SQUARE FUNCTION INEQUALITIES FOR MONOTONE BASES IN L^1

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Abstract. The paper contains the description of a novel method of handling general sharp square-function inequalities for monotone bases and contractive projections in L^1. The technique rests on the construction of an appropriate special function enjoying certain size and convexity-type properties. As an illustration, we establish a strong $L^1 \to L^1$ and a weak-type $L^1 \to L^{1,\infty}$ estimate for square functions.

1. Introduction

The purpose of this paper is to study certain square function estimates arising naturally in the context of monotone bases and contractive projections in L^1. The paper is a natural continuation of the research presented in [12], which concerned mainly the maximal estimates arising in the context (see below). Let us begin with the necessary background and notation. Suppose that the sequence $e = (e_n)_{n\geq 0}$, taking values in a given real Banach space X, is a Schauder basis: that is, for every $f \in X$ there exists a unique sequence $a = (a_n)_{n\geq 0} \subset \mathbb{R}$ satisfying $||f - \sum_{k=0}^n a_k e_k||_X \to 0$. This basis $(e_n)_{n\geq 0}$ is called unconditional, if for any $f \in X$ the corresponding series converges unconditionally. This can be equivalently expressed as the inequality $\sup\{||P_E|| : E \subset \mathbb{N} \text{ finite}\} < \infty$, where, for a given E, the symbol P_E denotes the associated projection defined by $P_E f = \sum_{k\in E} a_k e_k$. A basis is monotone if for each n the projection $P_n := P_{\{0,1,\ldots,n\}}$ is contractive; equivalently, for any nonnegative integer n and any real numbers $a_0, a_1, \ldots, a_n, a_{n+1}$, we have

$$\left|\left|\sum_{k=0}^{n} a_k e_k\right|\right|_X \leq \left|\left|\sum_{k=0}^{n+1} a_k e_k\right|\right|_X.$$

The primary interest of this paper lies in the properties of monotone bases in $L^p(\Omega, \mathcal{F}, \mu)$, where the underlying measure μ is assumed to be positive and non-atomic. Assume first that $1 < p < \infty$. Then, as observed by Ando [1], every non-vanishing contractive projection of L^p is isometrically equivalent to a conditional expectation. This argument can be pushed further to yield that every nondecreasing sequence $(P_n)_{n\geq 0}$ of contractive projections (i.e., satisfying $P_m P_n = P_{m\wedge n}$ for all m, n) gives rise to a sequence of conditional expectations with respect to a nondecreasing family of sub-σ-algebras. Clearly, this immediately connects the

2010 *Mathematics Subject Classification.* Primary: 46B15. Secondary: 46A35, 46B20.

Key words and phrases. Contractive projection, Haar system, square function, monotone basis.

subject with the theory of martingales. As shown by Dor and Odell [7], an application of estimates for martingale transforms (see Burkholder [2]) leads to the following.

Theorem 1.1. *Assume that* $(\Omega, \mathcal{F}, \mu)$ *is a positive measure space. Let* $P_{-1} = 0$, P_0, P_1, P_2, ... *be a nondecreasing sequence of contractive projections in* $L^p(\Omega, \mathcal{F}, \mu)$, $1 < p < \infty$. *If* $f \in L^p(\Omega, \mathcal{F}, \mu)$, *then for any sequence* ε_0, ε_1, $\varepsilon_2, \ldots$ *of signs,*

$$\left|\left|\sum_{k=0}^{\infty} \varepsilon_k (P_k - P_{k-1}) f\right|\right|_p \leq C_p ||f||_p, \tag{1.1}$$

for some universal constant C_p *which depends only on* p.

One can show that the optimal choice for the constant C_p in (1.1) equals $p^* - 1$, where $p^* = \max\{p, p/(p-1)\}$. This is a consequence of a related sharp inequality for martingales established by Burkholder in [3] (see also [4]). The theorem above implies that every monotone basis in L^p is unconditional provided $1 < p < \infty$. Further combination with the results of Olevskiĭ [9], [10] gives that the unconditional constant of any monotone basis e of L^p $(1 < p < \infty)$ equals $p^* - 1$. That is, for any n, any sequence a_0, a_1, a_2, ..., a_n of real numbers and any sequence ε_0, ε_1, ε_2, ..., ε_n of signs we have

$$\left|\left|\sum_{k=0}^{n} \varepsilon_k a_k e_k\right|\right|_p \leq (p^* - 1) \left|\left|\sum_{k=0}^{n} a_k e_k\right|\right|_p, \qquad 1 < p < \infty, \tag{1.2}$$

and the constant $p^* - 1$ cannot be improved. Consult also the paper of Choi [6], in which the unconditional constant is defined in a slightly different manner.

There is a very interesting question about the validity of the inequality (1.2) in the limit case $p = 1$. A well-known result, due to Paley [13] (consult also Marcinkiewicz [8]), states that the Haar basis, a fundamental monotone basis of $L^1([0,1], \mathcal{B}([0,1]), |\cdot|)$, is not unconditional. Thus there is a further question about an appropriate version for the inequality (1.2) for $p = 1$, which will serve as a substitute for the unconditionality. In [12], the following maximal estimate was established:

$$\left|\left|\sum_{k=0}^{n} \varepsilon_k a_k e_k\right|\right|_1 \leq \beta \left|\left|\sup_{n \geq 0} \left|\sum_{k=0}^{n} a_k e_k\right|\right|\right|_1, \tag{1.3}$$

where $\beta = 2.536\ldots$ is the unique solution to the equation

$$\beta = 3 - \exp\frac{1-\beta}{2}.$$

However, there is another very natural substitute of (1.2), which can be expressed in terms of square functions. The purpose of the paper is to pursue this line of research. Here is the statement of our main result.

Theorem 1.2. *Suppose that $e = (e_n)_{n\geq 0}$ is a monotone basis of $L^1(\Omega, \mathcal{F}, \mu)$. Then for any sequences a_0, a_1, a_2, ... of real numbers we have the inequalities*

$$\left|\left|\sum_{k=0}^{\infty} a_k e_k\right|\right|_1 \leq 2 \left|\left|\left(\sum_{k=0}^{n} |a_k e_k|^2\right)^{1/2}\right|\right|_1 \tag{1.4}$$

and

$$\left|\left|\left(\sum_{k=0}^{n} |a_k e_k|^2\right)^{1/2}\right|\right|_1 \leq \sqrt{3}\left|\left|\sup_{n\geq 0}\left|\sum_{k=0}^{n} a_k e_k\right|\right|\right|_1. \tag{1.5}$$

The constants 2 and $\sqrt{3}$ are the best possible. They are already optimal in the weak-type estimates

$$\left|\left|\sum_{k=0}^{\infty} a_k e_k\right|\right|_{1,\infty} \leq 2 \left|\left|\left(\sum_{k=0}^{n} |a_k e_k|^2\right)^{1/2}\right|\right|_1. \tag{1.6}$$

and

$$\left|\left|\left(\sum_{k=0}^{n} |a_k e_k|^2\right)^{1/2}\right|\right|_{1,\infty} \leq \sqrt{3}\left|\left|\sup_{n\geq 0}\left|\sum_{k=0}^{n} a_k e_k\right|\right|\right|_1. \tag{1.7}$$

Here, as usual, $||f||_{1,\infty} = \sup_{\lambda>0} \lambda\mu(\{\omega \in \Omega : |f(\omega)| \geq \lambda\})$ denotes the weak first norm of f. We would like to mention here that the method we plan to develop has its analogue in martingale theory (see Burkholder [5], Osękowski [11]), but the interplay between the two settings is non-trivial. Namely, if we compare the above statement to its version for martingale transforms, we have that the constants 2 and $\sqrt{3}$ are also optimal in the probabilistic counterparts of (1.4) and (1.5) (cf. [5], [11]); on contrary, quite surprisingly, the best constant in the martingale versions of (1.6) and (1.7) are strictly smaller.

A few words about the organization of the paper are in order. The next section contains the description of the structure of a monotone basis in L^1. Section 3 is devoted to the detailed presentation of the method which allows to study general maximal inequalities for a certain class of monotone bases. In Section 4 we construct the special function which yields the validity of the inequalities (1.4) and (1.5). In the final part of the paper we address the question about the optimality of the constants in (1.6) and (1.7).

2. On the structure of monotone bases in L^1

The contents of this short section is taken from the paper [7] of Dor and Odell. We have decided to include it here for the sake of completeness and the convenience of the reader. Our primary goal is to describe how to construct an isometry of $L^1(\Omega, \mathcal{F}, \mu)$ onto a certain $L^1(\Omega, \mathcal{F}, \nu)$, which sends a given monotone basis e onto a simple basis of the target space (the necessary definitions are introduced below), so that the L^1 norms of maximal function $\max_{0\leq m\leq n} |\sum_{k=0}^{m} a_k e_k|$ and the square function $(\sum_{k=0}^{n} |a_k e_k|^2)^{1/2}$, $n = 0, 1, 2, \ldots$, are preserved.

Definition 2.1. A system of sets $\{A_{n,i} : i = 1, 2, \ldots, 2^n, n = 0, 1, 2, \ldots\}$ is called a *dyadic tree* if for all n and $1 \leq i \leq 2^n$ we have

$$A_{n+1,2i-1} \cap A_{n+1,2i} = \emptyset \qquad \text{and} \qquad A_{n+1,2i-1} \cup A_{n+1,2i} = A_{n,i}.$$

Definition 2.2. Given a dyadic tree of sets satisfying $\mu(A_{n,i}) > 0$ for all n and i, we define the associated generalized Haar sequence $h = (h_k)_{k\geq 0}$ by $h_0 = h_{0,1} = \chi_{A_{0,1}}/||\chi_{A_{0,1}}||_1$ and

$$h_{2^{n-1}+i-1} = h_{n,i} = H_{n,i}/||H_{n,i}||_1,$$

where

$$H_{n,i} = \chi_{A_{n,2i-1}}/\mu(A_{n,2i-1}) - \chi_{A_{n,2i}}/\mu(A_{n,2i}), \qquad i \leq 2^n,\ n = 1, 2, \ldots.$$

If h forms a basis, it will be referred to as a *generalized Haar basis*.

It is easy to check that the generalized Haar sequence $(h_n)_{n\geq 0}$ is uniquely determined by a dyadic tree $\{A_{n,i}\}$ and the following condition: for each $n \geq 1$ and $1 \leq i \leq 2^n$, the function $h_{n,i}$ is a linear combination of $\chi_{A_{n,2i-1}}$ and $\chi_{A_{n,2i}}$, such that

$$||h_{n,i}||_1 = 1 \quad \text{and} \quad \int_\Omega h_{n,i} = 0 \quad \text{for } n \geq 1. \tag{2.1}$$

Note that if $\{A_{n,i}\}$ is the family of all dyadic subintervals of $[0, 1]$ and μ is Lebesgue's measure, then the generalized Haar basis reduces to the usual Haar system in L^1.

Definition 2.3. A basis $d = (d_k)_{k\geq 0}$ in $L^1(\Omega, \mathcal{F}, \nu)$ is called *simple*, if there is a sequence (possibly finite) of disjoint sets $E_n \in \mathcal{F}$ covering Ω, so that $(d_k)_{k\geq 0}$ is the union of disjoint subsequences $(d_i^n)_{i\geq 1}$, $n = 1, 2, \ldots$, satisfying the following two conditions.

(i) For each n the sequence $\chi_{E_n}/||\chi_{E_n}||_1$, d_2^n, d_3^n, $\ldots$ is a generalized Haar basis for $L^1(E_n)$.

(ii) For each n we have $d_1^n = c_n\chi_{E_n} + \psi_n$, where $||d_1^n||_1 = 1$, $||\psi_n||_1 \leq ||c_n\chi_{E_n}||_1$ and ψ_n is a combination of the elements of $(d_k)_{k\geq 0}$ which precede d_1^n.

Next, let recall Theorem 3.1 from [7], which asserts that monotone bases of L^1 are equivalent to simple bases.

Theorem 2.4. *Let $(e_k)_{k\geq 0}$ be a normalized monotone basis for $L^1(\Omega, \mathcal{F}, \mu)$. Then there is an isometry T of $L^1(\Omega, \mathcal{F}, \mu)$ onto some $L^1(\Omega, \mathcal{F}, \nu)$, which sends $(e_k)_{k\geq 0}$ to some simple basis $(d_k)_{k\geq 0}$.*

The proof of this statement, presented in [7], shows that one can take $\mathrm{d}\nu = |\varphi|\mathrm{d}\mu$ and $Tf = f/\varphi$ for an appropriately chosen measurable function $\varphi : \Omega \to \mathbb{R} \setminus \{0\}$. Thus, we see that for each nonnegative integer n and any numbers a_0, a_2, $\ldots$, a_n,

$$T\left(\max_{0\leq m\leq n}\left|\sum_{k=0}^m a_k e_k\right|\right) = \max_{0\leq m\leq n}\left|\sum_{k=0}^m a_k d_k\right|$$

and

$$T\left(\left(\sum_{k=0}^{n}|a_ke_k|^2\right)^{1/2}\right)=\left(\sum_{k=0}^{n}|a_kd_k|^2\right)^{1/2},$$

which implies

$$\left\|\max_{0\le m\le n}\left|\sum_{k=0}^{m}a_ke_k\right|\right\|_{L^1(\mu)}=\left\|\max_{0\le m\le n}\left|\sum_{k=0}^{m}a_kd_k\right|\right\|_{L^1(\nu)}$$

and

$$\left\|\left(\sum_{k=0}^{n}|a_ke_k|^2\right)^{1/2}\right\|_{L^1(\mu)}=\left\|\left(\sum_{k=0}^{n}|a_kd_k|^2\right)^{1/2}\right\|_{L^1(\nu)}$$

for each n. Therefore (1.4) and (1.5) follow in full generality if we establish them for simple bases only. We will now introduce a method which will enable us to study this special case efficiently.

3. A related boundary value problem

Throughout this section, $e=(e_k)_{k\ge 0}$ stands for a simple basis of $L^1(\Omega,\mathcal{F},\mu)$. For any $f=\sum_{k=0}^\infty a_ke_k$, we will use the notation $f_n=P_nf=\sum_{k=0}^n a_ke_k$ for the projection on the subspace generated by e_0, e_1, …, e_n. In addition, for any $n=0,1,2,\ldots$, we will write $f_n^*(\omega)=\max_{0\le k\le n}|f_k(\omega)|$ and $S_n(f)(\omega)=\left(\sum_{k=0}^n|a_ke_k(\omega)|^2\right)^{1/2}$, $\omega\in\Omega$, for the maximal and square functions of f.

We turn our attention to the method. Consider the set

$$D=\{(x,y,z)\in\mathbb{R}\times[0,\infty)\times[0,\infty):|x|\vee z>0\}\cup\{(0,0,0)\}$$

and suppose that $V:D\to\mathbb{R}$ is a fixed function, satisfying $V(0,0,0)=0$ and

$$V(x,y,z)=V(x,y,|x|\vee z),\qquad (x,y,z)\in D. \tag{3.1}$$

This function need not be Borel or even measurable. Suppose further that we want to establish the estimate

$$\int_\Omega V\left(f_n(\omega),S_n(f)(\omega),f_n^*(\omega)\right)\,\mathrm{d}\mu(\omega)\le 0,\qquad n=0,1,2,\ldots, \tag{3.2}$$

for all $f\in L^1(\Omega)$. To do this, we introduce the class $\mathcal{U}(V)$ which consists of all functions U which enjoy the following.

1° For all $(x,y,z)\in D$ we have

$$U(x,y,z)=U(x,y,|x|\vee z). \tag{3.3}$$

2° For all $(x,y,z)\in D$ we have

$$U(x,y,z)\ge V(x,y,z). \tag{3.4}$$

3° If $|x|\le z$ and $\alpha_1,\alpha_2\in(0,1)$, $t_1,t_2\in\mathbb{R}$ satisfy $\alpha_1+\alpha_2=1$, $\alpha_1t_1+\alpha_2t_2=0$, then

$$U(x,y,z)\ge\alpha_1U(x+t_1,(y^2+t_1^2)^{1/2},z)+\alpha_2U(x+t_2,(y^2+t_2^2)^{1/2},z). \tag{3.5}$$

4° If $|x| \leq z$ and t_1, $t_2 \in \mathbb{R}$, then

$$|t_2|U(x,y,z) \geq |t_2|U(x+t_1,(y^2+t_1^2)^{1/2},z)+|t_1|U(t_2,|t_2|,|t_2|). \tag{3.6}$$

Let us gain some intuition about these requirements. The property 1° is a technical assumption which handles the maximal function in an appropriate induction argument: see below. In a moment, we will show that the properties 1°, 3° and 4° yield (3.2), but with V replaced by U; therefore, the role of the majorization 2° is to enable the replacement of U by V. The condition 3° can be regarded as an appropriate concavity-type property. More precisely, 3° means that for any $(x,y,z) \in D$ with $|x| \leq z$, the function $\Phi(t) = U(x+t,(y^2+t^2)^{1/2},z)$, $t \in \mathbb{R}$, is majorized by a linear function Ψ satisfying $\Psi(0) = \Phi(0)$ (if U is of class C^1, then $\Psi(t) = U(x,y,z) + tU_x(x,y,z)$; otherwise one sets $\Psi(t) = U(x,y,z) + t\limsup_{s\to 0+}(U(x-s,(y^2+s^2)^{1/2},z) - U(x,y,z))/s)$. Finally, 4° can be understood as a uniform bound for the slopes of all such functions Ψ's.

Observe first that the above requirements enforce the conditions

$$U(0,0,0) = 0 \tag{3.7}$$

and

$$U(t,|t|,|t|) \leq 0, \qquad t \in \mathbb{R}. \tag{3.8}$$

Indeed, plugging $t_2 = 0$ into (3.6) gives $U(0,0,0) \leq 0$, while 1° together with $V(0,0,0) = 0$ implies the reverse bound. Thus (3.7) follows. To see (3.8), fix $t \neq 0$ and apply 4° to $x = y = z = 0$ and $t_1 = t_2 = t$. As the result, we get an estimate which is equivalent to $0 \geq U(t,|t|,0) + U(t,|t|,|t|)$ which, by (3.3), yields (3.8).

What is the relation between the inequality (3.2) and the class $\mathcal{U}(V)$? The answer is contained in the following statement.

Theorem 3.1. *If the class $\mathcal{U}(V)$ is nonempty, then (3.2) is valid.*

Proof. Since $(e_k)_{k\geq 0}$ is simple, we see that each term e_k is either a generalized Haar function, or it can be written in the form $c\chi_{E_k} + \psi_k$, where ψ_k is a combination of e_0, e_1, ..., e_{k-1}, the set E_k is disjoint from the union of the supports of these functions and $||\psi_k||_1 \leq ||c\chi_{E_k}||_1$. Take $f \in L^1(\Omega,\mathcal{F},\mu)$ and let a_0, a_1, a_2, ... be the coefficients appearing in the expansion of f. The main ingredient of the proof is to show that for any $n \geq 0$,

$$\int_\Omega U\left(f_n, S_n(f), f_n^*\right) \mathrm{d}\mu \geq \int_\Omega U\left(f_{n+1}, S_{n+1}(f), f_{n+1}^*\right) \mathrm{d}\mu. \tag{3.9}$$

To this end, fix $n \geq 0$ and suppose first that e_{n+1} is a generalized Haar function, and its support E is contained in the union of the supports of e_0, e_1, ..., e_n. Then the triples $(f_n, S_n(f), f_n^*)$, $(f_{n+1}, S_{n+1}(f), f_{n+1}^*)$ coincide outside E and hence it suffices to show that

$$\int_E U\left(f_n, S_n(f), f_n^*\right) \mathrm{d}\mu \geq \int_E U\left(f_{n+1}, S_{n+1}(f), f_{n+1}^*\right) \mathrm{d}\mu. \tag{3.10}$$

But f_n, $S_n(f)$ and f_n^* are constant on E, because of the structure of the simple basis e. Denoting the corresponding values by x, y and z, we see that $|x| \leq z$.

By 1°, we have

$$U\left(f_{n+1},S_{n+1}(f),f^*_{n+1}\right)=U\left(f_{n+1},S_{n+1}(f),f^*_n\right)\quad \text{on } \Omega,$$

which allows us to transform the previous estimate into

$$\frac{1}{\mu(E)}\int_E U(x+a_{n+1}e_{n+1},(y^2+|a_{n+1}e_{n+1}|^2)^{1/2},z)\mathrm{d}\mu\leq U(x,y,z).$$

This bound follows at once from 3°, because e_{n+1} is a generalized Haar function (see the second equation in (2.1)). Next, suppose that e_{n+1} is of the second type, that is, $e_{n+1}=c\chi_{E_{n+1}}+\psi_{n+1}$, for appropriate $c\neq 0$, E_{n+1} and ψ_{n+1}. Let E be the support of e_{n+1}. Again, the triples $(f_n,S_n(f),f^*_n)$, $(f_{n+1},S_{n+1}(f),f^*_{n+1})$ coincide outside E; furthermore, $U(f_n,S_n(f),f^*_n)=0$ on E_{n+1}, see (3.7). Consequently, (3.9) can be rewritten in the form

$$\int_{E\setminus E_{n+1}} U\left(f_n,S_n(f),f^*_n\right)\mathrm{d}\mu\geq \int_E U\left(f_{n+1},S_{n+1}(f),f^*_{n+1}\right)\mathrm{d}\mu. \tag{3.11}$$

The right-hand side above is equal to

$$\begin{aligned}&\int_{E_{n+1}} U(c,|c|,|c|)\mathrm{d}\mu+\int_{E\setminus E_{n+1}} U\left(f_n+\psi_{n+1},(S^2_n(f)+\psi^2_{n+1})^{1/2},f^*_{n+1}\right)\mathrm{d}\mu\\ &=\mu(E_{n+1})\,U(c,|c|,|c|)+\int_{E\setminus E_{n+1}} U\left(f_n+\psi_{n+1},(S^2_n(f)+\psi^2_{n+1})^{1/2},f^*_n\right)\mathrm{d}\mu,\end{aligned}$$

by virtue of 1°. Applying 4°, we get the pointwise estimate

$$\begin{aligned}U\left(f_n+\psi_{n+1},(S^2_n(f)+\psi^2_{n+1})^{1/2},f^*_n\right)\leq &\, U\left(f_n,S_n(f),f^*_n\right)\\ &-\frac{|\psi_{n+1}|}{c}U\left(c,|c|,|c|\right).\end{aligned}$$

By (3.8), we have the inequality $U(c,|c|,|c|)\leq 0$. Moreover, $||\psi_{n+1}||_1\leq c\mu(E_{n+1})$, which follows from the form of e_{n+1}. This yields

$$\int_{E\setminus E_{n+1}}\frac{|\psi_{n+1}|}{c}U(c,|c|,|c|)\mathrm{d}\mu\geq \mu(E_{n+1})U(c,|c|,|c|).$$

Combining the facts above gives (3.11) and hence the sequence

$$\left(\int_\Omega U\left(f_n,S_n(f),f^*_n\right)\mathrm{d}\mu\right)_{n\geq 0}$$

is nonincreasing. Consequently, by 2°, we obtain that for any $n\geq 0$,

$$\int_\Omega V\left(f_n,S_n(f),f^*_n\right)\mathrm{d}\mu\leq\int_\Omega U\left(f_n,S_n(f),f^*_n\right)\mathrm{d}\mu\leq\int_\Omega U\left(f_0,S_0(f),f^*_0\right)\mathrm{d}\mu.$$

It remains to note that $f^*_0=S_0(f)=|f_0|$ and use (3.8) to get the desired estimate (3.2). □

A very important phenomenon is that the implication of the above theorem can be reversed: the validity of the estimate (3.2) yields the existence of a special function satisfying the conditions 1°-4°. We need some additional notation to explain this fact. For a given measure space $(\Omega,\mathcal{F},\mu)$ with $\mu(\Omega)\geq 1$, we denote by $\mathcal{E}(\Omega,\mathcal{F},\mu)$ the family of all simple bases $(e_k)_{k\geq 0}$ of $L^1(\Omega,\mathcal{F},\mu)$ such that e_0 is the characteristic function of a set of measure 1. Of course, this family is nonempty. Next, for a given basis $e \in \mathcal{E}(\Omega, \mathcal{F}, \mu)$ and a real number x, we define $\mathcal{M}(x, e)$ to be the class of all functions f which admit the expansion $f = xe_0 + \sum_{k=1}^n a_k e_k$ for some n and some sequence $a_1, a_2, \ldots, a_n \in \mathbb{R}$.

Equipped with the above definitions, we are ready to introduce the abstract function $U^0 : D \to \mathbb{R} \cup \{\infty\}$ which will be shown to belong to the class $\mathcal{U}(V)$. Set

$$U^0(x, y, z) = \sup \left\{ \int_\Omega V(f, (y^2 e_0 - x^2 e_0 + S^2(f))^{1/2}, f^* \vee ze_0) \mathrm{d}\mu \right\}, \tag{3.12}$$

where the supremum is taken over all measurable spaces $(\Omega, \mathcal{F}, \mu)$ with $\mu(\Omega) \geq 1$, all $e \in \mathcal{E}(\Omega, \mathcal{F}, \mu)$ and all $f \in \mathcal{M}(x, e)$.

Theorem 3.2. *If the inequality (3.2) holds true, then the function U^0 belongs to the class $\mathcal{U}(V)$.*

Proof. The condition 1° is a consequence of the pointwise bound $f^* \geq |f_0| = |x|e_0$, since $f^* \vee ze_0 = f^* \vee ((|x| \vee z)e_0)$. To prove 2°, we compute the integral in (3.12) for $n = 0$ and some arbitrary basis $e \in \mathcal{E}(\Omega, \mathcal{F}, \mu)$. Because e_0 is the indicator function of some set of measure one, we see that the integral is equal to $V(x, y, |x| \vee z) = V(x, y, z)$, by (3.1). This implies the desired property 2°. To establish 3°, choose x, y, z, α_1, α_2, t_1 and t_2 as in the statement of the condition. Take two bases $e^1 \in \mathcal{E}(\Omega^1, \mathcal{F}^1, \mu^1)$, $e^2 \in \mathcal{E}(\Omega^2, \mathcal{F}^2, \mu^2)$ and two functions f^1, f^2 enjoying the following finite expansions in e^1 and e^2:

$$f^i = (x + t_i)e_0^i + \sum_{k=1}^n a_k^i e_k^i, \qquad i = 1, 2 \tag{3.13}$$

(we may assume that the length of the expansion is the same for both pairs, enlarging one of them by zeros if necessary). By the very definition of the square function,

$$S^2(f^i) = (x + t_i)^2 e_0^i + \sum_{k=1}^n |a_k^i e_k^i|^2. \tag{3.14}$$

Now suppose that Ω^1 and Ω^2 are disjoint and let us "glue" the measure spaces $(\Omega^i, \mathcal{F}^i, \mu^i)$ into one space $(\Omega, \mathcal{F}, \mu)$, with $\Omega = \Omega^1 \cup \Omega^2$, $\mathcal{F} = \sigma(\mathcal{F}^1, \mathcal{F}^2)$ and $\mu(A^1 \cup A^2) = \alpha_1 \mu^1(A^1) + \alpha_2 \mu^2(A^2)$ for all $A^i \in \mathcal{F}^i$, $i = 1, 2$. Next, we concatenate e^1 and e^2 into one basis $e \in \mathcal{E}(\Omega, \mathcal{F}, \mu)$, putting $e_0 = e_0^1 \chi_{\Omega^1} + e_0^2 \chi_{\Omega^2}$, $e_1 = \frac{1}{2}\alpha_1^{-1} e_0^1 \chi_{\Omega^1} - \frac{1}{2}\alpha_2^{-1} e_0^2 \chi_{\Omega^2}$ and, for $k \geq 1$,

$$e_{2k} = \alpha_1^{-1} e_k^1 \chi_{\Omega^1}, \quad e_{2k+1} = \alpha_2^{-1} e_k^2 \chi_{\Omega^2}.$$

It is evident that this new sequence forms a simple basis of $L^1(\Omega, \mathcal{F}, \mu)$. In addition, e_0 is the indicator function of a certain set of measure 1, so $e \in \mathcal{E}(\Omega, \mathcal{F}, \mu)$. Using (3.13), we check easily that the function

$$f = f^1 \chi_{\Omega^1} + f^2 \chi_{\Omega^2} \tag{3.15}$$

has the following expansion in the basis e:

$$f = xe_0 + 2\alpha_1 t_1 e_1 + \sum_{k=1}^{n} \big(\alpha_1 a_k^1 e_{2k} + \alpha_2 a_k^2 e_{2k+1}\big).$$

Therefore, using the equality $2\alpha_1 t_1 e_1 = t_1 e_0^1 \chi_{\Omega^1} + t_2 e_0^2 \chi_{\Omega^2}$, we have

$$S^2(f) = x^2 e_0 + t_1^2 e_0^1 \chi_{\Omega^1} + t_2^2 e_0^2 \chi_{\Omega^2} + \sum_{k=1}^{n} |a_k^1 e_k^1|^2 \chi_{\Omega^1} + \sum_{k=1}^{n} |a_k^2 e_k^2|^2 \chi_{\Omega^2}.$$

Consequently, by the definition of U^0 and the formula (3.15) for f, we have

$$\begin{aligned} &U^0(x,y,z)\\ &\geq \int_\Omega V(f, (y^2 e_0 - x^2 e_0 + S^2(f))^{1/2}, f^* \vee z e_0) \mathrm{d}\mu\\ &= \alpha_1 \int_{\Omega^1} V(f^1, ((y^2 + t_1^2) e_0^1 - (x+t_1)^2 e_0^1 + S^2(f^1))^{1/2}, (f^1)^* \vee z e_0^1) \mathrm{d}\mu^1\\ &\quad + \alpha_2 \int_{\Omega^2} V(f^2, ((y^2 + t_2^2) e_0^2 - (x+t_2)^2 e_0^2 + S^2(f^2))^{1/2}, (f^2)^* \vee z e_0^2) \mathrm{d}\mu^2. \end{aligned}$$

Take the supremum over all triples $(\Omega^i, \mathcal{F}^i, \mu^i)$, all n and all functions f^i to get (3.5). Finally, to show (3.6), we may assume that t_1, $t_2 \neq 0$. Pick two bases $e^i \in \mathcal{E}(\Omega^i, \mathcal{F}^i, \mu^i)$ with $(\Omega^i, \mathcal{F}^i, \mu^i)$ as above and two functions f^1, f^2 of the form

$$f^1 = (x+t_1) e_0^1 + \sum_{k=1}^{n} a_n^1 e_n^1, \qquad f^2 = t_2 e_0^2 + \sum_{k=1}^{n} a_n^2 e_n^2.$$

Set

$$\Omega = \Omega^1 \cup \Omega^2, \quad \mathcal{F} = \sigma(\mathcal{F}^1, \mathcal{F}^2), \quad \mu(A^1 \cup A^2) = \mu^1(A^1) + \frac{|t_1|}{|t_2|} \mu^2(A^2),$$

for all $A^1 \in \mathcal{F}^1$, $A^2 \in \mathcal{F}^2$. Furthermore, put $e_0 = e_0^1 \chi_{\Omega^1}$, $e_1 = \frac{1}{2} e_0^1 \chi_{\Omega^1} + \frac{t_2}{2t_1} e_0^2 \chi_{\Omega^2}$ and, for $k \geq 1$, define

$$e_{2k} = e_k^1 \chi_{\Omega^1} \quad \text{and} \quad e_{2k+1} = \frac{t_2}{t_1} e_k^2 \chi_{\Omega^2}.$$

It is straightforward to verify that e is a simple basis; this follows at once from the simplicity of e^1 and e^2. The only thing which needs to be checked is whether e_1 satisfies the condition (ii) of Definition 2.3. But this amounts to verifying the inequality

$$\left|\left| \frac{1}{2} e_0^1 \chi_{\Omega^1} \right|\right|_1 \leq \left|\left| \frac{t_2}{2t_1} e_0^2 \chi_{\Omega^2} \right|\right|_1,$$

which is trivial: actually, both sides are equal. Now, we easily see that the function f given by $f = f^1 \chi_{\Omega^1} + f^2 \chi_{\Omega^2}$ has the expansion

$$f = xe_0 + 2t_1 e_1 + \sum_{k=1}^{n} \left(a_k^1 e_{2k} + \frac{t_1 a_k^2}{t_2} e_{2k+1} \right).$$

Consequently, by the definition of U^0, we get

$$\begin{aligned} U^0(x,y,z) &\geq \int_\Omega V(f,(y^2e_0-x^2e_0+S^2(f))^{1/2}, f^*\vee ze_0)\mathrm{d}\mu \\ &= \int_{\Omega^1} V(f^1,((y^2+t_1^2)e_0^1-(x+t_1)^2+S^2(f^1))^{1/2},(f^1)^*\vee ze_0^1)\mathrm{d}\mu^1 \\ &\quad + \frac{|t_1|}{|t_2|}\int_{\Omega^2} V(f^2,S(f^2),(f^2)^*\vee 0)\mathrm{d}\mu^2. \end{aligned}$$

However, $(f^2)^* \geq |f_0^2|$, so

$$V(f^2,S(f^2),(f^2)^*\vee 0)=V(f^2,S(f^2),(f^2)^*\vee |f_0^2|)=V(f^2,g^2,(f^2)^*\vee |t_2|e_0^2),$$

so it suffices to take supremum over all f^i to obtain (3.6). □

We conclude this section by two important observations.

Remark 3.3. (i) If the maximal function does not appear in the estimate under investigation, then we may consider U, V defined on the appropriate two-dimensional domain. Simply remove the variable z corresponding to the non-existing maximal function. A similar argument applies to the case when the studied inequality does not involve the square function: then the variable y can be omitted. However, in general we *cannot* remove the variable x in the case when the inequality involves the square and maximal functions only (e.g., like in (1.7)). To understand the reason, the reader is urged to (3.12). If V does not depend on y or z, then the same is true for U^0 (in other words, there exists a special function of two variables only). However, if V does not depend on x, then in general we cannot say the same about U^0.

(ii) In certain cases, the function U^0 inherits some crucial properties from the function V, which in turn simplifies the search for its explicit formula. For example, if V is homogeneous of order p, then so is U^0. To see this, pick arbitrary $(\Omega,\mathcal{F},\mu)$ with $\mu(\Omega)\geq 1$, $e\in\mathcal{E}(\Omega,\mathcal{F},\mu)$, $f\in\mathcal{M}(x,e)$ and $\lambda>0$. Then $\lambda f\in\mathcal{M}(\lambda x,e)$ and hence

$$\begin{aligned} U^0(\lambda x,\lambda y,\lambda z) &\geq \int_\Omega V\big(\lambda f,\lambda S(f),\lambda(f^*\vee ze_0)\big)\mathrm{d}\mu \\ &= \lambda^p\int_\Omega V\big(f,S(f),f^*\vee ze_0\big)\mathrm{d}\mu. \end{aligned}$$

Taking the supremum over all the parameters gives the inequality

$$U^0(\lambda x,\lambda y,\lambda z)\geq \lambda^p U^0(x,y,z) \qquad \text{for } (x,y,z)\in D,$$

and switching from λ to λ^{-1} yields the reverse bound. Using a similar reasoning one can show that if V satisfies the symmetry condition

$$V(x,y,z)=V(-x,y,z)=V(x,-y,z) \qquad \text{for all } (x,y,z)\in D,$$

then the same is true for U^0.

4. Proofs of (1.4) and (1.5)

As an application of the method described in the previous section, let us present the proofs of the estimates formulated in the introduction. Obviously, it suffices to focus on the L^1-inequalities (1.4) and (1.5); then the weak-type bounds follow immediately by the use of Chebyshev's inequality. We start with (1.4). In view of Lebesgue's monotone convergence theorem and Fatou's lemma, it suffices to prove that for any monotone basis e of $L^1(\Omega, \mathcal{F}, \mu)$, any n and all a_0, a_1, a_2, $\ldots a_n \in \mathbb{R}$ we have

$$\int_\Omega \left|\sum_{k=0}^n a_k e_k(\omega)\right| \mathrm{d}\mu(\omega) \leq 2\int_\Omega \left(\sum_{k=0}^n |a_k e_k(\omega)|^2\right)^{1/2} \mathrm{d}\mu(\omega).$$

This can be rewritten in the more compact form

$$\int_\Omega V(f_n, S_n(f))\mathrm{d}\mu \leq 0, \tag{4.1}$$

where $V(x,y) = |x| - 2y$ and f_n, $S_n(f)$ are as previously. Thus the problem is of the form (3.2) and hence it can be treated by means of Theorems 3.1 and 3.2. Now, for any $(x,y) \in \mathbb{R} \times [0,\infty)$, define the special function by

$$U(x,y) = \begin{cases} -(2y^2 - x^2)^{1/2} & \text{if } |x| \leq y, \\ |x| - 2y & \text{if } |x| > y. \end{cases} \tag{4.2}$$

Theorem 4.1. *The function U belongs to $\mathcal{U}(V)$.*

Proof. We need to verify the conditions 1°-4°. The first of them is empty, since U does not depend on z. To show the majorization 2°, we may assume that $|x| \leq y$ (otherwise both sides are equal). Squaring both sides, we obtain the equivalent form $(|x| - y)^2 \geq 0$, which is obviously true. The main technical difficulty lies in proving the conditions 3° and 4°.

Proof of 3°. We will prove the estimate

$$U(x+t, (y^2+t^2)^{1/2}) \leq U(x,y) + U_x(x,y)t \tag{4.3}$$

for all $x \in \mathbb{R}$, $y > 0$ and $t \in \mathbb{R}$. This will yield the desired condition. Indeed, if we apply (4.3) to $t = t_1$ and multiply both sides by α_1, then apply (4.3) to $t = t_2$ and multiply throughout by α_2, and finally add the obtained estimates, we get the inequality from 3° (for $y > 0$, but the passage to nonnegative y is trivial, by continuity argument).

To show (4.3), note that we may assume that $x \geq 0$, by symmetry. We consider four cases. Suppose first that $x \geq y$ and $|x + t| \geq (y^2 + t^2)^{1/2}$. Then $t \geq -x$, since otherwise, squaring the previous estimate, we would get $x^2 + 2tx \geq y^2$, a contradiction. Hence, (4.3) reduces to the trivial bound

$$x + t - 2(y^2 + t^2)^{1/2} \leq x + t - 2y.$$

Suppose now that $x \geq y$ and $|x+t| \leq (y^2+t^2)^{1/2}$; then $t \leq t_0$, where $t_0 \leq 0$ is the unique number satisfying $x + t_0 = (y^2 + t_0^2)^{1/2}$. The inequality (4.3) is equivalent to $2y - x - t \leq (2y^2 + 2t^2 - (x+t)^2)^{1/2}$. Both sides are nonnegative, so squaring both sides we get the equivalent form

$$2y^2 + 2yx + x^2 + 2xt + 2yt \leq 0.$$

This bound is true for $t = t_0$, by the previous case. Hence, since $t \leq t_0$ and x, y are nonnegative, the inequality holds. Next, assume that $x \leq y$ and $|x+t| \leq (y^2+t^2)^{1/2}$. Then (4.3) reads

$$-(2y^2 + 2t^2 - (x+t)^2)^{1/2} \leq -(2y^2 - x^2)^{1/2} + xt(2y^2 - x^2)^{-1/2},$$

or, equivalently,

$$2y^2 - x^2 - xt \leq \left[(2y^2 - x^2)(2y^2 - x^2 + t^2 - 2xt)\right]^{1/2}.$$

If the left-hand side is negative, there is nothing to prove; otherwise, we square both sides and manipulate a little to obtain the equivalent estimate $x^2t^2 \leq (2y^2 - x^2)t^2$, which is true due to the assumption $x \leq y$. Finally, suppose that $x \leq y$ and $|x+t| \geq (y^2+t^2)^{1/2}$. Then $(x+t)^2 \leq y^2 + 2xt + t^2 \leq 2(y^2+t^2)$ and the calculations from the previous case can be repeated: we get

$$U(x,y) + U_x(x,y)t \geq -(2y^2 + 2t^2 - (x+t)^2)^{1/2},$$

and the expression on the right is not smaller than $|x+t| - 2(y^2+t^2)^{1/2}$, which can be easily checked by squaring.

Proof of 4°. If $t_2 = 0$, then both sides are equal; otherwise we divide both sides by $|t_2|$ and note that the inequality reduces to

$$U(x,y) \geq U(x+t, (y^2+t^2)^{1/2}) - |t|$$

for all x, y and t. Again, we consider four cases, depending on the interplay between x, y and $x+t$, $(y^2+t^2)^{1/2}$. If $|x| \geq y$ and $|x+t| \geq (y^2+t^2)^{1/2}$, then the desired bound is the sum of the obvious estimates $|x| \geq |x+t| - |t|$ and $-2y \geq -2(y^2+t^2)^{1/2}$. If $|x| \geq y$ and $|x+t| \leq (y^2+t^2)^{1/2}$, then

$$|x| - 2y + |t| \geq -y + |t| \geq -(y^2+t^2)^{1/2} \geq -(2y^2 + 2t^2 - (x+t)^2)^{1/2},$$

which is precisely the claim. If $|x| \leq y$ and $|x+t| \geq (y^2+t^2)^{1/2}$, then, using 2°,

$$-(2y^2 - x^2)^{1/2} \geq |x| - 2y \geq |x+t| - |t| - 2(y^2+t^2)^{1/2},$$

as desired. Finally, if $|x| \leq y$ and $|x+t| \leq (y^2+t^2)^{1/2}$, the estimate 4°, after squaring, is equivalent to

$$t^2 - xt + |t|(2(y^2+t^2) - (x+t)^2)^{1/2} \geq 0,$$

and follows from the estimates $t^2 \geq 0$ and $(2(y^2+t^2)-(x+t)^2)^{1/2} \geq (y^2+t^2)^{1/2} \geq y \geq |x|$. □

The proof of the estimate (1.5) will be similar (of course, a different special function will be used). Arguing as previously, it is enough to show that for any monotone basis e of $L^1(\Omega, \mathcal{F}, \mu)$, any n and all a_0, a_1, a_2, $\dots a_n \in \mathbb{R}$ we have

$$\int_\Omega \left(\sum_{k=0}^n |a_k e_k(\omega)|^2 \right)^{1/2} \mathrm{d}\mu(\omega) \leq \sqrt{3} \int_\Omega \max_{0 \leq m \leq n} \left| \sum_{k=0}^m a_k e_k(\omega) \right| \mathrm{d}\mu(\omega).$$

This is equivalent to saying that

$$\int_\Omega V(f_n, S_n(f), f_n^*)\mathrm{d}\mu \le 0, \tag{4.4}$$

where $V(x,y,z) = y - \sqrt{3}|x| \vee z$. This problem is of the form (3.2) and we can apply the machinery developed in the preceding section. The corresponding special function $U : D \to \mathbb{R}$ is given by

$$U(x,y,z) = \frac{y^2 - x^2 - 2(|x| \vee z)^2}{2\sqrt{3}(|x| \vee z)}, \tag{4.5}$$

when $|x| \vee z > 0$, and $U(0,0,0) = 0$.

Theorem 4.2. *The function U belongs to $\mathcal{U}(V)$.*

Proof. The condition 1° is evident. To show majorization 2°, observe that for $(x,y,z) \neq (0,0,0)$,

$$\begin{aligned} U(x,y,z) &= \frac{y^2 - x^2 - 2(|x| \vee z)^2}{2\sqrt{3}(|x| \vee z)} \\ &\ge \frac{y^2 - 3(|x| \vee z)^2}{2\sqrt{3}(|x| \vee z)} \\ &= \frac{(y - \sqrt{3}(|x| \vee z))^2 + 2\sqrt{3}y(|x| \vee z) - 6(|x| \vee z)^2}{2\sqrt{3}(|x| \vee z)} \ge V(x,y,z). \end{aligned}$$

Again, the most elaborate part of the proof is the study of 3° and 4°.

Proof of 3°. As previously, we will show the slightly stronger bound

$$U(x+t, (y^2+t^2)^{1/2}, z) \le U(x,y,z) + U_x(x,y,z)t \tag{4.6}$$

for any x, $t \in \mathbb{R}$ and y, $z > 0$ such that $|x| \le z$. By symmetry, we may and do assume that $t \ge 0$. If $|x+t| \le z$, then both sides are equal. If $|x+t| > z$, then $|x+t| \vee z = x+t$ and the estimate becomes

$$\frac{y^2 + t^2 - 3(x+t)^2}{x+t} \le \frac{y^2 - x^2 - 2xt - 2z^2}{z},$$

or $(y^2 + t^2 - (x+t)^2)(z - (x+t)) \le 2(x+t)z(x+t-z)$. Dividing both sides by $x+t-z$ we get the equivalent form $x^2 - 2xt - y^2 \le 2(x+t)z$, which is evident: we have $y^2 \ge 0$ and $x^2 + 2xt \le 2x(x+t) \le 2z(x+t)$.

Proof of 4°. If $t_2 = 0$, then the inequality is satisfied. If $t_2 \neq 0$, we divide both sides by t_2, put $t = t_1$ and obtain the equivalent form

$$U(x,y,z) \ge U(x+t, (y^2+t^2)^{1/2}, z) - \frac{|t|}{\sqrt{3}}.$$

By symmetry, we may assume that $t \ge 0$. If $|x+t| \le z$, then some simple manipulations transform the estimate into $-2xt - 2zt \le 0$, which is obviously true. Suppose then that $|x+t|>z$; then $|x+t|=x+t$ and, after some straightforward calculations,

$$U(x+t,(y^2+t^2)^{1/2},z)-\frac{|t|}{\sqrt{3}}=\frac{y^2+x^2}{2\sqrt{3}(x+t)}-\frac{4(x+t)}{2\sqrt{3}}$$
$$\leq \frac{y^2+x^2}{2\sqrt{3}z}-\frac{4z}{2\sqrt{3}}\leq U(x,y,z).$$

This completes the proof. □

5. Sharpness

In this section we address the optimality of the constants 2 and $\sqrt{3}$ involved in the estimates (1.4)-(1.7). Obviously, it is enough to show that these constants are the best in the weak-type estimates (1.6) and (1.7), respectively. Our argument will exploit Theorem 3.2.

Sharpness of (1.6). Suppose that the estimate (1.6) holds true with some constant β. Then for any n and any coefficients a_0, a_1, $\ldots\in\mathbb{R}$ we have

$$2\mu\left(\left|\sum_{k=0}^n a_ke_k\right|\geq 2\right)\leq \beta\int_\Omega\left(\sum_{k=0}^n |a_ke_k|^2\right)^{1/2}\mathrm{d}\mu.$$

This is equivalent to the estimate

$$\int_\Omega V(f_n,S_n(f))\mathrm{d}\mu\leq 0,$$

where V is given by $V(x,y)=2\chi_{\{|x|\geq 2\}}-\beta y$. By Theorem 3.2, the function

$$U^0(x,y)=\sup\left\{\int_\Omega V(f_n,S_n(f))\mathrm{d}\mu\right\},$$

with the supremum taken over appropriate parameters, belongs to the class $\mathcal{U}(V)$. Since $V(x,y)=V(-x,y)$ for all x, y, the function U^0 inherits this property. Apply 3° with $x=y=0$, $t_1=-t_2=1$, $\alpha_1=\alpha_2=1$ to get

$$0=U^0(0,0)\geq\frac{1}{2}U^0(1,1)+\frac{1}{2}U^0(-1,1)=U^0(1,1),$$

so in particular $U^0(1,1)$ is finite. Now, for given integers $0\leq k<n$, apply 4° with $x=1+k/n$, $y=(1+k/n^2)^{1/2}$, $t_1=1/n$ and $t_2=1$ to obtain

$$U^0\left(1+\frac{k}{n},\left(1+\frac{k}{n^2}\right)^{1/2}\right)\geq U^0\left(1+\frac{k+1}{n},\left(1+\frac{k+1}{n^2}\right)^{1/2}\right)+\frac{1}{n}U^0(1,1).$$

This, by induction, implies

$$U^0(1,1)\geq U^0\left(2,\left(1+\frac{1}{n}\right)^{1/2}\right)+U^0(1,1),$$

and hence, using 2°, we get

$$0 \geq U^0\left(2, \left(1+\frac{1}{n}\right)^{1/2}\right) \geq V\left(2, \left(1+\frac{1}{n}\right)^{1/2}\right) = 2 - \beta\left(1+\frac{1}{n}\right)^{1/2}.$$

Letting $n \to \infty$, we see that β must be at least 2. This proves the desired sharpness. □

Sharpness of (1.7). The argument is similar. We assume that the inequality (1.7) holds true with some constant γ. Then for any n and any coefficients $a_0, a_1, \ldots \in \mathbb{R}$ we have

$$\sqrt{3}\mu\left(\left(\sum_{k=0}^n |a_k e_k|^2\right) \geq \sqrt{3}\right) \leq \gamma \int_\Omega \max_{0\leq m\leq n}\left|\sum_{k=0}^m a_k e_k\right| \mathrm{d}\mu,$$

or $\int_\Omega V(f_n, S_n(f), f_n^*)\mathrm{d}\mu \leq 0$, where $V(x,y,z) = \sqrt{3}\chi_{\{y\geq\sqrt{3}\}} - \gamma|x| \vee z$. By the machinery developed in §3, the function

$$U^0(x,y,z) = \sup\left\{\int_\Omega V(f_n, S_n(f), f_n^*)\mathrm{d}\mu\right\}$$

(with the supremum taken over appropriate parameters), lies in the class $\mathcal{U}(V)$. Furthermore, we have $U^0(x,y,z) = U^0(-x,y,z)$, since the same property holds for V.

By 4°, applied to $x=y=z=0$, $t_1 = -t_2 = 1$ and $\alpha_1 = \alpha_2 = 1/2$, we get

$$0 = U^0(0,0,0) \geq \frac{1}{2}U^0(1,1,0) + \frac{1}{2}U^0(-1,1,0) = U^0(1,1,0) = U^0(1,1,1),$$

where in the last passage we have exploited 1°. Consequently, $U^0(1,1,1)$ is finite. Now apply 4° with $x=y=z=1$, $t_1=-1$ and $t_2=1$ to obtain

$$U^0(1,1,1) \geq U^0(0,\sqrt{2},1) + U^0(1,1,1),$$

or $U^0(0,\sqrt{2},1) \leq 0$, by the aforementioned finiteness of $U^0(1,1,1)$. Now use 3° with $x=0$, $y=\sqrt{2}$, $z=1$, $t_1=-t_2=1$ and $\alpha_1=\alpha_2=1/2$ to get

$$0 \geq U^0(0,\sqrt{2},1) \geq \frac{1}{2}U^0(1,\sqrt{3},1) + \frac{1}{2}U^0(-1,\sqrt{3},1) = U^0(1,\sqrt{3},1).$$

This, by 2°, implies $V(1,\sqrt{3},1) \leq 0$, or $\gamma \geq \sqrt{3}$. Hence $\sqrt{3}$ is indeed the best possible. □

Acknowledgments

The author would like to thank an anonymous Referee for the careful reading of the paper and several helpful suggestions. The research was supported by Narodowe Centrum Nauki (Poland) grant DEC-2014/14/E/ST1/00532.

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