SHARP WEAK-TYPE (p,p) ESTIMATES $(1 < p < \infty)$ FOR POSITIVE DYADIC SHIFTS

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Abstract. The paper contains the study of sharp weak-type estimates for positive dyadic shifts on $\mathbb{R}^d$. The proof exploits Bellman function method: the inequalities are deduced from the existence of certain associated special functions, enjoying appropriate majorization and concavity conditions.

1. Introduction

Assume that $Q \subset \mathbb{R}^d$ is a dyadic cube and let $\mathcal{D}(Q)$ stand for the grid of its dyadic subcubes. For a given sequence $\alpha = (\alpha_R)_{R\in\mathcal{D}(Q)}$ of nonnegative numbers, we define its Carleson constant by

$$\mathrm{Carl}(\alpha) = \sup_{R\in\mathcal{D}(Q)} \frac{1}{|R|} \sum_{R'\in\mathcal{D}(R)} \alpha_{R'}|R'|,$$

where $|A|$ is the Lebesgue measure of A. For any such sequence, one can introduce the associated dyadic shift $\mathcal{A}$ which acts on integrable functions $f : Q \to \mathbb{R}$ by

$$\mathcal{A}f = \sum_{R\in\mathcal{D}(Q)} \alpha_R \langle f\rangle_R \chi_R. \tag{1.1}$$

Here $\langle f\rangle_R = \frac{1}{|R|}\int_R f \mathrm{d}u$ denotes the average of f over the cube R. The class of positive dyadic shifts have appeared in many papers devoted to weighted inequalities for various „continuous" objects and their dyadic counterparts. In particular, it is well known that an efficient control over various norms of dyadic shifts yields the corresponding estimates for singular integral operators [10, 12]. As an example, we will describe in more detail the works of A. Lerner on the celebrated A_2 theorem. Assume that T is a Calderón-Zygmund operator on $\mathbb{R}^d$ and let $w : \mathbb{R}^d \to (0,\infty)$ be a weight satisfying Muckenhoupt's condition A_2. The so-called A_2 conjecture concerns the linear dependence of the norm $||T||_{L^2(w)\to L^2(w)}$ on $[w]_{A_2}$, the A_2 characteristic of w:

$$||Tf||_{L^2(w)} \leq C(T,d)[w]_{A_2}||f||_{L^2(w)}.$$

This question has gained a lot of interest in the recent literature (see e.g. [1, 4, 11, 16, 19, 20, 21, 22, 27]) and was finally answered in the positive by Hÿtonen [7], with the use of clever representation of T as an average of good dyadic shifts. Lerner [12] provided a more elementary proof of the A_2 theorem, which avoided the use of most of the complicated techniques in [7]. The idea was to exploit a general

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pointwise estimate for T in terms of positive dyadic operators, proven in [10]. This allowed to reduce the A_2 problem to a weighted result for the positive dyadic shifts, which had been already established in the earlier works [4], [5] and [9].

The above discussion motivates the question about various sharp-norm estimates for dyadic shifts. Using best-constant estimates for the dyadic maximal operator, one can show that any $\mathcal{A}$ from the class (1.1) is bounded on $L^p(Q)$, $1 < p < \infty$. Actually, one can identify the norm: we have

$$||\mathcal{A}f||_{L^p(Q)} \leq \frac{p^2}{p-1}\,\mathrm{Carl}(\alpha)||f||_{L^p(Q)} \tag{1.2}$$

and the multiplicative constant $p^2/(p-1)$ cannot be improved in general (i.e., there are Carleson sequences for which the L^p norm is precisely $p^2/(p-1)$). For $p = 1$ the L^p-boundedness does not hold, but, as proved by Rey and Reznikov [23] (for $d = 1$ only), we have the sharp weak-type bound

$$|\{x \in Q : \mathcal{A}f(x) \geq 1\}| \leq 2\,\mathrm{Carl}(\alpha)||f||_{L^1(Q)}.$$

See also [18] for a related logarithmic estimate. In this paper, we will identify the weak norms of $\mathcal{A}$ in the range $1 < p < \infty$. Namely, the main emphasis will be put on the weak-type inequality

$$\|\mathcal{A}f\|_{L^{p,\infty}(Q)} \leq C_p\|f\|_{L^p(Q)}, \qquad 1 < p < \infty, \tag{1.3}$$

where Q is a fixed dyadic cube in $\mathbb{R}^d$ and $\mathcal{A}$ is the shift associated with some sequence $(\alpha_Q)_{Q\in\mathcal{T}}$ of Carleson constant not exceeding 1. However, we will use a slightly different norming than that above. Namely, instead of the quasinorms $|||\varphi|||_{L^{p,\infty}(Q)} = \sup_{\lambda>0}\lambda|\{x \in Q : |\varphi(x)| \geq \lambda\}|^{1/p}$, we will use the alternative, equivalent weak norms

$$\|\varphi\|_{L^{p,\infty}(Q)} = \sup\left\{\frac{1}{|E|^{1-1/p}}\int_E |\varphi|\mathrm{d}u\right\},$$

where the supremum is taken over all measurable sets $E \subseteq Q$ of positive Lebesgue measure. The main result of the paper can be stated as follows.

Theorem 1.1. *For any $1 < p < \infty$, the optimal constant in* (1.3) *is given by*

$$C_p = \left(p - 1 + \int_1^\infty e^{1-u}u^{p/(p-1)}\,du\right)^{(p-1)/p}. \tag{1.4}$$

More precisely, for any dimension d, any dyadic cube $Q \subset \mathbb{R}^d$ and any $\varepsilon > 0$, there is a Carleson sequence $(\alpha_R)_{R\in\mathcal{D}(Q)}$ such that the corresponding dyadic shift has the weak L^p norm bigger than $\left(p - 1 + \int_1^\infty e^{1-u}u^{p/(p-1)}\,du\right)^{(p-1)/p} - \varepsilon$.

The nice feature of the above weak norming (and one of the main reasons why we have decided to work under it) is that the inequality (1.3) has a very convenient dual form. Note that it is enough to study this bound for nonnegative functions only (indeed, the passage from f to $|f|$ does not change the right-hand side, and can only make the left-hand side bigger). Since $\mathcal{A}$ is self-adjoint, we have

$$\int_E \mathcal{A}f\mathrm{d}u = \int_Q f\mathcal{A}\chi_E\mathrm{d}u \leq \|f\|_{L^p(Q)}\|\mathcal{A}\chi_E\|_{L^q(Q)},$$

where $q = p/(p-1)$ is the conjugate to p. Therefore, if c_q is the best constant in the inequality

$$\|\mathcal{A}\chi_E\|_{L^q(Q)} \leq c_q\|\chi_E\|_{L^q(Q)}, \tag{1.5}$$

then $C_p \leq c_q$. However, both constants are equal: the inequality (1.3) implies

$$\frac{\int_Q f\mathcal{A}\chi_E \mathrm{d}u}{\|f\|_{L^p(Q)}} \leq C_p\|\chi_E\|_{L^q(Q)}$$

for any set $E \subseteq Q$, and taking the supremum over f on the left-hand side yields (1.5) with the constant C_p.

So, it is enough to study the best constant in the inequality (1.5). Our approach will rest on the so-called Bellman function method. This technique enables to reduce the study of a given "dyadic" estimate to the investigation of a certain special function which possesses appropriate size and concavity properties (cf. e.g. [2, 15, 20, 25, 26, 28, 29] and consult references therein).

The rest of the paper is organized as follows. In the next section we present an informal reasoning which leads to the discovery of a Bellman function corresponding to our problem. Then, in Section 3, we exploit rigorously the properties of this object to prove the inequalities (1.3) and (1.5). In the final part of the paper we exploit further properties of the Bellman function to show that the constants in the estimates (1.3) and (1.5) cannot be improved.

2. A related Bellman function

As we have mentioned above, the proof of the weak-type inequality will depend heavily on a certain associated special function. This object will have a quite complicated explicit formula, and the purpose of this section is to present the analysis which leads to its discovery. Throughout this section, d is a fixed dimension.

2.1. Abstract definition of Bellman function and its structural properties.

The abstract Bellman function associated with (1.5) is the function $\mathbb{B} : [0,1]\times[0,1]\times[0,\infty) \to \mathbb{R}$ given by

$$\mathbb{B}(s,t,x) = \sup\left\{\frac{1}{|Q|}\int_Q \left(x + \mathcal{A}\chi_E\right)^q \mathrm{d}u\right\},$$

where Q is a fixed dyadic cube in $\mathbb{R}^d$. The supremum above is taken over all measurable sets $E \subseteq Q$ satisfying $\langle\chi_E\rangle_Q = s$ and all sequences $\alpha = (\alpha_R)_{R\in\mathcal{D}(Q)}$ of nonnegative numbers satisfying $\mathrm{Carl}(\alpha) \leq 1$ and

$$\frac{1}{|Q|}\sum_{R\in\mathcal{D}(Q)} \alpha_R|R| = t. \tag{2.1}$$

The relation between $\mathbb{B}$ and (1.5) is evident: the best constant in the latter estimate equals $\sup_{s,t}\left(\mathbb{B}(s,t,0)/s\right)^{1/q}$. So, if we manage to find the formula for $\mathbb{B}$, it will immediately lead us to the desired results. How to approach the problem? Let us start with some simple structural properties of $\mathbb{B}$ which are obvious from its very definition. First, the function does not depend on the underlying cube Q. Indeed, for any two dyadic cubes Q_1 and Q_2 in $\mathbb{R}^d$, affine mappings of one of them onto the other put the Carleson sequences satisfying (2.1) in one-to-one correspondence; furthermore, such a change of the variable preserves the averages. Our next observation is that

$$\begin{aligned}&\text{for any } s\in[0,1] \text{ and } x\geq 0, \text{ the function } t\mapsto \mathbb{B}(s,t,x) \text{ is nondecreasing,}\\ &\text{for any } t\in[0,1] \text{ and } x\geq 0, \text{ the function } s\mapsto \mathbb{B}(s,t,x) \text{ is nondecreasing.}\end{aligned} \tag{2.2}$$

Furthermore, we have

$$\mathbb{B}(s,t,x)\geq x^q, \text{ with equality if and only if } st=0. \tag{2.3}$$

These statements follow immediately from the definition of $\mathbb{B}$. The next property is less evident and amounts to saying that $\mathbb{B}$ satisfies an appropriate concavity.

Lemma 2.1. *For any s_1, s_2, …, s_{2^d}, t_1, t_2, …, $t_{2^d}\in[0,1]$ and $x\geq 0$ we have*

$$\mathbb{B}\left(\frac{1}{2^d}\sum_{j=1}^{2^d}s_j,\frac{1}{2^d}\sum_{j=1}^{2^d}t_j,x\right)\geq\frac{1}{2^d}\sum_{j=1}^{2^d}\mathbb{B}(s_j,t_j,x).$$

Proof. Let Q^1, Q^2, …, Q^{2^d} be the pairwise disjoint direct dyadic children of $[0,2]^d$. Take $E^j\subseteq Q^j$ and $(\alpha_R)_{R\in\mathcal{D}(Q^j)}$ as in the definitions of $\mathbb{B}(s_j,t_j,x)$. We glue these objects as follows: set $E=\bigcup_{j=1}^{2^d}E^j$ and let $(\alpha_R)_{R\in\mathcal{D}([0,2]^d)}=\bigcup_{j=1}^{2^d}(\alpha_R)_{R\in\mathcal{D}(Q^j)}$ be the collection of all $(\alpha_R)_{R\in\mathcal{D}(Q^j)}$, with the additional term $\alpha_{[0,2]^d}=0$. Of course, we have $\langle\chi_E\rangle_{[0,2]^d}=2^{-d}\sum_{j=1}^d\langle\chi_{E^j}\rangle_{Q^k}=2^{-d}\sum_{j=1}^{2^d}s_j$. Furthermore, it is easy to see that $(\alpha_R)_{R\in\mathcal{D}([0,2]^d)}$ has Carleson constant less or equal to 1 and satisfies $2^{-d}\sum_{R\in\mathcal{D}([0,2]^d)}\alpha_R|R|=2^{-d}\sum_{j=1}^{2^d}\sum_{R\in\mathcal{D}(Q^j)}\alpha_R|R|=2^{-d}\sum_{j=1}^{2^d}t_j$. Hence

$$\begin{aligned}\mathbb{B}\left(\frac{1}{2^d}\sum_{j=1}^{2^d}s_j,\frac{1}{2^d}\sum_{j=1}^{2^d}t_j,x\right)&\geq\frac{1}{2^d}\int_{[0,2]^d}\left(x+\sum_{Q\in\mathcal{D}([0,2]^d)}\alpha_R\langle\chi_E\rangle_R\chi_R\right)^q\mathrm{d}u\\&=\frac{1}{2^d}\sum_{j=1}^{2^d}\int_{R^j}\left(x+\sum_{R\in\mathcal{D}(Q^j)}\alpha_R\langle\chi_{E^j}\rangle_R\chi_R\right)^q\mathrm{d}u.\end{aligned}$$

Taking the supremum over all E^j and $(\alpha_R^j)_{R\in\mathcal{D}(Q^j)}$, we get the desired claim. □

In particular, the lemma above implies that for each $x\geq 0$, the function $\mathbb{B}(\cdot,\cdot,x)$ is concave. The following informal observation plays a crucial role.

Remark 2.2. Let $(s,t,x)\in[0,1]\times[0,1]\times[0,\infty)$ be a fixed point. Suppose that $E\subseteq[0,2]^d$ and $(\alpha_R)_{R\in\mathcal{D}([0,2]^d)}$ are the extremizers of $\mathbb{B}(s,t,x)$, that is,

$$\mathbb{B}(s,t,x)=\int_{[0,2]^d}\left(x+\sum_{R\in\mathcal{D}([0,2]^d)}\alpha_R\langle\chi_E\rangle_R\chi_R\right)^q\mathrm{d}u.$$

Let Q^1, Q^2, …, Q^{2^d} be the direct dyadic children of $[0,2]^d$ and let $s^j=2^d|E\cap Q^j|$, $t^j=2^d\sum_{R\in\mathcal{D}(Q^j)}\alpha_R|R|$. If $\alpha_{[0,2]^d}=0$, then we have

$$\mathbb{B}(s,t,x)=\frac{1}{2^d}\sum_{j=1}^{2^d}\int_{Q^j}\left(x+\sum_{R\in\mathcal{D}(Q^j)}\alpha_R\langle\chi_E\rangle_R\chi_R\right)^q\mathrm{d}u\leq\frac{1}{2^d}\mathbb{B}(s^j,t^j,x)$$

and hence, in the light of the lemma above, we actually have equality here. So, if at least one (s^j, t^j, x) is different from (s, t, x) and $\alpha_{[0,1]} = 0$ (which, as we might hope, should hold for most points (s, t, x)), then there is a line segment passing through (s, t, x) along which $\mathbb{B}$ is linear.

The second structural property of $\mathbb{B}$ is the following.

Lemma 2.3. *For any $s \in [0,1]$, $0 \le t < t+\delta \le 1$ and $x \ge 0$ we have*

$$\mathbb{B}(s, t+\delta, x) \ge \mathbb{B}(s, t, x+s\delta).$$

Proof. We proceed as in the previous lemma, with $s^j = s$ and $t^j = t$, and take the appropriate parameters E^j and $(\alpha_R)_{R\in\mathcal{D}(Q^j)}$. Then E^j are glued into one set E with the use of the same procedure, and the sequences $(\alpha_R)_{R\in\mathcal{D}(Q^j)}$ are combined into one sequence as before. The only difference is that the additional term $\alpha_{[0,2]^d}$ corresponding to the "full space" $[0,2]^d$ is set to be equal to δ. Then the Carleson constant of $(\alpha_R)_{R\in\mathcal{D}([0,2]^d)}$ is bounded by 1 and $\sum_{R\in\mathcal{D}([0,2]^d)} \alpha_R|R| = \frac{1}{2^d}\sum_{j=1}^{2^d} t^j + \delta = t+\delta$. Therefore, by the definition of $\mathbb{B}$,

$$\begin{aligned}\mathbb{B}\,(s, t+\delta, x) &\ge \frac{1}{2^d}\int_{[0,2]^d}\left(x + \sum_{R\in\mathcal{D}([0,2]^d)} \alpha_R\langle\chi_E\rangle_R\chi_R\right)^q \mathrm{d}u\\ &= \frac{1}{2^d}\sum_{j=1}^{2^d}\int_{Q^j}\left(x + \alpha_{[0,2]^d}\langle\chi_E\rangle_{[0,2]^d} + \sum_{R\in\mathcal{D}(Q^j)} \alpha_R\langle\chi_E\rangle_R\chi_R\right)^q \mathrm{d}u.\end{aligned}$$

However, $\alpha_{[0,2]^d}\langle\chi_E\rangle_{[0,2]^d} = s\delta$, so taking the supremum over all E^j and $(\alpha^j_R)_{R\in\mathcal{D}(Q^j)}$, we get the claim. □

2.2. **On the search for the explicit formula.** Equipped with the structural properties above, we are ready to construct a candidate for $\mathbb{B}$. This construction will depend on the additional assumption that the Bellman function is continuous on its domain and of class C^2 in its interior; furthermore, we will impose some additional conditions - we will assume that certain inequalities become equalities on some parts of the domain. Since this regularity and these equalities do not seem to follow from the abstract definition of $\mathbb{B}$ (actually, we will even see that the constructed object we will end up with will only be of class C^1), the candidate should be denoted by a different symbol; we will use the letter B. One should keep in mind that the primary goal of this section is to *discover the explicit formula*, which will be rigorously exploited later. We split the reasoning into a few intermediate steps.

Step 1. By Lemma 2.1 and the assumed regularity of B, we see that for each x the function

$$(s,t) \mapsto B(s,t,x)$$

is concave, so

$$D^2_{s,t}B(s,t,x) = \begin{bmatrix} B_{ss}(s,t,x) & B_{st}(s,t,x)\\ B_{st}(s,t,x) & B_{tt}(s,t,x)\end{bmatrix} \le 0.$$

Furthermore, in the light of Remark 2.2, it seems plausible to *assume* that the determinant of the above Hessian matrix is equal to zero. This is equivalent to saying that for any $x \ge 0$ the function $B(\cdot,\cdot,x)$ satisfies the Monge-Ampére equation. It follows from general theory of these equations (see a similar discussion in [29]) that

$[0,1]\times[0,1]$, the domain of $B(\cdot,\cdot,x)$, can be foliated, i.e., split into union of pairwise disjoint line segments along which this function is linear.

Step 2. It is not difficult to guess how these line segments (leaves of foliation) should look like. Indeed, fix $x\geq 0$ and suppose first that $0<s<t$. Let I be a line segment containing (s,t), with endpoints $\mathbf{a}$, $\mathbf{b}$ lying at the boundary of the square $[0,1]\times[0,1]$. By the concavity of $B(\cdot,\cdot,x)$, we have

$$B(s,t,x)\geq \gamma B(\mathbf{a},x)+(1-\gamma)B(\mathbf{b},x), \tag{2.4}$$

where $\gamma\in[0,1]$ is uniquely determined by the requirement $\gamma\mathbf{a}+(1-\gamma)\mathbf{b}=(s,t)$. Furthermore, if I is "the linearity segment" we search for, both sides above become equal. In the other words, the leaf of the foliation is the line segment for which the right-hand side of (2.4) is maximized. We claim that this is the case if one of the endpoints equals $(0,0)$ and the other is $(s/t,1)$: then the right-hand side equals

$$(1-t)B(0,0,x)+tB(s/t,1,x)=x^q+t\big(B(s/t,1,x)-x^q\big). \tag{2.5}$$

To see this, we consider several cases. If $\mathbf{a}=(a,0)$ and $\mathbf{b}=(0,b)$ for some a, $b\geq 0$, then the right-hand side of (2.4) is equal to x^q, which is less than (2.5) by virtue of (2.3). If $\mathbf{a}=(a,0)$ and $\mathbf{b}=(b,1)$, then $b\leq s/t$ and the right-hand side of (2.4) is equal to $(1-t)x^q+tB(b,1,x)$. By (2.2), this is not bigger than (2.4). The next case is $\mathbf{a}=(0,a)$ and $\mathbf{b}=(b,1)$. Then $b\geq s/t$ and the right-hand side of (2.4) equals

$$\left(1-\frac{s}{b}\right)x^q+\frac{s}{b}B(b,1,x)=x^q+s\cdot\frac{B(b,1,x)-B(0,1,x)}{b}.$$

But the function $u\mapsto B(u,1,x)$ is concave, so the above expression is not bigger than

$$x^q+s\cdot\frac{B(s/t,1,x)-B(0,1,x)}{s/t},$$

which is exactly (2.5). The final possibility is $\mathbf{a}=(0,a)$ and $\mathbf{b}=(1,b)$, for which we use (2.2) again: the right-hand side of (2.4) is given by

$$(1-s)x^q+sB(1,b,x)\leq(1-s)x^q+sB(1,1,x),$$

and the latter expression has been handled in the analysis of the preceding case. So, the line segments joining $\mathbf{a}=(0,0)$ with $\mathbf{b}=(b,1)$ belong to the foliation.

Step 3. Now we will identify the foliation of the set $\{(s,t)\in[0,1]\times[0,1]:s\geq t\}$. To this end, we will show that

$$\text{the function } s\mapsto B_s(s,s,x) \text{ is constant on } (0,1). \tag{2.6}$$

Indeed, by the previous step and continuity of B, the function $s\mapsto B(s,s,x)$ is linear and thus

$$B_{ss}(s,s,x)+2B_{st}(s,s,x)+B_{tt}(s,s,x)=0.$$

This, combined with the Monge-Ampére equation $B_{ss}B_{tt}=\big(B_{st}\big)^2$, gives

$$\left(B_{ss}(s,s,x)+B_{st}(s,s,x)\right)^2=0,$$

and hence (2.6) follows by simple differentiation. Let us now *assume* that the function $s\mapsto B_s(s,s,x)$ is constant on the full interval $[0,1]$; since $B_s(0,0,x)=0$ (which is trivial: $B(s,0,x)=x^q$ for all s), we get that $B_s(s,s,x)=0$ for all s. Combining this with (2.2) and the concavity of $B(\cdot,t,x)$, we see that for any t and x the function $s\mapsto B(s,t,x)$ must be constant on $[t,1]$. Equivalently, we must have the identity $B(s,t,x) = B(\min\{s,t\},t,x)$ on the domain of B. Now it is easy to see that the line segments joining $\mathbf{a} = (0,0)$ with $\mathbf{b} = (1,b)$ belong to the foliation.

Step 4. Now we will try to determine the values of B on the set $[0,1]\times\{1\}$. To this end, we will exploit the inequality of Lemma 2.3. Replacing $\mathbb{B}$ with B, we see that $B(s,t+\delta,x) - B(s,t,x+s\delta) \geq 0$ for all $\delta > 0$. Hence, in particular, if we divide both sides by δ and let $\delta \to 0$, we get

$$B_t(s,t,x) - sB_x(s,t,x) \geq 0.$$

Let us *assume* that we have equality if $t = 1$. Combining this with the equation

$$sB_s(s,1,x) + B_t(s,t,1) = B(s,1,x) - B(0,0,x)$$

(which is due to the linearity of the function $t \mapsto B(st,t,x)$ obtained in Step 2), we get

$$s\big(B_s(s,1,x) + B_x(s,1,x)\big) = B(s,1,x) - x^q. \tag{2.7}$$

To solve this equation, fix $u \geq -1$ and consider the function

$$\xi(s) = \xi^{(u)}(s) = B(s,1,u+s),$$

defined on $s \in [0,1] \cap [-u,\infty)$. By the previous equation, we see that $s\xi'(s) = \xi(s) - (u+s)^q$ and hence

$$B(s,1,u+s) = \xi(s) = s\int_s^1 \frac{(u+r)^q}{r^2}\mathrm{d}r + sA(u),$$

for some unknown function A to be found. The substitution $u = x - s$ yields

$$B(s,1,x) = s\int_s^1 \frac{(x-s+r)^q}{r^2}\mathrm{d}r + sA(x-s).$$

Plugging $s = 1$, we get that $A(x-1) = B(1,1,x)$ and hence the above identity can be rewritten in the form

$$B(s,1,x) = s\int_s^1 \frac{(x-s+r)^q}{r^2}\mathrm{d}r + sB(1,1,x-s+1). \tag{2.8}$$

Now we will use the equality $B_s(1,1,x) = 0$ we assumed at the previous step: plugging this into (2.7) gives

$$B_x(1,1,x) = B(1,1,x) - x^q,$$

which can be easily solved:

$$B(1,1,x) = e^x\left(\int_x^\infty e^{-r}r^q\mathrm{d}r + C\right),$$

for some absolute constant C. It is easy to see that $C = 0$; if $C < 0$, then it would mean that $B(1,1,x)$ is negative for sufficiently large x, which contradicts (2.3). The inequality $C > 0$ would imply the exponential growth of $B(1,1,x)$ as $x \to \infty$, which is impossible: directly from the definition of $\mathbb{B}$, we see that $\mathbb{B}(1,1,x) \leq 2^q\left(x^q + B(1,1,0)\right)$. Thus, we conclude that

$$B(1,1,x) = e^x\int_x^\infty e^{-r}r^q\mathrm{d}r,$$

and coming back to (2.8), we obtain

$$\begin{aligned} B(s,1,x) &= s\int_s^1 \frac{(x-s+r)^q}{r^2}\mathrm{d}r + se^{x-s+1}\int_{x-s+1}^\infty e^{-r}r^q\mathrm{d}r \\ &= s\int_s^\infty (x-s+r)^q\Psi(r)\mathrm{d}r, \end{aligned}$$

where the kernel Ψ is defined by the formula

$$\Psi(r) = \begin{cases} r^{-2} & \text{if } 0<r<1, \\ e^{1-r} & \text{if } r\geq 1. \end{cases}$$

Step 5. Putting all the above facts together, we obtain the following candidate for the function $B:[0,1]\times[0,1]\times[0,\infty)\to\mathbb{R}$: $B(s,0,x)=B(0,t,x)=x^q$ and

$$B(s,t,x) = \begin{cases} (1-t)x^q + s\displaystyle\int_{s/t}^\infty \left(x-\frac{s}{t}+r\right)^q \Psi(r)\mathrm{d}r & \text{if } 0<s\leq t, \\ (1-t)x^q + t\displaystyle\int_1^\infty (x-1+r)^q\Psi(r)\mathrm{d}r & \text{if } 0<t\leq s. \end{cases}$$

This function will be rigorously investigated in Section 3 and it will lead us to the main results of the paper.

2.3. **The inequality $B\leq\mathbb{B}$.** We are ready to address one of the major two steps in the study of the Bellman function $\mathbb{B}$. Namely, we will show that it is bounded from below by the function B discovered in the previous section. The reverse estimate is postponed to the next section. Again, we split the arguments into several intermediate steps.

Step 1. We have $B(s,t,x)\leq\mathbb{B}(s,t,x)$ if $st=0$. This follows from (2.3) and the equality $B(s,t,x)=x^q$.

Step 2. We have $B(1,1,x)\leq\mathbb{B}(1,1,x)$. By Lemma 2.3 and (2.2), we get, for small $\delta>0$,

$$\mathbb{B}(1,1,x)\geq\mathbb{B}(1,1-\delta,x+\delta)\geq\mathbb{B}(1-\delta,1-\delta,x+\delta). \tag{2.9}$$

Next, $\mathbb{B}$ is concave, so in particular it is concave along the line segment with end-points $(0,0,x+\delta)$ and $(1,1,x+\delta)$, containing $(1-\delta,1-\delta,x+\delta)$. Consequently,

$$\mathbb{B}(1-\delta,1-\delta,x+\delta)\geq\delta\mathbb{B}(0,0,x+\delta)+(1-\delta)\mathbb{B}(1,1,x+\delta).$$

Putting the two steps above together, we get

$$\begin{aligned} \mathbb{B}(1,1,x) &\geq \delta\mathbb{B}(0,0,x+\delta)+(1-\delta)\mathbb{B}(1,1,x+\delta) \\ &\geq \delta(x+\delta)^q+(1-\delta)\mathbb{B}(1,1,x+\delta), \end{aligned}$$

where the last inequality follows from the previous step. Iterating, we get

$$\begin{aligned} \mathbb{B}(1,1,x) &\geq \delta\sum_{k=0}^{N-1}(1-\delta)^k(x+(k+1)\delta)^q+(1-\delta)^N\mathbb{B}(1,1,x+N\delta) \\ &\geq \delta\sum_{k=0}^{N-1}(1-\delta)^k(x+k\delta)^q \end{aligned}$$

for each N. Now, if we fix $T > 0$, put $\delta = T/N$ and let $N \to \infty$, we get

$$\mathbb{B}(1,1,x) \geq \int_0^T e^{-u}(x+u)^q \mathrm{d}u.$$

Since T was arbitrary and $B(1,1,x) = \int_0^\infty e^{-u}(x+u)^q \mathrm{d}u$, the desired bound holds.

Step 3. We have $B(s,1,x) \leq \mathbb{B}(s,1,x)$ for $0 < s < 1$. Lemma 2.3 and the concavity of $\mathbb{B}$ imply that

$$\begin{aligned}\mathbb{B}(s,1,x) &\geq \mathbb{B}(s,1-\delta,x+s\delta)\\ &\geq \delta\mathbb{B}(0,0,x+s\delta) + (1-\delta)\mathbb{B}(s/(1-\delta),1,x+s\delta)\\ &\geq \delta(x+s\delta)^q + (1-\delta)\mathbb{B}(s/(1-\delta),1,x+s\delta)\end{aligned}$$

for small $\delta > 0$. Here in the last line we have exploited Step 1 above. This inequality can be iterated. If we pick a large integer N and set $\delta = 1 - s^{1/N}$ (so that $(1-\delta)^N = s$), we get

$$\begin{aligned}\mathbb{B}(s,1,x) &\geq \delta \sum_{k=0}^{N-1} (1-\delta)^k \left(x + s\delta + \frac{s\delta}{1-\delta} + \ldots + \frac{s\delta}{(1-\delta)^k}\right)\\ &\quad + (1-\delta)^N \mathbb{B}\left(1,1,x+s\delta+\frac{s\delta}{1-\delta}+\ldots+\frac{s\delta}{(1-\delta)^{N-1}}\right)\\ &= \delta \sum_{k=0}^{N-1} (1-\delta)^k \left(x + s(1-\delta)((1-\delta)^{-k-1} - 1\right)\\ &\quad + (1-\delta)^N \mathbb{B}(1,1,x+(1-s)(1-\delta))\\ &\geq \delta \sum_{k=0}^{N-1} (1-\delta)^k \left(x + s(1-\delta)((1-\delta)^{-k-1} - 1\right)\\ &\quad + (1-\delta)^N B(1,1,x+(1-s)(1-\delta)),\end{aligned}$$

where in the last line we have exploited the previous step. Letting $N \to \infty$, we obtain

$$\begin{aligned}\mathbb{B}(s,1,x) &\geq \int_0^{-\ln s} e^{-r}(x+(e^r-1)s)^q \mathrm{d}r + sB(1,1,x+1-s)\\ &= s\int_s^1 (x-s+r)^q r^{-2}\mathrm{d}r + s\int_1^\infty (x-s+r)^q e^{1-r}\mathrm{d}r = B(s,1,x).\end{aligned}$$

Step 4. We have the inequality $B(s,t,x) \leq \mathbb{B}(s,t,x)$ for $0 < t < 1$. If $s \leq t$, we use the concavity of $\mathbb{B}$ and the previous steps to obtain

$$\mathbb{B}(s,t,x) = (1-t)\mathbb{B}(0,0,x) + t\mathbb{B}(s/t,1,x) = (1-t)x^p + tB(s/t,1,x) = B(s,t,x),$$

as desired. If $s > t$, the argument is equally simple: we have

$$\begin{aligned}\mathbb{B}(s,t,x) &= (1-t)\mathbb{B}((s-t)/(1-t),0,x) + t\mathbb{B}(1,1,x)\\ &= (1-t)x^p + tB(1,1,x) = B(s,t,x).\end{aligned}$$

This completes the proof of the inequality $B \leq \mathbb{B}$.

2.4. **A lower bound for the weak-type constant.** As a by-product of the previous subsection, we will show that the weak-type constant in (1.3) is not smaller than the number in (1.4). As we have discussed in the introduction, this is equivalent to showing that the best constant in (1.5) (let us denote it by $\beta_{q,d}$) is at least the value in (1.4). To this end, fix a dyadic cube $Q \subset \mathbb{R}^d$ and a number $s \in (0,1]$. As we have already established above, there is a measurable set $E \subseteq Q$ satisfying $\langle \chi_E \rangle_Q = s$ and a Carleson sequence $(\alpha_R)_{R\in\mathcal{D}(Q)}$ with $\mathrm{Carl}(\alpha) \leq 1$ and $|Q|^{-1}\sum_{R\in\mathcal{D}(Q)} \alpha_R|R| = 1$, such that the associated dyadic shift $\mathcal{A}$ enjoys

$$\frac{1}{|Q|}\int_Q (\mathcal{A}\chi_E)^q \mathrm{d}u \geq \mathbb{B}(s,1,0) - s^2 \geq \mathbb{B}(s,1,0) - s^2.$$

Consequently,

$$\beta_{q,d}^q \geq \frac{1}{|E|}\int_Q (\mathcal{A}\chi_E)^q \mathrm{d}u \geq s^{-1}B(s,1,0) - s,$$

and since $\lim_{s\downarrow 0}\left[s^{-1}B(s,1,0) - s\right] = c_q^q$, the claim follows.

3. Proof of the inequality $\mathbb{B} \leq B$ and the weak-type bound (1.5)

Now we will study certain properties of the function B and show how they lead to the desired weak-type bound. We start with the appropriate regularity condition.

Lemma 3.1. *The function B is continuous on $[0,1]\times[0,1]\times[0,\infty)$, of class C^1 on $(0,1)\times(0,1)\times(0,\infty)$ and of class C^2 on $\big((0,1)\times(0,1)\times(0,\infty)\big)\setminus\{(s,t,x) : s = t\}$.*

Proof. The continuity is evident, maybe except for the points of the form $(0,0,x_0)$. If we let $(s,t,x) \to (0,0,x_0)$ along the region $\{(s,t,x) : s \geq t\}$, we clearly have $B(s,t,x) \to x_0^q$; on the other hand, if we let along the region $\{(s,t,x) : s < t\}$, then note that the expression

$$\frac{s}{t}\int_{s/t}^\infty \left(x - \frac{s}{t} + r\right)^q \Psi(r)\mathrm{d}r \leq \frac{s}{t}\int_{s/t}^\infty (x+r)^q\,\Psi(r)\mathrm{d}r$$

is uniformly bounded and hence $B(s,t,x) \to x_0^q$ also in this case. The postulated smoothness on the set $\big((0,1)\times(0,1)\times(0,\infty)\big)\setminus\{(s,t,x) : s = t\}$ is obvious, and it remains to note that for any $t_0 \in (0,1)$ and $x_0 > 0$,

$$\lim_{(s,t,x)\to(t_0,t_0,x_0)} B_s(s,t,x) = B_s(t_0,t_0,x_0) = 0,$$

$$\lim_{(s,t,x)\to(t_0,t_0,x_0)} B_t(s,t,x) = B_t(t_0,t_0,x_0) = -x_0^q + \int_1^\infty (x_0 - 1 + r)^q \Psi(r)\mathrm{d}r$$

(here we have used the integration by parts) and

$$\begin{aligned}\lim_{(s,t,x)\to(t_0,t_0,x_0)} B_x(s,t,x) &= B_x(t_0,t_0,x_0)\\ &= q(1-t_0)x_0^{q-1} + qt_0\int_1^\infty (x_0-1+r)^{q-1}\,\Psi(r)\mathrm{d}r.\end{aligned}$$

This proves the claim. ∎

The next step is to establish the following fact.

Lemma 3.2. *For any $x \geq 0$, the function $B(\cdot,\cdot,x)$ is concave on $[0,1]\times[0,1]$.*

Proof. In the light of the regularity established in the previous lemma, it is enough to show that the Hessian matrix $D^2_{s,t}B$ is nonpositive-definite on $\big((0,1)\times(0,1)\big)\setminus\{(s,t):s=t\}$. This is clear on the set $\{(s,t):s>t\}$ (the function does not depend on s and depends linearly on t), so we may focus on the region $\{(s,t):s<t\}$. Each point (s,t) belonging to this region is contained in a line segment along which B is linear (this follows from the construction), which implies that the determinant of the Hessian matrix is zero. Consequently, it is enough to check that $B_{tt}\le 0$. We compute directly that if $s<t$, then

$$B_t(s,t,x)=\frac{qs^2}{t^2}\int_{s/t}^\infty\left(x-\frac{s}{t}+r\right)^{q-1}\Psi(r)\mathrm{d}r \tag{3.1}$$

and, using integration by parts,

$$B_{tt}(s,t,x)=-\frac{2qs^2}{t^3}\int_{s/t}^\infty\left(x-\frac{s}{t}+r\right)^{q-1}\Psi(r)\mathrm{d}r-\frac{qs^3}{t^4}\int_{s/t}^\infty\left(x-\frac{s}{t}+r\right)^{q-1}\Psi'(r)\mathrm{d}r.$$

Now we split the integrals: we have

$$\begin{aligned}-\frac{2qs^2}{t^3}\int_{s/t}^1\left(x-\frac{s}{t}+r\right)^{q-1}\Psi(r)\mathrm{d}r-\frac{qs^3}{t^4}\int_{s/t}^1\left(x-\frac{s}{t}+r\right)^{q-1}\Psi'(r)\mathrm{d}r\\ =-\frac{2qs^2}{t^3}\int_{s/t}^1\left(x-\frac{s}{t}+r\right)^{q-1}r^{-2}\left(1-\frac{s}{tr}\right)\mathrm{d}r\le 0\end{aligned}$$

and

$$\begin{aligned}-\frac{2qs^2}{t^3}\int_1^\infty\left(x-\frac{s}{t}+r\right)^{q-1}\Psi(r)\mathrm{d}r-\frac{qs^3}{t^4}\int_1^\infty\left(x-\frac{s}{t}+r\right)^{q-1}\Psi'(r)\mathrm{d}r\\ =-\frac{2qs^2}{t^3}\int_1^\infty\left(x-\frac{s}{t}+r\right)^{q-1}e^{1-r}\left(1-\frac{s}{2t}\right)\mathrm{d}r\le 0.\end{aligned}$$

Summing the above two inequalities, we get the desired bound $B_{tt}(s,t,x)\le 0$. □

Next, we will verify the "infinitesimal version" of Lemma 2.3.

Lemma 3.3. *For any s, $t\in(0,1)$ and any $x>0$ we have*

$$B_t(s,t,x)\ge sB_x(s,t,x). \tag{3.2}$$

Proof. We consider two cases. If $s\ge t$, the estimate is equivalent to

$$-x^q+\int_1^\infty(x-1+r)^q\Psi(r)\mathrm{d}r\ge qs\left((1-t)x^{q-1}+t\int_1^\infty(x-1+r)^{q-1}\Psi(r)\mathrm{d}r\right).$$

The right hand side attains its maximal value for $s=t=1$, because

$$\int_1^\infty(x-1+r)^{q-1}\Psi(r)\mathrm{d}r>x^{q-1}\int_1^\infty\Psi(r)\mathrm{d}r=x^{q-1}$$

and hence it is enough to show that

$$-x^q+\int_1^\infty(x-1+r)^q\Psi(r)\mathrm{d}r\ge q\int_1^\infty(x-1+r)^{q-1}\Psi(r)\mathrm{d}r.$$

However, both sides are equal, which follows immediately from the integration by parts. If $s<t$, the inequality (3.2) is equivalent to

$$\frac{s}{t^2}\int_{s/t}^\infty\left(x-\frac{s}{t}+r\right)^{q-1}\Psi(r)\mathrm{d}r\ge(1-t)x^{q-1}+s\int_{s/t}^\infty\left(x-\frac{s}{t}+r\right)^{q-1}\Psi(r)\mathrm{d}r,$$

or

$$\int_{s/t}^{\infty}\left(x-\frac{s}{t}+r\right)^{q-1}\Psi(r)\mathrm{d}r\geq\frac{t^2x^{q-1}}{s(1+t)}.$$

However, we have

$$\int_{s/t}^{\infty}\left(x-\frac{s}{t}+r\right)^{q-1}\Psi(r)\mathrm{d}r\geq\int_{s/t}^{\infty}x^{q-1}\Psi(r)\mathrm{d}r=x^{q-1}\cdot\frac{t}{s}$$

and the assertion follows. □

Finally, we will establish the following majorization property.

Lemma 3.4. *For any* $(s,t,x)\in[0,1]\times[0,1]\times[0,\infty)$ *we have* $B(s,t,x)\geq x^q$ *and* $B(s,t,0)\leq c_q^q s$, *where* $c_q=C_{q/(q-1)}$ *is the constant defined in* (1.4).

Proof. The inequality $B(s,t,x)\geq x^q$ follows at once from the facts that B is continuous, $B_t\geq 0$ (see (3.1)) and $B(s,0,x)=x^q$. To show the second estimate, we use the bound $B_t\geq 0$ again and get

$$s^{-1}B(s,t,0)\leq s^{-1}B(s,1,0)=\int_s^{\infty}(-s+r)^q\Psi(r)\mathrm{d}r.$$

Obviously, the latter expression decreases as s increases, so we obtain

$$s^{-1}B(s,t,0)\leq\int_0^{\infty}r^q\Psi(r)\mathrm{d}r=c_q^q.\quad\square$$

As a consequence of the lemmas above, we get the following concavity-type property of B. It will be fundamental for the induction argument used in the proof of the inequality $\mathbb{B}\leq B$.

Lemma 3.5. *Let* $d\geq 1$ *be an integer, let* $x\geq 0$ *and suppose that* $(s_i,t_i)\in[0,1]\times[0,1]$, $i=1,2,\ldots,2^d$, *are arbitrary points. Set* $s=2^{-d}\sum_{j=1}^{2^d}s_j$ *and* $t=2^{-d}\sum_{j=1}^{2^d}$. *Then for any* $\tilde{t}\geq t$ *we have*

$$B(s,\tilde{t},x)\geq 2^{-d}\sum_{j=1}^{2^d}B\left(s_j,t_j,x+s(\tilde{t}-t)\right).\tag{3.3}$$

Proof. Denote $\tilde{x}=x+s\left(\tilde{t}-t\right)$. Since the function $B(\cdot,\cdot,\tilde{x})$ is concave, we obtain

$$2^{-d}\sum_{j=1}^{2^d}B\left(s_j,t_j,x+s\left(\tilde{t}-t\right)\right)\leq B(s,t,\tilde{x})=B\big(s,\tilde{t}-(\tilde{t}-t),x+s(\tilde{t}-t)\big).$$

However, by (3.2), the function $u\mapsto B(s,\tilde{t}-u,x+su)$ is decreasing on $[0,\tilde{t}]$. Therefore, (3.3) follows. □

We are ready to establish the weak-type estimate.

Proof of the inequality $\mathbb{B}\leq B$. Fix the dimension d and let Q be a given dyadic cube in $\mathbb{R}^d$. Fix $(s,t,x)\in[0,1]\times[0,1]\times[0,\infty)$. Pick a sequence $(\alpha_R)_{R\in\mathcal{D}(Q)}$ with a Carleson constant less or equal to 1 and satisfying $|Q|^{-1}\sum_{R\in\mathcal{D}(Q)}\alpha_R|R|=t$, and let E be a measurable set with $\langle\chi_E\rangle_Q=s$. We will use the following notation: for $R\in\mathcal{D}(Q)$ we will write

$$s_R=\langle\chi_E\rangle_R,\quad t_R=\frac{1}{|R|}\sum_{R'\in\mathcal{D}(R)}\alpha_{R'}|R'|,\quad x_R=x+\sum_{R'\supsetneq R,R'\in\mathcal{D}(Q)}\alpha_{R'}\langle\chi_E\rangle_{R'},$$

with the convention $x_Q = x$. Observe that if R^1, R^2, …, R^{2^d} are direct children of R, then

$$s_R = 2^{-d}\sum_{j=1}^{2^d} s_{R^j}, \qquad t_R = \alpha_R + 2^{-d}\sum_{j=1}^{2^d} t_{R^j} \quad \text{and} \quad x_{R^j} = x_R + \alpha_R s_R.$$

Therefore, if we apply the inequality (3.3) with $s_j = s_{R^j}$ and $t_j = t_{R^j}$, $j = 1, 2, \ldots, 2^d$, then we get

$$B(s_R, t_R, x_R) \geq 2^{-d}\sum_{j=1}^{2^d} B\big(s_{R^j}, t_{R^j}, x_R + \alpha_R s_R\big). \tag{3.4}$$

Multiply throughout by $|R|$ and sum the obtained inequalities over all $R \in \mathcal{D}^N$ (for some fixed $N \geq 0$; here $\mathcal{D}^N$ is the N-th dyadic generation of Q, i.e., the collection of all dyadic cubes contained in Q which have measure $2^{-Nd}|Q|$) to get

$$\sum_{R\in\mathcal{D}^N} B(s_R, t_R, x_R)|R| \geq \sum_{R\in\mathcal{D}^{N+1}} B(s_R, t_R, x_R)|R|.$$

This immediately implies that for any positive integer M,

$$B(s_Q, t_Q, x_Q) = \sum_{R\in\mathcal{D}^0} B(s_R, t_R, x_R)|R| \geq \sum_{R\in\mathcal{D}^M} B(s_R, t_R, x_R)|R|.$$

The left-hand side is equal to $B(s, t, x)$. Since $B(s_R, t_R, x_R) \geq x_R^p$ (see Lemma 3.4), the right-hand side is at least

$$\sum_{R\in\mathcal{D}^M} x_R^p|R| = \int_Q \left(x + \sum_{n=0}^{M-1}\sum_{R\in\mathcal{D}^n} \alpha_R\langle\chi_E\rangle_R\chi_R\right)^q \mathrm{d}u.$$

Therefore, letting $M \to \infty$ we obtain

$$\int_Q \big(x + \mathcal{A}\chi_E\big)^q \mathrm{d}u \leq B(s, t, x),$$

by Lebesgue's monotone convergence theorem. This implies $\mathbb{B}(s, t, x) \leq B(s, t, x)$, by the very definition of $\mathbb{B}$, since the set E and the Carleson sequence $(\alpha_R)_{R\in\mathcal{D}(Q)}$ were arbitrary. □

As an immediate corollary, we obtain our main weak-type inequality.

Proof of (1.5). For any measurable E and any Carleson sequence $(\alpha_R)_{R\in\mathcal{D}(Q)}$ with $\mathrm{Carl}(\alpha) \leq 1$, we have

$$\int_Q (\mathcal{A}\chi_E)^q \mathrm{d}u \leq \mathbb{B}\left(|E|, \sum_{R\in\mathcal{D}(Q)} \alpha_R|R|, 0\right) \leq c_q^q|E|.$$

Here in the last passage we have exploited Lemma 3.4. This is what we need. □

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