A Fefferman-Stein inequality for the martingale square and maximal functions

Adam Osękowski

Department of Mathematics, Informatics and Mechanics
University of Warsaw
Banacha 2, 02-097 Warsaw
Poland

Abstract

Suppose that f is a martingale and let $|f|^*$, $S(f)$ denote the associated maximal and square functions. We prove that for any weight w we have

$$|| \, |f|^* \, ||_{L^1(w)} \leq C ||S(f)||_{L^1(w^*)}$$

with $C = 16(\sqrt{2}+1) = 38.62742\ldots$. The proof rests on the construction of an appropriate special function, enjoying certain size and concavity conditions. Furthermore, we show that the term w^* on the right cannot be replaced by the r-maximal function of w for any $0 < r < 1$.

Keywords: Martingale, maximal function, square function, best constant
2010 MSC: 60G44, 60G42.

1. Introduction

The purpose of this note is to establish a weighted version of Davis' inequality between maximal and square functions of an arbitrary discrete-time martingales. In what follows, we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space equipped with a family $(\mathcal{F}_n)_{n\geq 0}$ of non-decreasing sub-σ-algebras of $\mathcal{F}$. Let $f = (f_n)_{n\geq 0}$ be an adapted martingale with the difference sequence $df = (df_n)_{n\geq 0}$ given by the equalities $df_0 = f_0$ and $df_n = f_n - f_{n-1}$ for $n \geq 1$. The maximal and square functions of f are defined by $|f|^* = \sup_{k\geq 0} |f_k|$ and $S(f) = \left(\sum_{k=0}^\infty df_k^2\right)^{1/2}$. We will also need truncated versions of these objects, given by $|f|_n^* = \max_{0\leq k\leq n} |f_k|$ and $S_n(f) = \left(\sum_{k=0}^n df_k^2\right)^{1/2}$, $n = 0, 1, 2, \ldots$. Furthermore, we will work with one-sided maximal function $f^* = \sup_{k\geq 0} f_k$ and let $f_n^* = \max_{0\leq k\leq n} f_k$ for $n \geq 0$.

The inequalities involving f, $|f|^*$ and $S(f)$ play a prominent role in probability and the theory of stochastic integration. For an overview of the results

Email address: ados@mimuw.edu.pl (Adam Osękowski)
URL: www.mimuw.edu.pl/∼ados (Adam Osękowski)

in this direction, we refer the interested reader to the survey by Burkholder (1991) or the monograph by Osękowski (2012). Our motivation comes from the classical result of Davis (1970) which asserts the existence of an absolute constant c such that

$$|| \, |f|^* \, ||_{L^1} \leq c||S(f)||_{L^1}$$

for all real-valued martingales f. We will be interested in the *weighted* version of this bound, where, in the probabilistic context, the word "weight" refers to a nonnegative and integrable random variable w. A short bibliographical comment is in order. The theory of weighted inequalities has developed rapidly during the last forty years, starting with the seminal analytic work of Muckenhoupt (1972) on the L^p-boundedness of Hardy-Littlewood maximal operator. Since then, numerous estimates for various classes of operators (e.g., integral, singular integral, fractional, area functionals) have been investigated. Many aspects of this theory can be carried over to the probabilistic setting: see Kazamaki (1994) for an overview of results in this direction.

We come back to the weighted version of Davis' inequality. Consider the estimate

$$|| \, |f|^* \, ||_{L^1(w)} \leq c_w||S(f)||_{L^1(w)},$$

where $||\xi||_{L^1(w)} = \mathbb{E}|\xi|w$ is the usual weighted L^1-norm and the constant c_w depends only on w. It is not difficult to show that in general, without any additional assumptions on w, this estimate fails to hold, no matter what c_w is. To see a simple example, suppose that f is a symmetric random walk over the integers, started at 1 and stopped upon reaching 0. Then the random variable $|f|^*/S(f)$ is unbounded and hence there is a nonnegative random variable w such that $|| \, |f|^* \, ||_{L^1(w)} = \infty$ and $|| \, S(f) \, ||_{L^1(w)} < \infty$.

Thus the inequality must be modified. Motivated by related results of Fefferman & Stein (1971) from harmonic analysis, we will replace the weight w on the right-hand side by its maximal function. That is, we consider the martingale $(w_n)_{n\geq 0} = (\mathbb{E}(w|\mathcal{F}_n))_{n\geq 0}$ induced by w and consider the space $L^1(|w|^*) = L^1(w^*)$ on the right. Here is the precise statement.

Theorem 1.1. *Suppose that w is a weight. Then for any martingale $f = (f_n)_{n\geq 0}$ we have*

$$|| \, |f|^* \, ||_{L^1(w)} \leq 16(\sqrt{2}+1)||S(f)||_{L^1(w^*)}. \tag{1}$$

Here the constant $16(\sqrt{2}+1) = 38.62742\ldots$ does not seem to be the best possible, but we believe it is not very far from the optimal one. We would like to comment that the above result complements the reverse estimate obtained in the recent paper by Osękowski (2017).

Our proof will exploit Burkholder's technique (called sometimes the Bellman function method in the literature). Namely, we will deduce the validity of (1) from the existence of a certain special function of four variables, enjoying appropriate majorization and concavity.

There is an interesting question whether the estimate (1) can be improved by decreasing the term w^* on the right. A natural candidate is the r-maximal function of w, where $0 < r < 1$ is a given parameter. Recall that the r-maximal function is given by the formula $[(w^r)^*]^{1/r}$, i.e., we take the martingale generated by the integrable random variable w^r and consider its maximal function raised to the power $1/r$. We will show that after such change, the weighted estimate fails to hold, even if we replace $|f|^*$ on the left by the terminal variable f of the martingale.

Theorem 1.2. *Let $0 < r < 1$. Then for any $c > 0$ there is a finite martingale $f = (f_n)_{n=0}^N$ and a weight w such that*

$$\mathbb{E}|f_N|w > c\mathbb{E}S(f)[(w^r)^*]^{1/r}.$$

We have organized the rest of this paper as follows. We establish Theorem 1.1 in the next section. The counterexamples of Theorem 1.2 are provided in the final part of this note.

2. Proof of Theorem 1.1

Throughout this paper, the letter C will stand for the constant $4(\sqrt{2}+1)$. Consider the special function $U : \mathbb{R} \times [0,\infty) \times [0,\infty) \times (0,\infty) \to \mathbb{R}$, given by

$$U(x, y, u, v) = (x^2 + y^2)^{1/2}u - Cyv + 4yv\ln(uv^{-1} + 1).$$

This function enjoys the following size conditions.

Lemma 2.1. *Take $(x, y, u, v) \in \mathbb{R} \times [0,\infty) \times [0,\infty) \times (0,\infty)$ satisfying $u \leq v$. Then we have*

$$U(x, y, v, v) \leq 0, \qquad \text{if } |x| \leq y, \tag{2}$$

$$U(x, y, u, v) \geq |x|u - Cyv \tag{3}$$

and

$$|U_x(x, y, u, v)| \leq v. \tag{4}$$

Proof. This is very straightforward, we leave the details to the reader.

The key property of U is the following concavity-type condition.

Lemma 2.2. *For any $(x, y, u, v) \in \mathbb{R} \times [0,\infty) \times [0,\infty) \times (0,\infty) \to \mathbb{R}$ satisfying $u \leq v$ and any $d \in \mathbb{R}$, $e \geq -u$ we have*

$$\begin{aligned} &U\left(x + d, (y^2 + d^2)^{1/2}, u + e, \max(u + e, v)\right) \\ &\leq U(x, y, u, v) + U_x(x, y, u, v)d + U_u(x, y, u, v)e. \end{aligned} \tag{5}$$

Proof. Fix x, y, u, v, d, e as in the statement. We may assume that $d \geq 0$, replacing x, d with $-x$ and $-d$, if necessary (this does not change the left and the right-hand side of (5)). It is convenient to split the reasoning into three separate parts.

Step 1: $u+e\le v$, $d\le y$. Consider the functions G, $H:[0,1]\to\mathbb{R}$ given by

$$\begin{aligned}G(t)&=U\left(x+td,(y^2+t^2d^2)^{1/2},u+te,v\right)\\&=\left((x+td)^2+y^2+t^2d^2\right)^{1/2}(u+te)-4\sqrt{2}\left(y^2+t^2d^2\right)^{1/2}v\\&\quad-4(y^2+t^2d^2)^{1/2}v+4(y^2+t^2d^2)^{1/2}v\ln((u+te)v^{-1}+1)\end{aligned}$$

and

$$\begin{aligned}H(t)=&\left((x+td)^2+y^2\right)^{1/2}(u+te)-4\sqrt{2}\left(y^2+t^2d^2\right)^{1/2}v\\&-4yv+4yv\ln((u+te)v^{-1}+1).\end{aligned}$$

The assertion is equivalent to $G(1)\le G(0)+G'(0)$. However, we see that $G(0)=H(0)$ and $G'(0)=H'(0)$. Furthermore, for any $t\in[0,1]$ we have $G(t)\le H(t)$, which follows from the estimate

$$\begin{aligned}&\frac{\partial}{\partial s}\Bigg[\left((x+td)^2+s^2\right)^{1/2}(u+te)\\&\qquad\qquad-4\sqrt{2}\left(y^2+t^2d^2\right)^{1/2}v-4sv+4sv\ln((u+te)v^{-1}+1)\Bigg]\\&=\frac{s(u+te)}{\left((x+td)^2+s^2\right)^{1/2}}-4v+4v\ln((u+te)v^{-1}+1)\le v-4v+4v\ln 2\le 0.\end{aligned}$$

Consequently, it suffices to show that $H(1)\le H(0)+H'(0)$ which, in turn, will immediately follow if we prove that H is concave. To this end, we derive that

$$H'(t)=\frac{(x+td)(u+te)d}{\left((x+td)^2+y^2\right)^{1/2}}+\left((x+td)^2+y^2\right)^{1/2}e-\frac{4\sqrt{2}td^2v}{(y^2+d^2)^{1/2}}+\frac{4yve}{u+te+v}$$

and $H''(t)$ is equal to

$$\begin{aligned}&\frac{y^2(u+te)d^2}{\left((x+td)^2+y^2\right)^{3/2}}+\frac{2(x+td)de}{\left((x+td)^2+y^2\right)^{1/2}}-\frac{4\sqrt{2}y^2d^2v}{(y^2+d^2)^{3/2}}-\frac{4yve^2}{(u+te+v)^2}\\&\le\frac{vd^2}{y}+2d|e|-\frac{4\sqrt{2}y^2d^2v}{(y^2+d^2)^{3/2}}-\frac{ye^2}{v},\end{aligned}$$

where we have used the bound $u+te\le v$. By the second assumption of this step, we have $d\le y$. This implies $y^2+d^2\le 2y^2$ and hence

$$H''(t)\le\frac{vd^2}{y}+2d|e|-\frac{2vd^2}{y}-\frac{ye^2}{v}\le 0.$$

Step 2: $d\ge y$, $u+e\le v$. Denote

$$\begin{aligned}I(d)=&\,U\left(x+d,(y^2+d^2)^{1/2},u+e,(u+e)\vee v\right)\\&-(U(x,y,u,v)+U_x(x,y,u,v)d+U_u(x,y,u,v)e)\end{aligned}$$

It follows from the previous step that $I(d) \leq 0$ if $d = y$. Therefore we will be done if we show that $I'(d) \leq 0$. A direct computation yields

$$
\begin{aligned}
I'(d) &= \frac{\big((x+d)+d\big)(u+te)}{((x+d)^2+y^2+d^2)^{1/2}} \\
&\quad - \frac{vd}{(y^2+d^2)^{1/2}}\left[4\sqrt{2}+4-4\ln((u+e)v^{-1}+1)\right] - \frac{xu}{(x^2+y^2)^{1/2}} \\
&\leq 2^{1/2}v - 2^{-1/2}v\left[4\sqrt{2}+4-4\ln 2\right] + v \leq 0,
\end{aligned}
$$

where we have used the bound $d/(y^2+d^2)^{1/2} \geq 2^{-1/2}$ which is a consequence of the assumption $d \geq y$.

Step 3. $u+e \geq v$. This assumption implies that e is a nonnegative number. Let

$$
\begin{aligned}
J(e) = U&\left(x+d, (y^2+d^2)^{1/2}, u+e, (u+e)\vee v\right) \\
&- \left(U(x,y,u,v) + U_x(x,y,u,v)d + U_u(x,y,u,v)e\right).
\end{aligned}
$$

This is precisely the expression which was denoted by $I(d)$ in the previous step; however, here we treat it as a function of e. It follows from Steps 1 and 2 that in the limit case $u+e=v$ we have $J(e) \leq 0$. Therefore it is enough to prove that $J'(e) \leq 0$. The derivative is equal to

$$
\begin{aligned}
&((x+d)^2+y^2+d^2)^{1/2} - (x^2+d^2)^{1/2} - (y^2+d^2)^{1/2}(4\sqrt{2}+4-4\ln 2) - 2y \\
&\leq \frac{2(x+d)d}{((x+d)^2+y^2+d^2)^{1/2}+(x^2+d^2)^{1/2}} - (y^2+d^2)^{1/2}(4\sqrt{2}+4-4\ln 2) \\
&\leq 2d - d(4\sqrt{2}+4-4\ln 2) \leq 0.
\end{aligned}
$$

This completes the proof of the lemma.

Proof of (1). Fix a martingale $f = (f_n)_{n\geq 0}$ and a weight w inducing the associated martingale $(w_n)_{n\geq 0}$. We may assume that the weight w is strictly positive, by a simple approximation argument (add a small ε to w and let $\varepsilon \to 0$ at the very end). Furthermore, we may and do assume that $\mathbb{E}S(f)w^* < \infty$, since otherwise there is nothing to prove. In particular, this implies that for any $n \geq 0$ we have $\mathbb{E}|df_n|w^* < \infty$ and hence also $\mathbb{E}|f_n|w^* < \infty$. For $n \geq 0$, denote $h_n = (f_n, S_n(f), w_n, w_n^*)$. The first step is to show that the process $(U(h_n))_{n\geq 0}$ is a supermartingale; note that the assumption on the positivity of the weight implies $w_n^* > 0$ for all n and hence h_n belongs to the domain of the function U. Fix $n \geq 0$ and apply (5) to $x = f_n$, $y = S_n(f)$, $u = w_n$, $v = w_n^*$, $d = df_{n+1}$ and $e = dw_{n+1}$, obtaining

$$U(h_{n+1}) \leq U(h_n) - U_x(h_n)df_{n+1} + U_u(h_n)dw_{n+1}.$$

Now, observe that the expressions above are integrable. This easily follows from (4) and the estimates $\mathbb{E}S(f)w^* < \infty$, $\mathbb{E}|df_n|w^* < \infty$ and $\mathbb{E}|f_n|w^* < \infty$ mentioned at the beginning of the proof. Thus, taking the conditional expectation with respect to $\mathcal{F}_n$ yields the desired supermartingale property of $(U(h_n))_{n\geq 0}$. Hence, by (2) and (3), we obtain

$$\mathbb{E}|f_n|w_n - C\mathbb{E}S_n(f)w_n^* \leq \mathbb{E}U(h_n) \leq \mathbb{E}U(h_0) = \mathbb{E}U(f_0, |f_0|, w_0, w_0) \leq 0. \quad (6)$$

This gives an upper bound for the weighted norm of f. To pass to $|f|^*$, consider another sequence $k_n = (\max(f_n^*, 0) - f_n, S_n(f), w_n, w_n^*)$ (here f_n^* is the one-sided maximal function of f). We will prove that the process $(U(k_n))_{n\geq 0}$ is a supermartingale. Fix $n \geq 0$ and observe first that $(\max(f_{n+1}^*, 0) - f_{n+1})^2 \leq (\max(f_n^*, 0) - f_{n+1})^2$. Indeed, if $\max(f_{n+1}^*, 0) = \max(f_n^*, 0)$, then both sides are equal; otherwise we must have $f_{n+1}^* = f_{n+1} > 0$ and then the bound is evident. Consequently,

$$U(k_{n+1}) \leq U(\max(f_n^*, 0) - f_{n+1}, S_{n+1}(f), w_{n+1}, w_{n+1}^*).$$

Now we apply (5) to $x = \max(f_n^*, 0) - f_n$, $y = S_n(f)$, $u = w_n$, $v = w_n^*$, $d = -df_{n+1}$ and $e = dw_{n+1}$, obtaining

$$\begin{aligned} U(k_{n+1}) &\leq U(\max(f_n^*, 0) - f_{n+1}, S_{n+1}(f), w_{n+1}, w_{n+1}^*) \\ &\leq U(k_n) - U_x(k_n)df_{n+1} + U_u(k_n)dw_{n+1}. \end{aligned}$$

Taking the conditional expectation with respect to $\mathcal{F}_n$, we get the supermartingale property of $(U(k_n))_{n\geq 0}$. Therefore, by (2) and (3),

$$\mathbb{E}(\max(f_n^*, 0) - f_n)w_n - C\mathbb{E}S_n f w_n^* \leq \mathbb{E}U(k_n) \leq \mathbb{E}U(k_0) \leq 0$$

and hence, using (6),

$$\mathbb{E}\max(f_n^*, 0)w_n \leq \mathbb{E}f_n w_n + C\mathbb{E}S_n f w_n^* \leq 2C\mathbb{E}S_n(f)w_n^*.$$

Applying this estimate to $-f$, we get $\mathbb{E}\max((-f)_n^*, 0)w_n \leq 2C\mathbb{E}S_n(f)w_n^*$ and adding it to the previous estimate, we finally get

$$\mathbb{E}|f_n|^* w_n \leq 4C\mathbb{E}S_n(f)w_n^*.$$

Letting $n \to \infty$ and using standard limiting theorems, we get the desired claim.

3. Proof of Theorem 1.2

Fix an arbitrary positive integer N and a parameter $r \in (0, 1)$. Define the martingale $f = (f_n)_{n=0}^N$ by $df_0 \equiv 0$ and requiring that df_1, df_2, $\ldots$, df_N are independent Rademacher variables (let $(\mathcal{F}_n)_{n=0}^N$ be the natural filtration of this sequence). Consider the weight

$$w = ((1 + df_0)(1 + df_1)\ldots(1 + df_N))^{1/r}$$

and introduce the events

$$A_n = \{df_1 = df_2 = \ldots = df_n = 1, \, df_{n+1} = -1\}, \qquad n = 0, 1, 2, \ldots, N - 1,$$

and $A_N = \{df_1 = df_2 = \ldots = df_N = 1\}$. We have

$$\mathbb{E}|f_N|w \geq \mathbb{E}|f_N|w1_{A_N} = N \cdot 2^{N/r} \cdot \mathbb{P}(A_N) = N2^{N(1/r-1)}. \tag{7}$$

To compute $\mathbb{E}S(f)[(w^r)^*]^{1/r}$, let us identify $[(w^r)^*]^{1/r}$ first. The martingale generated by w^r is given by

$$(w^r)_n = \mathbb{E}(w^r|\mathcal{F}_n) = (1+df_0)(1+df_1)\ldots(1+df_n), \qquad n = 0, 1, 2, \ldots, N,$$

since $1+df_0$, $1+df_1$, …, $1+df_N$ are independent mean-one random variables. Thus, on the set A_n we have $(w^r)_0 = 1$, $(w^r)_1 = 2$, …, $(w^r)_n = 2^n$ and $(w^r)_{n+1} = (w^r)_{n+2} = \ldots = (w^r)_N = 0$, so $[(w^r)^*]^{1/r} = 2^{n/r}$ there. Since $S(f)$ is identically $\sqrt{N}$, we get

$$\begin{aligned}
\mathbb{E}S(f)[(w^r)^*]^{1/r} &= \sqrt{N} \cdot \sum_{n=0}^{N} \mathbb{E}[(w^r)^*]^{1/r} 1_{A_n} \\
&= \sqrt{N} \left[\sum_{n=0}^{N-1} 2^{n/r} \cdot 2^{-n-1} + 2^{N/r} \cdot 2^{-N} \right] \\
&= \sqrt{N} \left[\frac{1}{2} \frac{2^{N(1/r-1)} - 1}{2^{1/r-1} - 1} + 2^{N(1/r-1)} \right] \leq \kappa \cdot \sqrt{N} 2^{N(1/r-1)},
\end{aligned}$$

where κ depends only on r. Combining this with (7), we see that if N is sufficiently large, then the ratio

$$\frac{\mathbb{E}|f_N|w}{\mathbb{E}S(f)[(w^r)^*]^{1/r}}$$

can be made arbitrarily large. This is precisely the assertion of Theorem 1.2.

Acknowledgments

The research was supported by Narodowe Centrum Nauki (Poland) grant DEC-2014/14/E/ST1/00532.

References

Burkholder, D. L., Explorations in martingale theory and its applications. École d'Ete de Probabilités de Saint-Flour XIX—1989, 1–66, Lecture Notes in Math., 1464, Springer, Berlin.

Davis, B., 1970. On the integrability of the martingale square function. Israel J. Math. 8, 187–190.

Fefferman, C. and Stein, E. M., 1971. Some maximal inequalities. Amer. J. Math. 93, 107–115.

Kazamaki, N., 1994. Continuous exponential martingales and BMO. Lecture Notes in Math. 1579, Springer-Verlag, Berlin, Heidelberg.

Muckenhoupt, B., 1972. Weighted norm inequalities for the Hardy maximal function. Trans. Amer. Math. Soc 165, 207–226.

Osękowski, A., 2012. Sharp martingale and semimartingale inequalities. Monografie Matematyczne 72, Birkhäuser Basel.

Osękowski, A., 2017. Weighted inequalities for the martingale square and maximal functions. Statist. Probab. Lett. 120, 95–100.