WEAK TYPE INEQUALITY FOR THE MARTINGALE SQUARE FUNCTION AND A RELATED CHARACTERIZATION OF HILBERT SPACES

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ABSTRACT. Let f be a martingale taking values in a Banach space $\mathcal{B}$ and let $S(f)$ be its square function. We show that if $\mathcal{B}$ is a Hilbert space, then

$$\mathbb{P}(S(f) \geq 1) \leq \sqrt{e}||f||_1$$

and the constant $\sqrt{e}$ is the best possible. This extends the result of Cox, who established this bound in the real case. Next, we show that this inequality characterizes Hilbert spaces in the following sense: if $\mathcal{B}$ is not a Hilbert space, then there is a martingale f for which the above weak-type estimate does not hold.

1. INTRODUCTION

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, filtered by $(\mathcal{F}_n)_{n\geq 0}$, a non-decreasing sequence of sub-σ-fields of $\mathcal{F}$. Let $f = (f_n)_{n\geq 0}$, $g = (g_n)_{n\geq 0}$ be adapted martingales taking values in a certain separable Banach space $(\mathcal{B}, ||\cdot||)$. The difference sequences $df = (df_n)_{n\geq 0}$, $dg = (dg_n)_{n\geq 0}$ of the martingales f and g are defined by $df_0 = f_0$ and $df_n = f_n - f_{n-1}$ for $n \geq 1$, and similarly for dg_n. We say that g is a ± 1 transform of f, if there is a deterministic sequence $\varepsilon = (\varepsilon_n)_{n\geq 0}$ of signs such that $dg_n = \varepsilon_n df_n$ for each n.

It is well-known that martingale inequalities reflect the geometry of Banach spaces in which the martingales take values: see e.g. [1], [2], [3], [4] and [7]. We shall mention here only one fact, closely related to the result studied in the present paper. As proved by Burkholder in [2], if f takes values in a separable Hilbert space and g is its ± 1-transform, then

$$\mathbb{P}(\sup_n ||g_n|| \geq 1) \leq 2||f||_1 \tag{1.1}$$

and the constant 2 is the best possible (here, as usual, $||f||_1 = \sup_n ||f_n||_1$). In fact, the implication can be reversed: if $\mathcal{B}$ is a separable Banach space with the property that (1.1) holds for any $\mathcal{B}$-valued martingale f and its ± 1 transform g, then $\mathcal{B}$ is a Hilbert space. For details, see Burkholder [2] and Lee [6].

In this paper we shall study a related problem and characterize the class of Hilbert spaces by another weak-type estimate. Let us introduce the *square function*

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of f by the formula

$$S(f)=\left(\sum_{k=0}^{\infty}||df_k||^2\right)^{1/2}.$$

We shall also use the notation

$$S_n(f)=\left(\sum_{k=0}^{n}||df_k||^2\right)^{1/2}$$

for the truncated square function, $n=0,1,2,\ldots$. Suppose that $\mathcal{B}$ is a given and fixed separable Banach space and let $\beta(\mathcal{B})$ denote the least extended real number β such that for any martingale f taking values in $\mathcal{B}$,

$$\mathbb{P}(S(f)\geq 1)\leq \beta(\mathcal{B})||f||_1.$$

Using the method of moments, Cox [5] showed that $\beta(\mathbb{R})=\sqrt{e}$: consequently, $\beta(\mathcal{B})\geq\sqrt{e}$ for any non-degenerate $\mathcal{B}$. We will extend this result to the following.

Theorem 1.1. *We have $\beta(\mathcal{B})=\sqrt{e}$ if and only if $\mathcal{B}$ is a Hilbert space.*

Let us sketch the proof. To show that for any martingale f taking values in a Hilbert space $(\mathcal{H},|\cdot|)$ we have

$$\mathbb{P}(S(f)\geq 1)\leq\sqrt{e}||f||_1, \tag{1.2}$$

we may restrict ourselves to the class of simple martingales. Recall that f is simple if for any n the random variable f_n takes only a finite number of values and there is a deterministic N such that $f_N=f_{N+1}=f_{N+2}=\ldots$. We must prove that

$$\mathbb{E}V(f_n,S_n(f))\leq 0,\qquad n=0,1,2,\ldots,$$

where $V(x,y)=1_{\{y\geq 1\}}-\sqrt{e}|x|$ for $x\in\mathcal{H}$ and $y\in[0,\infty)$. To do this, we use Burkholder's method and construct a function $U:\mathcal{H}\times[0,\infty)\to\mathbb{R}$, which satisfies the following three properties.

1° We have the majorization $U\geq V$.
2° For any $x\in\mathcal{H}$, $y\geq 0$ and any simple mean-zero random variable T taking values in $\mathcal{H}$ we have $\mathbb{E}U(x+T,\sqrt{y^2+|T|^2})\leq U(x,y)$.
3° For any $x\in\mathcal{H}$ we have $U(x,|x|)\leq 0$.

Then (1.2) follows. To see this, apply 2° conditionally on $\mathcal{F}_n$, with $x=f_n$, $y=S_n(f)$ and $T=df_{n+1}$. As the result, we obtain the inequality

$$\mathbb{E}\left[U(f_{n+1},S_{n+1}(f))|\mathcal{F}_n\right]\leq U(f_n,S_n(f)),$$

so, in other words, the process $(U(f_n,S_n(f)))_{n\geq 0}$ is a supermartingale. Hence, by 1° and 3°,

$$\mathbb{E}V(f_n,S_n(f))\leq\mathbb{E}U(f_n,S_n(f))\leq\mathbb{E}U(f_0,S_0(f))=\mathbb{E}U(f_0,|f_0|)\leq 0$$

and we are done. The special function U is constructed and studied in the next section. In Section 3 we prove the remaining part of Theorem 1.1: we shall show that the validity of (1.2) for all $\mathcal{B}$-valued martingales implies the parallelogram identity.

2. A special function

Let $\mathcal{H}$ be a separable Hilbert space: in fact we may and do assume that $\mathcal{H} = \ell^2$. The corresponding norm and scalar product will be denoted by $|\cdot|$ and $\cdot$, respectively. Introduce $U : \mathcal{H} \times [0, \infty) \to \mathbb{R}$ by the formula

$$U(x,y) = \begin{cases} 1 - (1-y^2)^{1/2} \exp\left[\frac{|x|^2}{2(1-y^2)}\right] & \text{if } |x|^2 + y^2 < 1, \\ 1 - \sqrt{e}|x| & \text{if } |x|^2 + y^2 \geq 1. \end{cases} \tag{2.1}$$

In the lemma below, we study the properties of U and V.

Lemma 2.1. *The function U satisfies the properties 1°, 2° and 3°.*

Proof. To show the majorization, we may assume that $|x|^2 + y^2 < 1$. Then the inequality takes the form

$$\exp\left[\frac{|x|^2}{2(1-y^2)}\right] \leq \sqrt{e}\frac{|x|}{\sqrt{1-y^2}} + \frac{1}{\sqrt{1-y^2}}$$

and follows immediately from an elementary bound $e^{s^2/2} \leq \sqrt{e}s + 1$, $s \in [0,1]$, applied to $s = |x|/\sqrt{1-y^2}$. To check 2°, we introduce an auxiliary function

$$A(x,y) = \begin{cases} -\frac{x}{\sqrt{1-y^2}} \exp\left[\frac{|x|^2}{2(1-y^2)}\right] & \text{if } |x|^2 + y^2 < 1, \\ -\sqrt{e}x' & \text{if } |x|^2 + y^2 \geq 1, \end{cases}$$

where $x' = x/|x|$ for $x \neq 0$, and $x' = 0$ otherwise. We shall establish a pointwise estimate

$$U(x + d, \sqrt{y^2 + |d|^2}) \leq U(x,y) + A(x,y) \cdot d \tag{2.2}$$

for all x, $d \in \mathcal{H}$ and $y \geq 0$. Observe that this inequality immediately yields 2°, simply by putting $d = T$ and taking expectation of both sides.

To prove (2.2), note first that $U(x,y) \leq 1 - \sqrt{e}|x|$ for all $x \in \mathcal{H}$ and $y \geq 0$. This is trivial for $|x|^2 + y^2 \geq 1$, while for the remaining pairs (x, y), it can be transformed into an equivalent inequality

$$\frac{|x|^2}{1-y^2} \leq \exp\left(\frac{|x|^2}{1-y^2} - 1\right),$$

which is obvious. Consequently, when $|x|^2 + y^2 \geq 1$, we have

$$U(x+d, \sqrt{y^2 + |d|^2}) \leq 1 - \sqrt{e}|x+d| \leq 1 - \sqrt{e}|x| + A(x,y) \cdot d = U(x,y) + A(x,y) \cdot d.$$

Now suppose that $|x|^2 + y^2 < 1$ and $|x + d|^2 + y^2 + |d|^2 \leq 1$. Observe that for X, $D \in \mathcal{H}$ with $|D| < 1$ we have

$$\begin{aligned} \exp\left[\frac{|D|^2|X|^2 + 2X \cdot D + |D|^2}{1 - |D|^2}\right] &\geq \exp\left[\frac{(X \cdot D)^2 + 2X \cdot D + |D|^2}{1 - |D|^2}\right] \\ &\geq \frac{(X \cdot D)^2 + 2X \cdot D + |D|^2}{1 - |D|^2} + 1 \\ &= \frac{(1 + X \cdot D)^2}{1 - |D|^2}. \end{aligned}$$

It suffices to plug $X = x/\sqrt{1-y^2}$ and $D = d/\sqrt{1-y^2}$ to obtain (2.2). Finally, if $|x|^2 + y^2 < 1 < |x+d|^2 + y^2 + |d|^2$, then substituting X and D as previously, we have $|X| < 1$, $|X+D|^2 + |D|^2 > 1$ and (2.2) can be written in the form

$$\exp\left(\frac{|X|^2-1}{2}\right)(1 + X\cdot D) \leq |X+D|$$

or

$$\exp\left(\frac{|X|^2-1}{2}\right)\left(1 + \frac{|X+D|^2 - |X|^2 - |D|^2}{2}\right) \leq |X+D|.$$

Now we fix $|X|$, $|X+D|$ and maximize the left-hand side over D. Consider two cases. If $|X+D|^2 + (|X+D| - |X|)^2 < 1$, then there is $D' \in \mathcal{H}$ satisfying $|X+D| = |X+D'|$ and $|X+D'|^2 + |D'|^2 = 1$. Consequently,

$$\begin{aligned}&\exp\left(\frac{|X|^2-1}{2}\right)\left(1 + \frac{|X+D|^2 - |X|^2 - |D|^2}{2}\right)\\ &\leq \exp\left(\frac{|X|^2-1}{2}\right)\left(1 + \frac{|X+D'|^2 - |X|^2 - |D'|^2}{2}\right) \leq |X+D'| = |X+D|.\end{aligned}$$

Here the first passage is due to $|D'| < |D|$, while in the second we have applied (2.2) to $x = X$, $y = 0$ and $d = D'$ (for these x, y and d we have already established the bound). Suppose then, that $|X+D|^2 + (|X+D| - |X|)^2 \geq 1$. This inequality is equivalent to

$$|X+D| \geq \frac{1-|X|^2}{\sqrt{2-|X|^2} - |X|}$$

and hence

$$\begin{aligned}&\exp\left(\frac{|X|^2-1}{2}\right)\left(1 + \frac{|X+D|^2 - |X|^2 - |D|^2}{2}\right) - |X+D|\\ &\leq \exp\left(\frac{|X|^2-1}{2}\right)\left(1 + \frac{|X+D|^2 - |X|^2 - (|X+D| - |X|)^2}{2}\right) - |X+D|\\ &= \exp\left(\frac{|X|^2-1}{2}\right)(1-|X|^2) + \left\{\exp\left(\frac{|X|^2-1}{2}\right)|X| - 1\right\}|X+D|\\ &\leq \frac{1-|X|^2}{\sqrt{2-|X|^2} - |X|}\left[\exp\left(\frac{|X|^2-1}{2}\right)\sqrt{2-|X|^2} - 1\right].\end{aligned}$$

It suffices to observe that the expression in the square brackets is nonpositive, which follows from the estimate $\exp(1-|X|^2) \geq 2 - |X|^2$. This completes the proof of 2°. Finally, 3° is a consequence of (2.2): $U(x, |x|) \leq U(0,0) + A(0,0)\cdot x = 0$. □

Thus, by the reasoning presented in Introduction, the inequality (1.2) holds true. The constant $\sqrt{e}$ is optimal even in the real case: see Cox [5]. In fact, we shall reprove this in the next section: see Remark 3.3 below.

3. Characterization of Hilbert spaces

Let $(\mathcal{B}, ||\cdot||)$ be a separable Banach space and recall the number $\beta(\mathcal{B})$ defined in the first section. Thus, for any $\mathcal{B}$-valued martingale f we have

$$\mathbb{P}(S(f) \geq 1) \leq \beta(\mathcal{B})||f||_1. \tag{3.1}$$

For $x \in \mathcal{B}$ and $y \geq 0$, let $M(x, y)$ denote the class of all simple martingales f given on the probability space $([0,1], \mathbb{B}(0,1), |\cdot|)$, such that f is $\mathcal{B}$-valued, $f_0 \equiv x$ and

$$y^2 - ||x||^2 + S^2(f) \geq 1 \qquad \text{almost surely.} \tag{3.2}$$

Here the filtration may vary. The key object in our further considerations is the function $U^0 : \mathcal{B} \times [0, \infty) \to \mathbb{R}$, given by

$$U^0(x, y) = \inf\{\mathbb{E}||f_n||\},$$

where the infimum is taken over all n and all $f \in M(x, y)$. We will prove that U^0 satisfies appropriate versions of the conditions 1°–3°.

Lemma 3.1. *The function U^0 enjoys the following properties.*

1°' *For any $x \in \mathcal{B}$ and $y \geq 0$ we have $U^0(x, y) \geq ||x||$.*

2°' *For any $x \in \mathcal{B}$, $y \geq 0$ and any simple centered $\mathcal{B}$-valued random variable T,*

$$\mathbb{E}U^0(x + T, \sqrt{y^2 + ||T||^2}) \geq U^0(x, y).$$

3°' *For any $x \in \mathcal{B}$ we have $U^0(x, ||x||) \geq \beta(\mathcal{B})^{-1}$.*

Proof. The property 1°' is obvious: when $f \in M(x, y)$, then $||f_n||_1 \geq ||f_0||_1 = ||x||$ for all n. To establish 2°', we use a modification of the so-called "splicing argument": see e.g. [1]. Let T be as in the statement and let $\{x_1, x_2, \ldots, x_k\}$ be the set of its values: $\mathbb{P}(T = x_j) = p_j > 0$, $\sum_{j=1}^k p_j = 1$. For any $1 \leq j \leq k$, pick a martingale f^j from the class $M(x+x_j, \sqrt{y^2 + ||x_j||^2})$. Let $a_0 = 0$ and $a_j = \sum_{\ell=1}^j p_\ell$, $j = 1, 2, \ldots, k$. Define a simple sequence f on $([0,1], \mathbb{B}(0,1), |\cdot|)$ by $f_0 \equiv x$ and

$$f_n(\omega) = f^j_{n-1}((\omega - a_{j-1})/(a_j - a_{j-1})), \qquad n \geq 1,$$

when $\omega \in (a_{j-1}, a_j]$. Then f is a martingale with respect to its natural filtration and, when $\omega \in (a_{j-1}, a_j]$,

$$y^2 - ||x||^2 + S^2(f)(\omega) = y^2 + ||x_j||^2 - ||x + x_j||^2 + S^2(f^j)((\omega - a_{j-1})/(a_j - a_{j-1})) \geq 1,$$

unless ω belongs to a set of measure 0. Therefore (3.2) holds, so by the definition of U^0,

$$||f_n||_1 \geq U^0(x, y).$$

However, the left hand side equals

$$\sum_{j=1}^k \int_{a_{j-1}}^{a_j} |f_n(\omega)| \mathrm{d}\omega = \sum_{j=1}^k p_j \int_0^1 |f^j_{n-1}(\omega)| \mathrm{d}\omega,$$

which, by the proper choice of n and f^j, $j = 1, 2, \ldots, k$, can be made arbitrarily close to $\sum_{j=1}^k p_j U^0(x + x_j, \sqrt{y^2 + ||x_j||^2}) = \mathbb{E}U^0(x + T, \sqrt{y^2 + ||T||^2})$. This gives 2°'. Finally, the condition 3°' follows immediately from (3.1) and the definition of U^0. □

The further properties are described in the next lemma.

Lemma 3.2. *(i) The function U^0 satisfies the symmetry condition*

$$U^0(x, y) = U^0(-x, y)$$

for all $x \in \mathcal{B}$ and $y \geq 0$.

(ii) The function U^0 has the homogeneity-type property

$$U^0(x,y)=\sqrt{1-y^2}\,U^0\left(\frac{x}{\sqrt{1-y^2}},0\right)$$

for all $x\in\mathcal{B}$ and $y\in[0,1)$.

(iii) If $z\in\mathcal{B}$ satisfies $||z||=1$ and $0\leq s<t\leq 1$, then

$$U^0(sz,0)\leq U^0(tz,0)\exp((s^2-t^2)||z||^2/2). \tag{3.3}$$

Proof. (i) It suffices to use the equivalence $f\in M(x,y)$ if and only if $-f\in M(-x,y)$.

(ii) This follows immediately from the fact that $f\in M(x,y)$ if and only if $f/\sqrt{1-y^2}\in M(x/\sqrt{1-y^2},0)$.

(iii) Fix $x\in\mathcal{B}$ with $0<||x||<1$ and $\delta>0$ such that $||x+\delta x||\leq 1$. Apply 2°' to $y=0$ and a centered random variable T which takes two values: δx and $-2x/(1+||x||^2)$. We get

$$\begin{aligned}U^0(x,0)\leq{}&\frac{\delta||x||(1+||x||^2)}{2||x||+\delta||x||(1+||x||^2)}U^0\left(-\frac{x(1-||x||^2)}{1+||x||^2},\frac{2||x||}{1+||x||^2}\right)\\&+\frac{2||x||}{2||x||+\delta||x||(1+||x||^2)}U^0\left(x+\delta x,\delta||x||\right).\end{aligned}$$

By (i) and (ii), the first term on the right equals

$$\frac{\delta||x||\big|1-||x||^2\big|}{2||x||+\delta||x||(1+||x||^2)}U^0(x,0).$$

The second summand can be bounded from above by

$$\frac{2||x||}{2||x||+\delta||x||(1+||x||^2)}U^0\left(x+\delta x,0\right),$$

because $M(x+\delta x,0)\subset M(x+\delta x,\delta||x||)$. Plugging these two facts into the inequality above and using the assumption $||x||\leq 1$ (so $|1-||x||^2|=1-||x||^2$) yields

$$\frac{U^0(x+\delta x,0)}{U^0(x,0)}\geq 1+\delta||x||^2. \tag{3.4}$$

This gives

$$\frac{U^0(x(1+k\delta),0)}{U^0(x(1+(k-1)\delta),0)}\geq 1+\delta(1+(k-1)\delta)||x||^2,$$

provided $||x(1+k\delta)||\leq 1$. Consequently, if N is an integer such that $||x(1+N\delta)||\leq 1$, then

$$\frac{U^0(x(1+N\delta),0)}{U^0(x,0)}\geq\prod_{k=1}^{N}(1+\delta(1+(k-1)\delta)||x||^2). \tag{3.5}$$

Now we turn to (3.3). Assume first that $s>0$. Put $x=sz$, $\delta=(t/s-1)/N$ and let $N\to\infty$ in the inequality above to obtain

$$\frac{U^0(tz,0)}{U^0(sz,0)}\geq\exp\left(\frac{1}{2}||z||^2(t^2-s^2)\right),$$

which is the claim. Next, suppose that $s=0$. For any $0<s'<t$ we have, by 2°',

$$U^0(0,0) \leq \frac{1}{2}U^0(s'z,||s'z||)+\frac{1}{2}U^0(-s'z,||s'z||)=U^0(s'z,||s'z||)\leq U^0(s'z,0),$$

where in the latter passage we have used the inclusion $M(s'z,0)\subset M(s'z,||s'z||)$. Thus,

$$\frac{U^0(tz,0)}{U^0(0,0)} \geq \frac{U^0(tz,0)}{U^0(s'z,0)} \geq \exp\left(\frac{1}{2}||z||^2(t^2-(s')^2)\right)$$

and it remains to let $s'\to 0$. □

Remark 3.3. Suppose that $\mathcal{B}=\mathbb{R}$. It is easy to see that $U^0(1,0)\leq 1$: consider f starting from 1 and satisfying $\mathbb{P}(df_1=-1)=\mathbb{P}(df_1=1)=1/2$, $df_2=df_3=\ldots\equiv 0$. Thus, by 3°' and (3.3) we have

$$\beta(\mathbb{R})^{-1}\leq U^0(0,0)\leq U^0(1,0)/\sqrt{e}\leq 1/\sqrt{e},$$

that is, $\beta(\mathbb{R})\geq\sqrt{e}$. This implies the sharpness of (1.2) in the Hilbert-space-valued setting.

Now we will work under the assumption $\beta(\mathcal{B})=\sqrt{e}$. Then we are able to derive the explicit formula for U^0.

Lemma 3.4. *If $\beta(\mathcal{B})=\sqrt{e}$, then*

$$U^0(x,y)=\begin{cases}\sqrt{1-y^2}\exp\left(\frac{||x||^2}{2(1-y^2)}-\frac{1}{2}\right) & \text{if } ||x||^2+y^2<1,\\ ||x|| & \text{if } ||x||^2+y^2\geq 1.\end{cases}$$

Proof. First let us focus on the set $\{(x,y):||x||^2+y^2\geq 1\}$. By 1°' we have $U^0(x,y)\geq||x||$. To get the reverse estimate, consider a martingale f such that $f_0\equiv x$, df_1 takes values $-x$ and x, and $df_2=df_3=\ldots\equiv 0$. Then $y^2-||x||^2+S^2(f)=y^2+||x||^2\geq 1$ (so $f\in M(x,y)$) and $||f||_1=||x||$, which implies $U^0(x,y)\leq||x||$ by the definition of U^0. Now suppose that $||x||^2+y^2<1$. Using the second and third part of the previous lemma, we may write

$$U^0(x,y)=\sqrt{1-y^2}U^0\left(\frac{x}{\sqrt{1-y^2}},0\right)\geq U^0(0,0)\sqrt{1-y^2}\exp\left(\frac{||x||^2}{2(1-y^2)}\right),$$

so, by 3°',

$$U^0(x,y)\geq\sqrt{1-y^2}\exp\left(\frac{||x||^2}{2(1-y^2)}-\frac{1}{2}\right).$$

To get the reverse bound, we use the homogeneity of U^0 and (3.3) again:

$$\begin{aligned}U^0(x,y)&=\sqrt{1-y^2}U^0\left(\frac{x}{\sqrt{1-y^2}},0\right)\\&\leq\sqrt{1-y^2}U^0\left(\frac{x}{|x|},0\right)\exp\left(\frac{1}{2}\left(\frac{||x||^2}{1-y^2}-1\right)\right)\\&=\sqrt{1-y^2}\exp\left(\frac{||x||^2}{2(1-y^2)}-\frac{1}{2}\right),\end{aligned}$$

where in the last line we have used the equality $U^0(\overline{x},0)=||\overline{x}||$ valid for $\overline{x}$ of norm 1 (we have just established this in the first part of the proof). For completeness, let us mention here that if $x=0$, then $x/|x|$ should be replaced above by any vector of norm one. □

Lemma 3.5. *Suppose that $\beta(\mathcal{B})=\sqrt{e}$ and assume that x, $y\in\mathcal{B}$ and $\alpha>0$ satisfy*

$$||x||<1,\quad ||x+\alpha x+y||^2+||\alpha x+y||^2<1 \text{ and } ||x+\alpha x-y||^2+||\alpha x-y||^2<1.$$

Then

$$\begin{aligned}2+2\alpha||x||^2\leq&\sqrt{1-||\alpha x+y||^2}\exp\left[\frac{||x+\alpha x+y||^2}{2(1-||\alpha x+y||^2)}-\frac{||x||^2}{2}\right]\\ &+\sqrt{1-||\alpha x-y||^2}\exp\left[\frac{||x+\alpha x-y||^2}{2(1-||\alpha x-y||^2)}-\frac{||x||^2}{2}\right].\end{aligned}\tag{3.6}$$

Proof. Consider a random variable T such that

$$\mathbb{P}\left(T=-\frac{2x}{1+||x||^2}\right)=p,\quad \mathbb{P}(T=\alpha x+y)=\mathbb{P}(T=\alpha x-y)=\frac{1-p}{2},$$

where $p\in(0,1)$ is chosen so that $\mathbb{E}T=0$. That is,

$$p=\frac{\alpha(1+||x||^2)}{2+\alpha(1+||x||^2)}.$$

By 2°', we have $U^0(x,0)\leq\mathbb{E}U^0(x+T,||T||)$. Since $||x+T||^2+||T||^2<1$ almost surely, the previous lemma implies that this can be rewritten in the equivalent form

$$\begin{aligned}\exp\left[\frac{||x||^2}{2}\right]\leq& p\sqrt{1-\left(\frac{2||x||}{1+||x||^2}\right)^2}\exp\left[\frac{\left|\left|x\left(\frac{-1+||x||^2}{1+||x||^2}\right)\right|\right|^2}{2\left(1-\left(\frac{2||x||}{1+||x||^2}\right)^2\right)}\right]\\ &+\frac{1-p}{2}\sqrt{1-||\alpha x+y||^2}\exp\left[\frac{||x+\alpha x+y||^2}{2(1-||\alpha x+y||^2)}\right]\\ &+\frac{1-p}{2}\sqrt{1-||\alpha x-y||^2}\exp\left[\frac{||x+\alpha x-y||^2}{2(1-||\alpha x-y||^2)}\right].\end{aligned}$$

However, the first term on the right equals

$$\frac{\alpha(1-||x||^2)}{2+\alpha(1+||x||^2)}\exp\left[\frac{||x||^2}{2}\right]$$

and, in addition, $(1-p)/2=(2+\alpha(1+||x||^2))^{-1}$. Consequently, it suffices to multiply both sides of the inequality above by $(2+\alpha(1+||x||^2))\exp\left[-||x||^2/2\right]$; the claim follows. □

Now we are ready to complete the proof of Theorem 1.1. Suppose that a, b belong to the unit ball K of $\mathcal{B}$ and take $\varepsilon\in(0,1/2)$. Applying (3.6) to $x=\varepsilon a$, $y=\varepsilon^2 b$ and $\alpha=\varepsilon$ gives

$$\begin{aligned}2+2\varepsilon^3||a||^2\leq&\sqrt{1-\varepsilon^4||a+b||^2}\exp(m(a,b))\\ &+\sqrt{1-\varepsilon^4||a-b||^2}\exp(m(a,-b)),\end{aligned}\tag{3.7}$$

where

$$\begin{aligned}m(a,b)&=\frac{\varepsilon^2||a+\varepsilon(a+b)||^2}{2(1-\varepsilon^4||a+b||^2)}-\frac{\varepsilon^2||a||^2}{2}\\ &=\frac{\varepsilon^2}{2}(||a+\varepsilon(a+b)||^2-||a||^2)+\frac{\varepsilon^6||a+\varepsilon(a+b)||^2||a+b||^2}{2(1-\varepsilon^4||a+b||^2)}.\end{aligned}$$

It is easy to see that there exists an absolute constant M_1 such that

$$\sup_{a,b\in K} |m(a,b)| \leq M_1\varepsilon^3.$$

Consequently, there is a universal $M_2 > 0$ such that if ε is sufficiently small, then

$$\exp(m(a,b)) \leq 1 + m(a,b) + m(a,b)^2 \leq 1 + \frac{\varepsilon^2}{2}(||a+\varepsilon(a+b)||^2 - ||a||^2) + M_2\varepsilon^6$$

for any a, $b \in K$. Since $\sqrt{1-x} \leq 1 - x/2$ for $x \in (0,1)$, the inequality (3.7) implies

$$\begin{aligned} 2 + 2\varepsilon^3||a||^2 \leq &(1-\varepsilon^4||a+b||^2/2)\left(1 + \frac{\varepsilon^2}{2}(||a+\varepsilon(a+b)||^2 - ||a||^2) + M_2\varepsilon^6\right) \\ &+ (1-\varepsilon^4||a-b||^2/2)\left(1 + \frac{\varepsilon^2}{2}(||a+\varepsilon(a-b)||^2 - ||a||^2) + M_2\varepsilon^6\right). \end{aligned}$$

This, after some manipulations, leads to

$$\begin{aligned} &||a+\varepsilon(a+b)||^2 + ||a+\varepsilon(a-b)||^2 - 2||a(1+\varepsilon)||^2 \\ &\qquad \geq \varepsilon^2(||a+b||^2 + ||a-b||^2 - 2||a||^2) - 2\varepsilon^4 M_3, \end{aligned}$$

where M_3 is a positive constant not depending on ε, a and b. Equivalently,

$$\begin{aligned} &\left|\left|a + \frac{\varepsilon}{1+\varepsilon}b\right|\right|^2 + \left|\left|a - \frac{\varepsilon}{1+\varepsilon}b\right|\right|^2 - 2||a||^2 - 2\left|\left|\frac{\varepsilon}{1+\varepsilon}b\right|\right|^2 \\ &\geq \frac{\varepsilon^2}{(1+\varepsilon)^2}(||a+b||^2 + ||a-b||^2 - 2||a||^2 - 2||b||^2) - 2\frac{\varepsilon^4}{(1+\varepsilon)^2}M_3. \end{aligned}$$

Next, let $c \in \mathcal{B}$, $\gamma > 0$ and substitute $a = \gamma c$; we assume that γ is small enough to ensure that $a \in K$. If we divide both sides by γ^2 and substitute $\delta = \varepsilon(1+\varepsilon)^{-1}\gamma^{-1}$, we obtain

$$\begin{aligned} &||c+\delta b||^2 + ||c-\delta b||^2 - 2||c||^2 - 2||\delta b||^2 \\ &\geq \delta^2(||\gamma c + b||^2 + ||\gamma c - b||^2 - 2||\gamma c||^2 - 2||b||^2) - 2\varepsilon^2\delta^2 M_3 \\ &\geq \delta^2(||\gamma c + b||^2 + ||\gamma c - b||^2 - 2||\gamma c||^2 - 2||b||^2) - 2\delta^4 M_3 \end{aligned}$$

Let γ and ε go to 0 so that δ remains fixed. As the result, we obtain that for any $\delta > 0$, $b \in K$ and $c \in \mathcal{B}$,

$$||c+\delta b||^2 + ||c-\delta b||^2 - 2||c||^2 - 2||\delta b||^2 \geq -2\delta^4 M_3. \tag{3.8}$$

Now let N be a large positive integer and consider a symmetric random walk $(g_n)_{n\geq 0}$ over integers, starting from 0. Let $\tau = \inf\{n : |g_n| = N\}$. The inequality (3.8), applied to $\delta = N^{-1}$, implies that for any $a \in \mathcal{B}$ and $b \in K$, the process

$$(\xi_n)_{n\geq 0} = \left(\left|\left|a + \frac{bg_{\tau\wedge n}}{N}\right|\right|^2 - \left\{\frac{||b||^2}{N^2} - \frac{M_3}{N^4}\right\}(\tau\wedge n)\right)_{n\geq 0}$$

is a submartingale. Since $\mathbb{E}(\tau\wedge n) = \mathbb{E}g_{\tau\wedge n}^2$, we obtain

$$\mathbb{E}\left[\left|\left|a + \frac{bg_{\tau\wedge n}}{N}\right|\right|^2 - \left\{\frac{||b||^2}{N^2} - \frac{M_3}{N^4}\right\}g_{\tau\wedge n}^2\right] = \mathbb{E}\xi_n \geq \mathbb{E}\xi_0 = ||a||^2.$$

Letting $n \to \infty$ and using Lebesgue's dominated convergence theorem gives

$$\frac{1}{2}\left[||a+b||^2 + ||a-b||^2\right] - ||b||^2 + \frac{M_3}{N^2} \geq ||a||^2.$$

It suffices to let N go to ∞ to obtain

$$||a+b||^2 + ||a-b||^2 \geq 2||a||^2 + 2||b||^2.$$

We have assumed that b belongs to the unit ball K, but, by homogeneity, the above estimate extends to any $b \in \mathcal{B}$. Putting $a+b$ and $a-b$ in the place of a and b, respectively, we obtain the reverse estimate

$$||a+b||^2 + ||a-b||^2 \leq 2||a||^2 + 2||b||^2.$$

This implies that the parallelogram identity is satisfied and hence $\mathcal{B}$ is a Hilbert space.

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