A NOTE ON BURKHOLDER-ROSENTHAL INEQUALITY

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Abstract. Let df be a Hilbert-space-valued martingale difference sequence. The paper is devoted to a new, elementary proof of the estimate

$$\left|\left|\sum_{k=0}^{\infty} df_k\right|\right|_p \leq C_p \left\{ \left|\left| \left(\sum_{k=0}^{\infty} \mathbb{E}(|df_k|^2|\mathcal{F}_{k-1})\right)^{1/2}\right|\right|_p + \left|\left|\left(\sum_{k=0}^{\infty}|df_k|^p\right)^{1/p}\right|\right|_p\right\},$$

with $C_p = O(p/\ln p)$ as $p \to \infty$.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, filtered by $(\mathcal{F}_n)_{n\geq 0}$, a nondecreasing family of sub-σ-algebras of $\mathcal{F}$. Assume that f is an adapted martingale, taking values in a certain separable Hilbert space $\mathcal{H}$ with norm $|\cdot|$ and scalar product $\langle\cdot,\cdot\rangle$. Then $df = (df_n)_{n\geq 0}$, the difference sequence of f, is given by $df_0 = f_0$ and $df_n = f_n - f_{n-1}$, $n \geq 1$. We define the conditional square function of f by

$$s(f) = \left[\sum_{k=0}^{\infty} \mathbb{E}(|df_k|^2|\mathcal{F}_{k-1})\right]^{1/2}$$

(here and below, $\mathcal{F}_{-1} = \mathcal{F}_0$) and use the notation

$$s_n(f) = \left[\sum_{k=0}^{n} \mathbb{E}(|df_k|^2|\mathcal{F}_{k-1})\right]^{1/2}, \qquad n = 0, 1, 2, \ldots,$$

for the truncated conditional square function of f.

The purpose of this note is to investigate Burkholder-Rosenthal inequality

$$||f||_p \leq c_p \left(||s(f)||_p + \left|\left|\left(\sum_{k=0}^{\infty}|df_k|^p\right)^{1/p}\right|\right|_p\right) \tag{1.1}$$

where $p \geq 2$ and c_p is a constant depending only on p. The special case in which the martingale f is a sum of independent mean-zero random variables forms an important extension of Khintchine inequality and was studied by Rosenthal in the 60's . The proof from [11] gives the constant c_p which grows exponentially in p as $p \to \infty$. Johnson, Schechtman and Zinn [4] refined the reasoning and showed that the optimal order of c_p as $p \to \infty$ (still in the independent case) is $p/\ln p$. Applying difficult isoperimetric techniques, Talagrand [12] extended this statement to the case of independent Banach-space-valued random variables. Using hypercontractivity

2010 *Mathematics Subject Classification.* Primary: 60G42.
Key words and phrases. martingale, conditional square function.
Partially supported by MNiSW Grant N N201 397437.

methods, Kwapień and Szulga [7] gave a completely elementary proof of Talagrand's result.

The inequality (1.1) for general real martingales (and some c_p) was established by Burkholder in [1]. The validity of this estimate with $c_p = O(p/\ln p)$ was proved by Hitczenko [5] (see also [6]). This result was further generalized to vector-valued setting by Pinelis [10]. Consult also Nagaev [8] for a yet another approach.

The purpose of this paper is to present a new and elementary proof of (1.1) with $c_p = O(p/\ln p)$. Precisely, we will establish the following statement.

Theorem 1.1. *If f is a Hilbert-space-valued martingale, then for $p \geq 4$ we have*

$$||f||_p \leq C_p \left(||s(f)||_p^p + \left|\left| \left(\sum_{k=0}^{\infty} |df_k|^p \right)^{1/p} \right|\right|_p^p \right)^{1/p}, \tag{1.2}$$

where

$$C_p = 2\sqrt{2} \left(\frac{p}{4} + 1 \right)^{1/p} \left(1 + \frac{p}{\ln(p/2)} \right).$$

In fact, using Davis' decomposition, we will be able to prove a slightly stronger estimate: see (2.10) and Remark 2.5 below.

A few words about the proof are in order. Hitczenko [5], [6] and Pinelis [10] apply the extrapolation method (good λ-inequality) of Burkholder and Gundy, combined with appropriate version of Prokhorov "arcsinh" estimate for martingales. Nagaev [8] first establishes a certain exponential bound for the tail of f and deduces Burkholder-Rosenthal estimate using a standard integration argument. Our approach is entirely different and exploits the properties of a certain special function; this type of proof can be regarded as an application of Burkholder's method (see [2] and [9] for more on the subject).

2. Proof of Theorem 1.1

The starting point is the following technical estimate proved by Kwapień and Szulga [7].

Lemma 2.1. *Let $p \geq 4$ and put*

$$\eta = \eta(p) := \frac{\ln(p/2)/p}{1 + \ln(p/2)/p}.$$

Then for any $t \geq 0$ we have

$$(1 + t\eta)^p - pt\eta \leq 1 + \left(\frac{p}{2} - 1 \right) t^2 + t^p. \tag{2.1}$$

We shall require the following vector-valued version of this bound. From now on, we assume that $p \geq 4$ and that $\sigma = \sigma(p) = \eta(p)/\sqrt{2}$.

Lemma 2.2. *For any y, $d \in \mathcal{H}$ we have*

$$|y + \sqrt{2}\sigma d|^p - p|y|^{p-2}\langle y, \sqrt{2}\sigma d\rangle \leq |y|^p + \frac{p}{2}|y|^{p-2}|d|^2 + |d|^p. \tag{2.2}$$

Proof. The left-hand side can be rewritten in the form $F(\langle y, \sqrt{2}\sigma d\rangle)$, where

$$F(s) = \left| |y|^2 + 2\sigma^2|d|^2 + 2s \right|^{p/2} - p|y|^{p-2}s, \qquad s \in \mathbb{R}.$$

Now keep $|y|$ and $|d|$ fixed; since the function F is convex, it suffices to prove the estimate for $\langle y, \sqrt{2}\sigma d\rangle = \pm\sqrt{2}\sigma|y||d|$, i.e. in the case when y and d are linearly dependent. If $\langle y, \sqrt{2}\sigma d\rangle = \sqrt{2}\sigma|y||d|$, then (2.2) follows directly from (2.1); on the other hand, if $\langle y, \sqrt{2}\sigma d\rangle = -\sqrt{2}\sigma|y||d|$, we have

$$\begin{aligned}|y+\sqrt{2}\sigma d|^p - p|y|^{p-2}\langle y, \sqrt{2}\sigma d\rangle &= \big||y| - \sqrt{2}\sigma|d|\big|^p + p\sqrt{2}\sigma|y|^{p-1}|d| \\ &\leq \big||y| + \sqrt{2}\sigma|d|\big|^p - p\sqrt{2}\sigma|y|^{p-1}|d|,\end{aligned}$$

so the claim again follows from (2.1). □

The key ingredient of the proof is the special function $U : [0,\infty)\times\mathcal{H}\times[0,\infty) \to \mathbb{R}$, given by

$$U(x,y,z) = \begin{cases} (|y|^2 - x^2)^{p/2} - cx^p - z & \text{if } |y| \geq \sqrt{2}x, \\ |y|^p - (2^{p/2} - 1 + c)x^p - z & \text{if } |y| < \sqrt{2}x, \end{cases}$$

where

$$c = p\,2^{p/2-2} + 1.$$

Let us list some properties of this function.

Lemma 2.3. *(i) For any $(x,y,z) \in [0,\infty)\times\mathcal{H}\times[0,\infty)$ we have*

$$U(x,y,z) = \min\left\{\big||y|^2 - x^2\big|^{p/2} - cx^p - z, |y|^p - (2^{p/2}-1+c)x^p - z\right\}. \tag{2.3}$$

(ii) For any $x \geq 0$ and $y \in \mathcal{H}$ we have

$$U(x,\sigma y, |y|^p) \leq 0. \tag{2.4}$$

(iii) For all $(x,y,z) \in [0,\infty)\times\mathcal{H}\times[0,\infty)$ we have

$$U(x,y,z) \geq 2^{-p/2}\left[|y|^p - \sigma^p C_p^p(x^p + z)\right]. \tag{2.5}$$

Proof. (i) For fixed x, $z \geq 0$, the function

$$F(s) = s^p - (2^{p/2}-1+c)x^p - z - \left(|s^2 - x^2|^{p/2} - cx^p - z\right), \qquad s \geq 0,$$

vanishes at $s = \sqrt{2}x$ and is strictly increasing:

$$F'(s) = ps\left[s^{p-2} - |s^2-x^2|^{(p-2)/2}\operatorname{sgn}(s^2-x^2)\right].$$

This yields (2.3).

(ii) This is obvious, since $\sigma \leq 1$.

(iii) Using the definitions of C_p and σ, we see that we must prove the bound

$$U(x,y,z) \geq 2^{-p/2}\left[|y|^p - \left(\frac{p}{4}+1\right)2^p(x^p+z)\right].$$

Now, for $|y| < \sqrt{2}x$, we have

$$U(x,y,z) = |y|^p - \left(\frac{p}{4}+1\right)2^{p/2}x^p - z \geq 2^{-p/2}\left[|y|^p - \left(\frac{p}{4}+1\right)2^p(x^p+z)\right].$$

On the other hand, if $|y| \geq \sqrt{2}x$, then $|y|^2 - x^2 \geq |y|^2/2$ and hence

$$U(x,y,z) \geq 2^{-p/2}\left[|y|^p - 2^{p/2}cx^p - 2^{p/2}z\right],$$

so the majorization is clear. □

We turn to the key property of the function U.

Lemma 2.4. *For any x, $z \geq 0$, $y \in \mathcal{H}$ and any $\mathcal{H}$-valued, mean-zero random variable d with $||d||_p < \infty$ we have*

$$\mathbb{E}U\left(\sqrt{x^2 + \mathbb{E}|d|^2}, y+\sigma d, z + |d|^p\right) \leq U(x,y,z). \tag{2.6}$$

Proof. We consider three cases separately.

1° The case $|y|^2 \le 2x^2$. By (2.3), we have

$$\begin{aligned}&\mathbb{E}U\left(\sqrt{x^2+\mathbb{E}|d|^2}, y+\sigma d, z+|d|^p\right)\\ &\le \mathbb{E}|y+\sigma d|^p - (2^{p/2}-1+c)(x^2+\mathbb{E}|d|^2)^{p/2} - z - \mathbb{E}|d|^p\\ &= \mathbb{E}\left\{|y+\sigma d|^p - p|y|^{p-2}\langle y,\sigma d\rangle - |d|^p\right\} - (2^{p/2}-1+c)(x^2+\mathbb{E}|d|^2)^{p/2} - z.\end{aligned}$$

By (2.2), the expression in the parentheses does not exceed $|y|^p + p|y|^{p-2}|d|^2/2$; furthermore, we have

$$\begin{aligned}(2^{p/2}-1+c)(x^2+\mathbb{E}|d|^2)^{p/2} &\ge (2^{p/2}-1+c)\left(x^p + \frac{p}{2}x^{p-2}\mathbb{E}|d|^2\right)\\ &\ge (2^{p/2}-1+c)x^p + \frac{p}{2}2^{(p-2)/2}x^{p-2}\mathbb{E}|d|^2\\ &\ge (2^{p/2}-1+c)x^p + \frac{p}{2}|y|^{p-2}\mathbb{E}|d|^2.\end{aligned}$$

Combining these estimates, we obtain

$$\mathbb{E}U\left(\sqrt{x^2+\mathbb{E}|d|^2}, y+\sigma d, z+|d|^p\right) \le |y|^p - (2^{p/2}-1+c)x^p - z,$$

which is precisely the desired bound.

2° The case $2x^2 < |y|^2 \le 2(x^2+\mathbb{E}|d|^2)$. We start as previously: by (2.3) and then (2.2),

$$\begin{aligned}&\mathbb{E}U\left(\sqrt{x^2+\mathbb{E}|d|^2}, y+\sigma d, z+|d|^p\right)\\ &\le |y|^p + \frac{p}{2}|y|^{p-2}\mathbb{E}|d|^2 - (2^{p/2}-1+c)(x^2+\mathbb{E}|d|^2)^{p/2} - z.\end{aligned}$$

The latter expression decreases as $\mathbb{E}|d|^2$ increases; indeed, the function

$$F(s) = |y|^p + \frac{p}{2}|y|^{p-2}s - (2^{p/2}-1+c)(x^2+s)^{p/2} - z, \qquad s \ge \frac{|y|^2}{2} - x^2,$$

satisfies

$$F'(s) \le \frac{p}{2}\left[|y|^{p-2} - 2^{p/2-1}(x^2+s)^{p/2-1}\right] \le 0.$$

In consequence, we have

$$\begin{aligned}&\mathbb{E}U\left(\sqrt{x^2+\mathbb{E}|d|^2}, y+\sigma d, z+|d|^p\right)\\ &\le F\left(\frac{|y|^2}{2} - x^2\right)\\ &= \frac{p}{2}|y|^{p-2}\left(\frac{|y|^2}{2} - x^2\right) - (c-1)\left(\frac{y^2}{2}\right)^{p/2} - z\\ &= -\frac{p}{2}|y|^{p-2}x^2 - z\\ &= \left(\frac{|y|^2}{2}\right)^{p/2-1}x^2 - \left(\frac{p}{2} + 2^{1-p/2}\right)|y|^{p-2}x^2 - z\\ &\le \left(\frac{|y|^2}{2}\right)^{p/2} - c\left(\frac{|y|^2}{2}\right)^{p/2-1}x^2 - z\\ &\le (|y|^2 - x^2)^{p/2} - cx^p - z,\end{aligned}$$

and we are done.

3° The case $|y|^2 > 2(x^2 + \mathbb{E}|d|^2)$. Here the reasoning is a bit more complicated. First we show the pointwise estimate

$$\begin{aligned} &\big||y+\sigma d|^2 - x^2 - \mathbb{E}|d|^2\big|^{p/2} - p\big||y|^2 - x^2 - \mathbb{E}|d|^2\big|^{p/2-1}\langle y, \sigma d\rangle \\ &\leq \left(\big||y|^2 - x^2 - \mathbb{E}|d|^2\big|^{1/2} + \sqrt{2}\sigma|d|\right)^p - p\big||y|^2 - x^2 - \mathbb{E}|d|^2\big|^{(p-1)/2}\sqrt{2}\sigma|d|. \end{aligned} \tag{2.7}$$

In fact, we will establish a slightly stronger inequality:

$$\begin{aligned} &\big||y+\sigma d|^2 - x^2 - \mathbb{E}|d|^2\big|^{p/2} - p\big||y|^2 - x^2 - \mathbb{E}|d|^2\big|^{p/2-1}\langle y, \sigma d\rangle \\ &\quad\leq \left(|y|^2 - x^2 - \mathbb{E}|d|^2 + \sigma^2|d|^2 + 2\sqrt{2}\big||y|^2 - x^2 - \mathbb{E}|d|^2\big|^{1/2}\sigma|d|\right)^{p/2} \\ &\quad\quad - p\big||y|^2 - x^2 - \mathbb{E}|d|^2\big|^{(p-1)/2}\sqrt{2}\sigma|d|. \end{aligned}$$

To do this, divide throughout by $||y|^2 - x^2 - \mathbb{E}|d|^2|^{p/2}$ and substitute

$$A^2 = \frac{|y|^2 - x^2 - \mathbb{E}|d|^2 + \sigma^2|d|^2}{|y|^2 - x^2 - \mathbb{E}|d|^2}, \quad Y = \frac{y}{||y|^2 - x^2 - \mathbb{E}|d|^2|^{1/2}}$$

and

$$D = \frac{d}{||y|^2 - x^2 - \mathbb{E}|d|^2|^{1/2}}.$$

The estimate becomes

$$\big|A^2 + 2\langle Y, \sigma D\rangle\big|^{p/2} - p\langle Y, \sigma D\rangle \leq \big|A^2 + 2\sqrt{2}\sigma|D|\big|^p - p\sqrt{2}\sigma|D|. \tag{2.8}$$

However, the reasoning presented in the proof of (2.2) gives

$$\big|A^2 + 2\langle Y, \sigma D\rangle\big|^{p/2} - p\langle Y, \sigma D\rangle \leq \big(A^2 + 2\sigma|Y||D|\big)^{p/2} - p\sigma|Y||D|.$$

It suffices to use the bounds $|Y| \leq \sqrt{2}$ and $A^2 \geq 1$ to obtain (2.8), because the function $s \mapsto (A^2 + 2s)^{p/2} - ps$ is increasing on $[0, \infty)$. Thus (2.7) follows. We turn to (2.6): applying (2.3), we get

$$\begin{aligned} &\mathbb{E}U\left(\sqrt{x^2 + \mathbb{E}|d|^2}, y + \sigma d, z + |d|^p\right) \\ &\leq \mathbb{E}\big||y+\sigma d|^2 - x^2 - \mathbb{E}|d|^2\big|^{p/2} - c(x^2 + \mathbb{E}|d|^2)^{p/2} - z - \mathbb{E}|d|^p \\ &= \mathbb{E}\left\{\big||y+\sigma d|^2 - x^2 - \mathbb{E}|d|^2\big|^{p/2} - p\big||y|^2 - x^2 - \mathbb{E}|d|^2\big|^{p/2-1}\langle y, \sigma d\rangle\right\} \\ &\quad - c(x^2 + \mathbb{E}|d|^2)^{p/2} - z - \mathbb{E}|d|^p \\ &\leq \mathbb{E}\left\{\left(\big||y|^2 - x^2 - \mathbb{E}|d|^2\big|^{1/2} + \sqrt{2}\sigma|d|\right)^p - p\big||y|^2 - x^2 - \mathbb{E}|d|^2\big|^{(p-1)/2}\sqrt{2}\sigma|d|\right\} \\ &\quad - cx^p - z - \mathbb{E}|d|^p. \end{aligned}$$

Now we apply (2.2) (in the real case) to obtain

$$\begin{aligned} &\mathbb{E}U\left(\sqrt{x^2 + \mathbb{E}|d|^2}, y + \sigma d, z + |d|^p\right) \\ &\leq \big||y|^2 - x^2 - \mathbb{E}|d|^2\big|^{p/2} + \frac{p}{2}\big||y|^2 - x^2 - \mathbb{E}|d|^2\big|^{p/2-1}\mathbb{E}|d|^2 - cx^p - z \\ &\leq \big||y|^2 - x^2\big|^{p/2} - cx^p - z = U(x, y, z). \end{aligned}$$

This completes the proof. □

Proof of (1.2). It suffices to prove that for any nonnegative integer n,

$$\mathbb{E}|f_n|^p \leq C_p^p \mathbb{E}\left(s_n^p(f) + \sum_{k=0}^n |df_k|^p\right). \tag{2.9}$$

Of course, we may assume that df_0, df_1, ..., df_n (and hence also f_n) belong to L^p, since otherwise there is nothing to prove. The key observation is that the process

$$\left(U\Big(s_n(f), \sigma f_n, \sum_{k=0}^n |df_k|^p\Big)\right)_{n\geq 0}$$

is a supermartingale with respect to $(\mathcal{F}_n)_{n\geq 0}$. Indeed, the integrability follows from the above assumption on df; furthermore, for any $n \geq 0$ we have

$$\begin{aligned}
&\mathbb{E}\left[U\left(s_{n+1}(f), \sigma f_{n+1}, \sum_{k=0}^{n+1} |df_k|^p\right)\Bigg|\mathcal{F}_n\right]\\
&= \mathbb{E}\left[U\left(\sqrt{s_n^2(f) + \mathbb{E}(|df_{n+1}|^2|\mathcal{F}_n)}, \sigma f_n + \sigma df_{n+1}, \sum_{k=0}^n |df_k|^p + |df_{n+1}|^p\right)\Bigg|\mathcal{F}_n\right],
\end{aligned}$$

which does not exceed $U(s_n(f), \sigma f_n, \sum_{k=0}^n |df_k|^p)$, by (2.6) applied conditionally with respect to $\mathcal{F}_{n-1}$. Next, we have $U(s_0(f), \sigma f_0, |df_0|^p) \leq 0$, in view of (2.4). Combining these two facts with (2.5) yields the claim:

$$\mathbb{E}\left[|f_n|^p - C_p^p\left(s_n^p(f) + \sum_{k=0}^n |df_k|^p\right)\right] \leq \frac{2^{p/2}}{\sigma^p}\mathbb{E}U\left(s_n(f), \sigma f_n, \sum_{k=0}^n |df_k|^p\right) \leq 0. \quad \square$$

Remark 2.5. Using Davis' decomposition (see e.g. Davis [3] or Burkholder [1]), one can deduce a slightly stronger form of (1.2). Namely, for all f as in the statement of Theorem 1.1 and $p \geq 4$ we have

$$||f||_p \leq 2C_p\left(||s(f)||_p^p + ||df^*||_p^p\right)^{1/p}, \tag{2.10}$$

where $df^* = \sup_{n\geq 0}|df_n|$. Indeed, fix a martingale f and consider the random variables $d_n' = df_n 1_{\{|df_n|<2df_{n-1}^*\}}$, $d_n'' = df_n 1_{\{|df_n|\geq 2df_{n-1}^*\}}$, $n = 0, 1, 2, \ldots$. Here, as usual, $df_{-1}^* \equiv 0$ and $df_n^* = \max_{0\leq k\leq n}|df_k|$. Note that on the set $\{|df_n| \geq 2df_{n-1}^*\}$ we have

$$(2^p - 1)|d_n''|^p + (2df_{n-1}^*)^p \leq (2|df_n|)^p \leq (2df_n^*)^p,$$

which implies

$$\sum_{k=0}^n |d_k''|^p \leq \frac{2^p}{2^p - 1}(df_n^*)^p, \qquad n = 0, 1, 2, \ldots.$$

Next, observe that for any n,

$$\begin{aligned}
\mathbb{E}\sum_{k=0}^n |d_k'|^p &\leq \mathbb{E}\sum_{k=0}^n |df_k|^2(2df_{k-1}^*)^{p-2}\\
&= \mathbb{E}\sum_{k=0}^n \mathbb{E}(|df_k|^2|\mathcal{F}_{k-1})(2df_{k-1}^*)^{p-2}\\
&\leq \mathbb{E}s_n^2(f)(2df_n^*)^{p-2}\\
&\leq \frac{2}{p}||s_n(f)||_p^p + \frac{p-2}{p}||2df_n^*||_p^p,
\end{aligned}$$

where in the last line we have exploited Young's inequality. Combining the above estimates for the sums of d'_n and d''_n we get

$$\begin{aligned}\mathbb{E}\sum_{k=0}^{n}|df_k|^p &\leq \frac{2}{p}||s_n(f)||_p^p + 2^p\left(\frac{1}{2^p-1}+\frac{p-2}{p}\right)||df_n^*||_p^p \\ &\leq \frac{2}{p}||s_n(f)||_p^p + 2^p||df_n^*||_p^p.\end{aligned}$$

Plugging this into (2.9) and using the fact that n is an arbitrary nonnegative integer, we obtain (2.10).

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