WEAK TYPE INEQUALITY FOR THE MARTINGALE SQUARE FUNCTION AND A RELATED CHARACTERIZATION OF HILBERT SPACES

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Abstract. Let $f$ be a martingale taking values in a Banach space $B$ and let $S(f)$ be its square function. We show that if $B$ is a Hilbert space, then

$$\mathbb{P}(S(f) \geq 1) \leq \sqrt{e} ||f||_1$$

and the constant $\sqrt{e}$ is the best possible. This extends the result of Cox, who established this bound in the real case. Next, we show that this inequality characterizes Hilbert spaces in the following sense: if $B$ is not a Hilbert space, then there is a martingale $f$ for which the above weak-type estimate does not hold.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, filtered by $(\mathcal{F}_n)_{n \geq 0}$, a non-decreasing sequence of sub-$\sigma$-fields of $\mathcal{F}$. Let $f = (f_n)_{n \geq 0}$, $g = (g_n)_{n \geq 0}$ be adapted martingales taking values in a certain separable Banach space $(B, ||\cdot||)$. The difference sequences $df = (df_n)_{n \geq 0}$, $dg = (dg_n)_{n \geq 0}$ of the martingales $f$ and $g$ are defined by $df_0 = f_0$ and $df_n = f_n - f_{n-1}$ for $n \geq 1$, and similarly for $dg_n$. We say that $g$ is a $\pm 1$ transform of $f$, if there is a deterministic sequence $\varepsilon = (\varepsilon_n)_{n \geq 0}$ of signs such that $dg_n = \varepsilon_n df_n$ for each $n$.

It is well-known that martingale inequalities reflect the geometry of Banach spaces in which the martingales take values: see e.g. [1], [2], [3], [4] and [7]. We shall mention here only one fact, closely related to the result studied in the present paper. As proved by Burkholder in [2], if $f$ takes values in a separable Hilbert space and $g$ is its $\pm 1$-transform, then

$$(1.1) \quad \mathbb{P}(\sup_n ||g_n|| \geq 1) \leq 2||f||_1$$

and the constant 2 is the best possible (here, as usual, $||f||_1 = \sup_n ||f_n||_1$). In fact, the implication can be reversed: if $B$ is a separable Banach space with the property that (1.1) holds for any $B$-valued martingale $f$ and its $\pm 1$ transform $g$, then $B$ is a Hilbert space. For details, see Burkholder [2] and Lee [6].

In this paper we shall study a related problem and characterize the class of Hilbert spaces by another weak-type estimate. Let us introduce the square function

2000 Mathematics Subject Classification. Primary: 60G42. Secondary: 60G44.
Key words and phrases. Martingale, square function, weak type inequality, Banach space, Hilbert space.
Partially supported by MNiSW Grant N N201 397437.
of $f$ by the formula

$$S(f) = \left( \sum_{k=0}^{\infty} ||df_k||^2 \right)^{1/2}.$$ 

We shall also use the notation

$$S_n(f) = \left( \sum_{k=0}^{n} ||df_k||^2 \right)^{1/2}$$

for the truncated square function, $n = 0, 1, 2, \ldots$. Suppose that $\mathcal{B}$ is a given and fixed separable Banach space and let $\beta(\mathcal{B})$ denote the least extended real number $\beta$ such that for any martingale $f$ taking values in $\mathcal{B}$,

$$\mathbb{P}(S(f) \geq 1) \leq \beta(\mathcal{B})||f||_1.$$ 

Using the method of moments, Cox [5] showed that $\beta(\mathbb{R}) = \sqrt{e}$; consequently, $\beta(\mathcal{B}) \geq \sqrt{e}$ for any non-degenerate $\mathcal{B}$. We will extend this result to the following.

**Theorem 1.1.** We have $\beta(\mathcal{B}) = \sqrt{e}$ if and only if $\mathcal{B}$ is a Hilbert space.

Let us sketch the proof. To show that for any martingale $f$ taking values in a Hilbert space $(\mathcal{H}, ||\cdot||)$ we have

$$\mathbb{P}(S(f) \geq 1) \leq \sqrt{e}||f||^2_1,$$ 

we may restrict ourselves to the class of simple martingales. Recall that $f$ is simple if for any $n$ the random variable $f_n$ takes only a finite number of values and there is a deterministic $N$ such that $f_N = f_{N+1} = f_{N+2} = \ldots$. We must prove that

$$EV(f_n, S_n(f)) \leq 0,$$ 

where $V(x, y) = 1_{\{y \geq 1\}} - \sqrt{e}|x|$ for $x \in \mathcal{H}$ and $y \in [0, \infty)$. To do this, we use Burkholder’s method and construct a function $U : \mathcal{H} \times [0, \infty) \to \mathbb{R}$, which satisfies the following three properties.

1° We have the majorization $U \geq V$.

2° For any $x \in \mathcal{H}$, $y \geq 0$ and any simple mean-zero random variable $T$ taking values in $\mathcal{H}$ we have $EU(x + T, \sqrt{y^2 + |T|^2}) \leq U(x, y)$.

3° For any $x \in \mathcal{H}$ we have $U(x, |x|) \leq 0$.

Then (1.2) follows. To see this, apply 2° conditionally on $\mathcal{F}_n$, with $x = f_n$, $y = S_n(f)$ and $T = df_{n+1}$. As the result, we obtain the inequality

$$E[U(f_{n+1}, S_{n+1}(f))|\mathcal{F}_n] \leq U(f_n, S_n(f)),$$

so, in other words, the process $(U(f_n, S_n(f)))_{n \geq 0}$ is a supermartingale. Hence, by 1° and 3°,

$$EV(f_n, S_n(f)) \leq EU(f_n, S_n(f)) \leq EU(f_0, S_0(f)) = EU(f_0, |f_0|) \leq 0$$

and we are done. The special function $U$ is constructed and studied in the next section. In Section 3 we prove the remaining part of Theorem 1.1: we shall show that the validity of (1.2) for all $\mathcal{B}$-valued martingales implies the parallelogram identity.
2. A SPECIAL FUNCTION

Let $\mathcal{H}$ be a separable Hilbert space: in fact we may and do assume that $\mathcal{H} = \ell^2$. The corresponding norm and scalar product will be denoted by $\| \cdot \|$ and $\cdot \cdot$, respectively. Introduce $U : \mathcal{H} \times [0, \infty) \to \mathbb{R}$ by the formula

$$(2.1) \quad U(x,y) = \begin{cases} 1 - (1 - y^2)^{1/2} \exp \left[ \frac{|x|^2}{2(1-y^2)} \right] & \text{if } |x|^2 + y^2 < 1, \\ 1 - \sqrt{c} |x| & \text{if } |x|^2 + y^2 \geq 1. \end{cases}$$

In the lemma below, we study the properties of $U$ and $V$.

**Lemma 2.1.** The function $U$ satisfies the properties $i^o$, $2^o$ and $3^o$.

**Proof.** To show the majorization, we may assume that $|x|^2 + y^2 < 1$. Then the inequality takes the form

$$\exp \left[ \frac{|x|^2}{2(1-y^2)} \right] \leq \sqrt{c} \frac{|x|}{\sqrt{1-y^2}} + \frac{1}{\sqrt{1-y^2}}$$

and follows immediately from an elementary bound $e^{x^2/2} \leq \sqrt{e} s + 1$, $s \in [0,1]$, applied to $s = |x|/\sqrt{1-y^2}$. To check $2^o$, we introduce an auxiliary function

$$A(x,y) = \begin{cases} -\frac{x}{\sqrt{1-y^2}} \exp \left[ \frac{|x|^2}{2(1-y^2)} \right] & \text{if } |x|^2 + y^2 < 1, \\ -\sqrt{c} x' & \text{if } |x|^2 + y^2 \geq 1, \end{cases}$$

where $x' = x/|x|$ for $x \neq 0$, and $x' = 0$ otherwise. We shall establish a pointwise estimate

$$(2.2) \quad U(x + d, \sqrt{y^2 + |d|^2}) \leq U(x,y) + A(x,y) \cdot d$$

for all $x, d \in \mathcal{H}$ and $y \geq 0$. Observe that this inequality immediately yields $2^o$, simply by putting $d = T$ and taking expectation of both sides.

To prove (2.2), note first that $U(x,y) \leq 1 - \sqrt{c} |x|$ for all $x \in \mathcal{H}$ and $y \geq 0$. This is trivial for $|x|^2 + y^2 \geq 1$, while for the remaining pairs $(x,y)$, it can be transformed into an equivalent inequality

$$\frac{|x|^2}{1-y^2} \leq \exp \left( \frac{|x|^2}{1-y^2} - 1 \right),$$

which is obvious. Consequently, when $|x|^2 + y^2 \geq 1$, we have $U(x + d, \sqrt{y^2 + |d|^2}) \leq 1 - \sqrt{c} |x + d| \leq 1 - \sqrt{c} |x| + A(x,y) \cdot d = U(x,y) + A(x,y) \cdot d$.

Now suppose that $|x|^2 + y^2 < 1$ and $|x + d|^2 + y^2 + |d|^2 \leq 1$. Observe that for $X, D \in \mathcal{H}$ with $|D| < 1$ we have

$$\exp \left[ \frac{|D|^2 |X|^2 + 2X \cdot D + |D|^2}{1 - |D|^2} \right] \geq \exp \left[ \frac{(X \cdot D)^2 + 2X \cdot D + |D|^2}{1 - |D|^2} \right]$$

$$\geq \frac{(X \cdot D)^2 + 2X \cdot D + |D|^2}{1 - |D|^2} + 1$$

$$= \frac{(1 + X \cdot D)^2}{1 - |D|^2}.$$
It suffices to plug $X = x/\sqrt{1 - y^2}$ and $D = d/\sqrt{1 - y^2}$ to obtain (2.2). Finally, if $|x|^2 + y^2 < 1 < |x + d|^2 + y^2 + |d|^2$, then substituting $X$ and $D$ as previously, we have $|X| < 1$, $|X + D|^2 + |D|^2 > 1$ and (2.2) can be written in the form
\[
\exp\left(\frac{|X|^2 - 1}{2}\right)(1 + X \cdot D) \leq |X + D|
\]
or
\[
\exp\left(\frac{|X|^2 - 1}{2}\right)\left(1 + \frac{|X + D|^2 - |X|^2 - |D|^2}{2}\right) \leq |X + D|.
\]
Now we fix $|X|, |X + D|$ and maximize the left-hand side over $D$. Consider two cases. If $|X + D|^2 + (|X + D| - |X|)^2 < 1$, then there is $D' \in \mathcal{H}$ satisfying $|X + D| = |X + D'|$ and $|X + D'|^2 + |D'|^2 = 1$. Consequently,
\[
\exp\left(\frac{|X|^2 - 1}{2}\right)\left(1 + \frac{|X + D|^2 - |X|^2 - |D|^2}{2}\right) \leq \exp\left(\frac{|X|^2 - 1}{2}\right)\left(1 + \frac{|X + D'|^2 - |X|^2 - |D'|^2}{2}\right) \leq |X + D'| = |X + D|.
\]
Here the first passage is due to $|D'| < |D|$, while in the second we have applied (2.2) to $x = X$, $y = 0$ and $d = D'$ (for these $x$, $y$ and $d$ we have already established the bound). Suppose then, that $|X + D|^2 + (|X + D| - |X|)^2 \geq 1$. This inequality is equivalent to
\[
|X + D| \geq \frac{1 - |X|^2}{\sqrt{2 - |X|^2 - |X|}}
\]
and hence
\[
\exp\left(\frac{|X|^2 - 1}{2}\right)\left(1 + \frac{|X + D|^2 - |X|^2 - |D|^2}{2}\right) - |X + D|
\]
\[
\leq \exp\left(\frac{|X|^2 - 1}{2}\right)\left(1 + \frac{|X + D|^2 - |X|^2 - (|X + D| - |X|)^2}{2}\right) - |X + D|
\]
\[
= \exp\left(\frac{|X|^2 - 1}{2}\right)(1 - |X|^2) + \left\{\exp\left(\frac{|X|^2 - 1}{2}\right)|X| - 1\right\}|X + D|
\]
\[
\leq \frac{1 - |X|^2}{\sqrt{2 - |X|^2 - |X|}} \left[\exp\left(\frac{|X|^2 - 1}{2}\right)\sqrt{2 - |X|^2 - |X|}\right].
\]
It suffices to observe that the expression in the square brackets is nonpositive, which follows from the estimate $\exp(1 - |X|^2) \geq 2 - |X|^2$. This completes the proof of 2\(^\circ\). Finally, 3\(^\circ\) is a consequence of (2.2): $U(x, |x|) \leq U(0, 0) + A(0, 0) \cdot x = 0$. \(\square\)

Thus, by the reasoning presented in Introduction, the inequality (1.2) holds true. The constant $\sqrt{\pi}$ is optimal even in the real case: see Cox [5]. In fact, we shall reprove this in the next section: see Remark 3.3 below.

3. Characterization of Hilbert spaces

Let $(\mathcal{B}, \| \cdot \|)$ be a separable Banach space and recall the number $\beta(\mathcal{B})$ defined in the first section. Thus, for any $\mathcal{B}$-valued martingale $f$ we have
\[
(3.1) \quad \mathbb{P}(S(f) \geq 1) \leq \beta(\mathcal{B})\|f\|_1.
\]
For \( x \in \mathcal{B} \) and \( y \geq 0 \), let \( M(x, y) \) denote the class of all simple martingales \( f \) given on the probability space \( ([0, 1], \mathcal{B}(0, 1), | \cdot |) \), such that \( f \) is \( \mathcal{B} \)-valued, \( f_0 \equiv x \) and

\[
y^2 - ||x||^2 + S^2(f) \geq 1 \quad \text{almost surely.}
\]

Here the filtration may vary. The key object in our further considerations is the function \( U^0 : \mathcal{B} \times [0, \infty) \rightarrow \mathbb{R} \), given by

\[
U^0(x, y) = \inf \{ \mathbb{E}||f_n|| \},
\]

where the infimum is taken over all \( n \) and all \( f \in M(x, y) \). We will prove that \( U^0 \) satisfies appropriate versions of the conditions 1°–3°.

**Lemma 3.1.** The function \( U^0 \) enjoys the following properties.

1°. For any \( x \in \mathcal{B} \) and \( y \geq 0 \) we have \( U^0(x, y) \geq ||x|| \).

2°. For any \( x \in \mathcal{B} \), \( y \geq 0 \) and any simple centered \( \mathcal{B} \)-valued random variable \( T \),

\[
\mathbb{E} U^0(x + T, \sqrt{y^2 + ||T||^2}) \geq U^0(x, y).
\]

3°. For any \( x \in \mathcal{B} \) we have \( U^0(x, ||x||) \geq \beta(\mathcal{B})^{-1} \).

**Proof.** The property 1° is obvious: when \( f \in M(x, y) \), then \( ||f_n||_1 \geq ||f_0||_1 = ||x|| \) for all \( n \). To establish 2°, we use a modification of the so-called "splicing argument": see e.g. [1]. Let \( T \) be as in the statement and let \( \{x_1, x_2, \ldots, x_k\} \) be the set of its values: \( \mathbb{P}(T = x_j) = p_j > 0 \), \( \sum_{j=1}^k p_j = 1 \). For any \( 1 \leq j \leq k \), pick a martingale \( f^j \) from the class \( M(x + x_j, \sqrt{y^2 + ||x_j||^2}) \). Let \( a_0 = 0 \) and \( a_j = \sum_{\ell=1}^j p_\ell \), \( j = 1, 2, \ldots, k \). Define a simple sequence \( f \) on \( ([0, 1], \mathcal{B}(0, 1), | \cdot |) \) by \( f_0 \equiv x \) and

\[
f_n(\omega) = f^j_{n-1}(\omega)(\omega - a_{j-1})/(a_j - a_{j-1}), \quad n \geq 1,
\]

when \( \omega \in (a_{j-1}, a_j) \). Then \( f \) is a martingale with respect to its natural filtration and, when \( \omega \in (a_{j-1}, a_j) \),

\[
y^2 - ||x||^2 + S^2(f)(\omega) = y^2 + ||x||^2 - ||x + x_j||^2 + S^2(f^j)((\omega - a_{j-1})/(a_j - a_{j-1})) \geq 1,
\]

unless \( \omega \) belongs to a set of measure 0. Therefore (3.2) holds, so by the definition of \( U^0 \),

\[
||f||_1 \geq U^0(x, y).
\]

However, the left hand side equals

\[
\sum_{j=1}^k \int_{a_{j-1}}^{a_j} |f_n(\omega)| d\omega = \sum_{j=1}^k p_j \int_0^1 |f^j_{n-1}(\omega)| d\omega,
\]

which, by the proper choice of \( n \) and \( f^j \), \( j = 1, 2, \ldots, k \), can be made arbitrarily close to \( \sum_{j=1}^k p_j U^0(x + x_j, \sqrt{y^2 + ||x_j||^2}) = \mathbb{E} U^0(x + T, \sqrt{y^2 + ||T||^2}) \). This gives 2°. Finally, the condition 3° follows immediately from (3.1) and the definition of \( U^0 \).

The further properties are described in the next lemma.

**Lemma 3.2.** (i) The function \( U^0 \) satisfies the symmetry condition

\[
U^0(x, y) = U^0(-x, y)
\]

for all \( x \in \mathcal{B} \) and \( y \geq 0 \).
(ii) The function \( U^0 \) has the homogeneity-type property

\[
U^0(x, y) = \sqrt{1 - y^2} U^0 \left( \frac{x}{\sqrt{1 - y^2}}, 0 \right)
\]

for all \( x \in \mathcal{B} \) and \( y \in [0, 1) \).

(iii) If \( z \in \mathcal{B} \) satisfies \( \|z\| = 1 \) and \( 0 \leq s < t \leq 1 \), then

\[
U^0(sz, 0) \leq U^0(tz, 0) \exp((s^2 - t^2)\|z\|^2/2).
\]

Proof. (i) It suffices to use the equivalence \( f \in M(x, y) \) if and only if \( -f \in M(-x, y) \).

(ii) This follows immediately from the fact that \( f \in M(x, y) \) if and only if \( f/\sqrt{1 - y^2} \in M(x/\sqrt{1 - y^2}, 0) \).

(iii) Fix \( x \in \mathcal{B} \) with \( 0 < \|x\| < 1 \) and \( \delta > 0 \) such that \( \|x + \delta x\| \leq 1 \). Apply \( 2^\circ \) to \( y = 0 \) and a centered random variable \( T \) which takes two values: \( \delta x \) and \( -2x/(1 + \|x\|^2) \). We get

\[
U^0(x, 0) \leq \frac{\delta \|x\|(1 + \|x\|^2)}{2\|x\| + \delta \|x\|(1 + \|x\|^2)} U^0 \left( \frac{x(1 - \|x\|^2)}{1 + \|x\|^2}, \frac{2\|x\|}{1 + \|x\|^2} \right)
\]

by (i) and (ii), the first term on the right equals

\[
\frac{\delta \|x\|(1 - \|x\|^2)}{2\|x\| + \delta \|x\|(1 + \|x\|^2)} U^0(x, 0).
\]

The second summand can be bounded from above by

\[
\frac{2\|x\|}{2\|x\| + \delta \|x\|(1 + \|x\|^2)} U^0(x + \delta x, 0),
\]

because \( M(x + \delta x, 0) \subset M(x + \delta x, \delta \|x\|) \). Plugging these two facts into the inequality above and using the assumption \( \|x\| \leq 1 \) (so \( 1 - \|x\|^2 = 1 - \|x\|^2 \)) yields

\[
\frac{U^0(x + \delta x, 0)}{U^0(x, 0)} \geq 1 + \delta \|x\|^2.
\]

This gives

\[
\frac{U^0(x(1 + k\delta), 0)}{U^0(x(1 + (k - 1)\delta), 0)} \geq 1 + \delta(1 + (k - 1)\delta)\|x\|^2,
\]

provided \( \|x(1 + k\delta)\| \leq 1 \). Consequently, if \( N \) is an integer such that \( \|x(1 + N\delta)\| \leq 1 \), then

\[
\frac{U^0(x(1 + N\delta), 0)}{U^0(x, 0)} \geq \prod_{k=1}^{N} (1 + \delta(1 + (k - 1)\delta)\|x\|^2).
\]

Now we turn to (3.3). Assume first that \( s > 0 \). Put \( x = sz, \delta = (t/s - 1)/N \) and let \( N \to \infty \) in the inequality above to obtain

\[
\frac{U^0(tz, 0)}{U^0(sz, 0)} \geq \exp \left( \frac{1}{2}\|z\|^2(t^2 - s^2) \right).
\]
which is the claim. Next, suppose that $s = 0$. For any $0 < s' < t$ we have, by $2^\circ$, 

$$U^0(0, 0) \leq \frac{1}{2} U^0(s'z, ||s'z||) + \frac{1}{2} U^0(-s'z, ||s'z||) = U^0(s'z, ||s'z||) \leq U^0(s'z, 0),$$

where in the latter passage we have used the inclusion $M(s'z, 0) \subset M(s'z, ||s'z||)$. Thus,

$$\frac{U^0(tz, 0)}{U^0(0, 0)} \geq \frac{\exp \left( \frac{1}{2} ||z||^2 (t^2 - (s')^2) \right)}{\exp \left( \frac{1}{2} ||z||^2 (s'^2) \right)}$$

and it remains to let $s' \to 0$. \hfill $\Box$

**Remark 3.3.** Suppose that $\mathcal{B} = \mathbb{R}$. It is easy to see that $U^0(1, 0) \leq 1$: consider $f$ starting from 1 and satisfying $\mathbb{P}(df_1 = -1) = \mathbb{P}(df_1 = 1) = 1/2$, $df_2 = df_3 = \ldots = 0$. Thus, by $3^\circ$ and (3.3) we have

$$\beta(\mathbb{R})^{-1} \leq U^0(0, 0) \leq U^0(1, 0)/\sqrt{e} \leq 1/\sqrt{e},$$

that is, $\beta(\mathbb{R}) \geq \sqrt{e}$. This implies the sharpness of (1.2) in the Hilbert-space-valued setting.

Now we will work under the assumption $\beta(\mathcal{B}) = \sqrt{e}$. Then we are able to derive the explicit formula for $U^0$.

**Lemma 3.4.** If $\beta(\mathcal{B}) = \sqrt{e}$, then

$$U^0(x, y) = \begin{cases} \sqrt{1 - y^2} \exp \left( \frac{||x||^2}{2(1 - y^2)} - \frac{1}{2} \right) & \text{if } ||x||^2 + y^2 < 1, \\ ||x|| & \text{if } ||x||^2 + y^2 \geq 1. \end{cases}$$

**Proof.** First let us focus on the set $\{(x, y) : ||x||^2 + y^2 \geq 1\}$. By $1^\circ$, we have $U^0(x, y) \geq ||x||$. To get the reverse estimate, consider a martingale $f$ such that $f_0 \equiv x$, $df_1$ takes values $-x$ and $x$, and $df_2 = df_3 = \ldots = 0$. Then $y^2 - ||x||^2 + S^2(f) = y^2 + ||x||^2 \geq 1$ (so $f \in M(x, y)$) and $||f||_1 = ||x||$, which implies $U^0(x, y) \leq ||x||$ by the definition of $U^0$. Now suppose that $||x||^2 + y^2 < 1$. Using the second and third part of the previous lemma, we may write

$$U^0(x, y) = \sqrt{1 - y^2} U^0 \left( \frac{x}{\sqrt{1 - y^2}}, 0 \right) \geq U^0(0, 0) \sqrt{1 - y^2} \exp \left( \frac{||x||^2}{2(1 - y^2)} \right),$$

so, by $3^\circ$,

$$U^0(x, y) \geq \sqrt{1 - y^2} \exp \left( \frac{||x||^2}{2(1 - y^2)} - \frac{1}{2} \right).$$

To get the reverse bound, we use the homogeneity of $U^0$ and (3.3) again:

$$U^0(x, y) = \sqrt{1 - y^2} U^0 \left( \frac{x}{\sqrt{1 - y^2}}, 0 \right)$$

$$\leq \sqrt{1 - y^2} U^0 \left( \frac{x}{||x||}, 0 \right) \exp \left( \frac{1}{2} \left( \frac{||x||^2}{1 - y^2} - 1 \right) \right)$$

$$= \sqrt{1 - y^2} \exp \left( \frac{||x||^2}{2(1 - y^2)} - \frac{1}{2} \right),$$

where in the last line we have used the equality $U^0(\overline{x}, 0) = ||\overline{x}||$ valid for $\overline{x}$ of norm 1 (we have just established this in the first part of the proof). For completeness, let us mention here that if $x = 0$, then $x/||x||$ should be replaced above by any vector of norm one. \hfill $\Box$
Lemma 3.5. Suppose that $\beta(B) = \sqrt{e}$ and assume that $x, y \in B$ and $\alpha > 0$ satisfy $||x|| < 1$, $||x + \alpha x + y||^2 + ||\alpha x + y||^2 < 1$ and $||x + \alpha x - y||^2 + ||\alpha x - y||^2 < 1$. Then

$$2 + 2\alpha||x||^2 \leq \sqrt{1 - ||\alpha x + y||^2} \exp \left[ \frac{||x + \alpha x + y||^2}{2(1 - ||\alpha x + y||^2)} - \frac{||x||^2}{2} \right]$$

$$+ \sqrt{1 - ||\alpha x - y||^2} \exp \left[ \frac{||x + \alpha x - y||^2}{2(1 - ||\alpha x - y||^2)} - \frac{||x||^2}{2} \right].$$

(3.6)

Proof. Consider a random variable $T$ such that

$$\mathbb{P} \left( T = -\frac{2x}{1 + ||x||^2} \right) = p, \quad \mathbb{P}(T = \alpha x + y) = \mathbb{P}(T = \alpha x - y) = \frac{1 - p}{2},$$

where $p \in (0, 1)$ is chosen so that $\mathbb{E}T = 0$. That is,

$$p = \frac{\alpha(1 + ||x||^2)}{2 + \alpha(1 + ||x||^2)}.$$ 

By $2^{a^2}$, we have $U^0(x, 0) \leq EU^0(x + T, ||T||)$. Since $||x + T||^2 + ||T||^2 < 1$ almost surely, the previous lemma implies that this can be rewritten in the equivalent form

$$\exp \left[ \frac{||x||^2}{2} \right] \leq p \sqrt{1 - \left( \frac{2||x||}{1 + ||x||^2} \right)^2} \exp \left[ \frac{||x \left( \frac{-1 + ||x||^2}{1 + ||x||^2} \right)||^2}{2 \left( 1 - \left( \frac{2||x||}{1 + ||x||^2} \right)^2 \right)} \right]$$

$$+ \frac{1 - p}{2} \sqrt{1 - ||\alpha x + y||^2} \exp \left[ \frac{||x + \alpha x + y||^2}{2(1 - ||\alpha x + y||^2)} \right]$$

$$+ \frac{1 - p}{2} \sqrt{1 - ||\alpha x - y||^2} \exp \left[ \frac{||x + \alpha x - y||^2}{2(1 - ||\alpha x - y||^2)} \right].$$

However, the first term on the right equals

$$\frac{\alpha(1 + ||x||^2)}{2 + \alpha(1 + ||x||^2)} \exp \left[ \frac{||x||^2}{2} \right]$$

and, in addition, $(1 - p)/2 = (2 + \alpha(1 + ||x||^2))^{-1}$. Consequently, it suffices to multiply both sides of the inequality above by $(2 + \alpha(1 + ||x||^2)) \exp \left[ -||x||^2/2 \right]$; the claim follows. 

Now we are ready to complete the proof of Theorem 1.1. Suppose that $a, b$ belong to the unit ball $K$ of $B$ and take $\varepsilon \in (0, 1/2)$. Applying (3.6) to $x = \varepsilon a$, $y = \varepsilon^2 b$ and $\alpha = \varepsilon$ gives

$$2 + 2\varepsilon^3||a||^2 \leq \sqrt{1 - \varepsilon^4}||a + b||^2 \exp(m(a, b))$$

$$+ \sqrt{1 - \varepsilon^4}||a - b||^2 \exp(m(a, -b)),$$

where

$$m(a, b) = \frac{\varepsilon^2||a + \varepsilon(a + b)||^2}{2(1 - \varepsilon^4)||a + b||^2} - \frac{\varepsilon^2||a||^2}{2}$$

$$= \frac{\varepsilon^2}{2}||a + \varepsilon(a + b)||^2 - ||a||^2 + \frac{\varepsilon^6||a + \varepsilon(a + b)||^2||a + b||^2}{2(1 - \varepsilon^4)||a + b||^2}.$$
It is easy to see that there exists an absolute constant $M_1$ such that
\[
\sup_{a,b \in K} |m(a, b)| \leq M_1 \varepsilon^3.
\]

Consequently, there is a universal $M_2 > 0$ such that if $\varepsilon$ is sufficiently small, then
\[
\exp(m(a, b)) \leq 1 + m(a, b) + m(a, b)^2 \leq 1 + \frac{\varepsilon^2}{2} (||a + \varepsilon(a + b)||^2 - ||a||^2) + M_2 \varepsilon^6
\]
for any $a, b \in K$. Since $\sqrt{1 - x} \leq 1 - x/2$ for $x \in (0, 1)$, the inequality (3.7) implies
\[
2 + 2\varepsilon^3 ||a||^2 \leq (1 - \varepsilon^4 ||a + b||^2/2) \left(1 + \frac{\varepsilon^2}{2} (||a + \varepsilon(a + b)||^2 - ||a||^2) + M_2 \varepsilon^6 \right) + (1 - \varepsilon^4 ||a - b||^2/2) \left(1 + \frac{\varepsilon^2}{2} (||a + \varepsilon(a - b)||^2 - ||a||^2) + M_2 \varepsilon^6 \right).
\]

This, after some manipulations, leads to
\[
||a + \varepsilon(a + b)||^2 + ||a + \varepsilon(a - b)||^2 - 2||a(1 + \varepsilon)||^2 \geq \varepsilon^2 (||a + b||^2 + ||a - b||^2 - 2||a||^2) - 2\varepsilon^4 M_3,
\]
where $M_3$ is a positive constant not depending on $\varepsilon$, $a$ and $b$. Equivalently,
\[
\left||a + \frac{\varepsilon}{1+\varepsilon} b\right|^2 + \left||a - \frac{\varepsilon}{1+\varepsilon} b\right|^2 - 2||a||^2 - 2 \left|\frac{\varepsilon}{1+\varepsilon} b\right|^2 \geq \frac{\varepsilon^2}{(1+\varepsilon)^2}(||a+b||^2 + ||a-b||^2 - 2||a||^2 - 2||b||^2) - 2\varepsilon^4 \frac{M_3}{(1+\varepsilon)^2}.
\]

Next, let $c \in B$, $\gamma > 0$ and substitute $a = \gamma c$; we assume that $\gamma$ is small enough to ensure that $a \in K$. If we divide both sides by $\gamma^2$ and substitute $\delta = \varepsilon(1 + \varepsilon)^{-1} \gamma^{-1}$, we obtain
\[
||c + \delta b||^2 + ||c - \delta b||^2 - 2||c||^2 - 2||\delta b||^2
\geq \delta^2 (||\gamma c + b||^2 + ||\gamma c - b||^2 - 2||\gamma c||^2 - 2||b||^2) - 2\varepsilon^2 \delta^2 M_3
\geq \delta^2 (||\gamma c + b||^2 + ||\gamma c - b||^2 - 2||\gamma c||^2 - 2||b||^2) - 2\delta^4 M_3
\]
Let $\gamma$ and $\varepsilon$ go to 0 so that $\delta$ remains fixed. As the result, we obtain that for any $\delta > 0$, $b \in K$ and $c \in B$,
\[
||c + \delta b||^2 + ||c - \delta b||^2 - 2||c||^2 - 2||\delta b||^2 \geq -2\delta^4 M_3.
\]

Now let $N$ be a large positive integer and consider a symmetric random walk $(g_n)_{n \geq 0}$ over integers, starting from 0. Let $\tau = \inf\{n : |g_n| = N\}$. The inequality (3.8), applied to $\delta = N^{-1}$, implies that for any $a \in B$ and $b \in K$, the process
\[
(\xi_n)_{n \geq 0} = \left(\left|a + \frac{bg_{\tau \wedge n}}{N}\right|^2 - \left\{\frac{||b||^2}{N^2} - \frac{M_3}{N^4}\right\}(\tau \wedge n)\right)_{n \geq 0}
\]
is a submartingale. Since $E(g_{\tau \wedge n}) = E g_{\tau \wedge n}^2$, we obtain
\[
E \left[\left|a + \frac{bg_{\tau \wedge n}}{N}\right|^2 - \left\{\frac{||b||^2}{N^2} - \frac{M_3}{N^4}\right\} g_{\tau \wedge n}^2\right] = E \xi_n \geq E \xi_0 = ||a||^2.
\]

Letting $n \to \infty$ and using Lebesgue’s dominated convergence theorem gives
\[
\frac{1}{2} \left[||a + b||^2 + ||a - b||^2 - ||b||^2 + \frac{M_3}{N^2}\right] \geq ||a||^2.
\]
It suffices to let $N$ go to $\infty$ to obtain
\[ \|a + b\|^2 + \|a - b\|^2 \geq 2\|a\|^2 + 2\|b\|^2. \]
We have assumed that $b$ belongs to the unit ball $K$, but, by homogeneity, the above estimate extends to any $b \in \mathcal{B}$. Putting $a + b$ and $a - b$ in the place of $a$ and $b$, respectively, we obtain the reverse estimate
\[ \|a + b\|^2 + \|a - b\|^2 \leq 2\|a\|^2 + 2\|b\|^2. \]
This implies that the parallelogram identity is satisfied and hence $\mathcal{B}$ is a Hilbert space.

References


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