sigma^N}{N^2} = \sum_{k=0}^{\sigma^N-1} |X_{\tau^N_{k+1}} - X_{\tau^N_k}|^2 \to [X,X]_\tau = \tau$$

in probability as $N \to \infty$ (see Dellacherie and Meyer [4]). Thus, by (3.6) and Fatou's lemma,

$$\mathbb{E}\tau^{p/2} \leq c_p^p.$$

However, the left hand side is precisely $v(0,0)$, by virtue of (3.2), and this shows that $[v(0,0)]^{1/p}$ is indeed the optimal constant in (1.5). □

Proof of (1.6). By (1.5), for any nonnegative integer n we have

$$\mathbb{E}S^{2n}(\varphi) \leq \frac{\sqrt{\pi}}{2^n\Gamma\left(\frac{2n+1}{2}\right)} \int_{\mathbb{R}} |x|^{2n} \frac{e^{\pi x/2}}{1+e^{\pi x}}\,\mathrm{d}x,$$

see the second line in (3.4). In consequence, for any $0 < c < \pi^2/8$ we may write

$$\begin{aligned}
\mathbb{E}\exp(cS^2(\varphi)) &\leq \sum_{n=0}^\infty \frac{\sqrt{\pi}c^n}{2^n\Gamma\left(\frac{2n+1}{2}\right)n!} \int_{\mathbb{R}} |x|^{2n} \frac{e^{\pi x/2}}{1+e^{\pi x}}\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \sum_{n=0}^\infty \frac{(\sqrt{2c}|x|)^{2n}}{(2n)!} \frac{e^{\pi x/2}}{1+e^{\pi x}}\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \cosh(\sqrt{2c}x)\frac{e^{\pi x/2}}{1+e^{\pi x}}\,\mathrm{d}x.
\end{aligned}$$

Using the residue theorem, one easily verifies that for any $a \in (0,\pi)$ we have

$$\int_{\mathbb{R}} \frac{e^{ax}}{1+e^{\pi x}}\mathrm{d}x = (\sin a)^{-1}.$$

Plugging this above yields the desired estimate. □

Sharpness. This can be shown exactly in the same manner as in the case of (1.5). We leave the details to the reader. □

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