A PROPHET INEQUALITY FOR L^p-BOUNDED DEPENDENT RANDOM VARIABLES

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Abstract. Let $X = (X_n)_{n\geq 1}$ be a sequence of arbitrarily nonnegative dependent random variables satisfying the boundedness condition

$$\sup_\tau \mathbb{E}X_\tau^p \leq t,$$

where $t > 0$, $1 < p < \infty$ are fixed numbers and the supremum is taken over all finite stopping times of X. Let $M = \mathbb{E}\sup_n X_n$ and $V = \sup_\tau \mathbb{E}X_\tau$ denote the expected supremum and the optimal expected return of the sequence X, respectively. We establish the prophet inequality

$$M \leq V + \frac{V}{p-1}\log\left(\frac{te}{V^p}\right)$$

and show that the bound on the right is the best possible.

1. Introduction

The purpose of the paper is to establish a sharp estimate between the expected supremum of a sequence $X = (X_n)_{n\geq 1}$ of L^p-bounded random variables and the optimal expected return (i.e., optimal stopping value) of X. Such comparisons are called "prophet inequalities" in the literature and play a distinguished role in the theory of optimal stopping, as evidenced in the papers of Allaart [1], [2], Boshuizen [3], [4], Hill [6], [7], Hill and Kertz [8], [9], [10], Kennedy [11], [12], Kertz [13], Krengel, Sucheston and Garling [14], [15], [16], Tanaka [21], [22] and many others.

We start with the necessary background and notation. Assume that $X = (X_n)_{n\geq 1}$ is a sequence of (possibly dependent) random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. With no loss of generality, we may assume that this probability space is the interval $[0, 1]$ equipped with its Borel subsets and Lebesgue measure. Let $(\mathcal{F}_n)_{n\geq 1} = (\sigma(X_1, X_2, \ldots, X_n))_{n\geq 1}$ be the natural filtration of X. The problem can be generally stated as follows: under some boundedness condition on X, find universal inequalities which compare $M = \mathbb{E}\sup_n X_n$, the expected supremum of the sequence, with $V = \sup_\tau \mathbb{E}X_\tau$, the optimal stopping value of the sequence; here τ runs over the class $\mathcal{T}$ of all finite stopping times adapted to $(\mathcal{F}_n)_{n\geq 1}$. The term "prophet inequality" arises from the optimal-stopping interpretation of M, which is the optimal expected return of a player endowed with complete foresight; this player observes the sequence X and may stop whenever he wants, incurring a reward equal to the variable at the time of stopping. With complete foresight, such a player obviously stops always when the largest value is

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observed, and on the average, his reward is equal to M. On the other hand, the quantity V corresponds to the optimal return of the non-prophet player.

Let us mention here several classical results in this direction; for an excellent exposition on the subject, we refer the interested reader to the survey [10] by Hill and Kertz. The first universal prophet inequality is due to Krengel, Sucheston and Garling [14], [15]: if $X_1, X_2, \ldots$ are independent and nonnegative, then

$$M \leq 2V$$

and the constant 2 is the best possible. The next result, coming from [6] and [8], states that if $X_1, X_2, \ldots$ are independent and take values in $[0,1]$, then

$$M - V \leq \frac{1}{4}$$

and

$$M \leq V - V^2.$$

Both estimates are sharp: equalities may hold for some non-trivial sequences X. Analogous inequalities for other types of variables $X_1, X_2, \ldots$ (e.g., arbitrarily-dependent and uniformly bounded, i.i.d., averages of independent r.v.'s, exchangeable r.v.'s, etc.), as well as for other stopping options (for instance, stopping with partial recall, stopping several times, using only threshold stopping rules, etc.) have been studied intensively in the literature. We refer the reader to the papers cited at the beginning.

The motivation for the results obtained in this paper comes from the following statement proved by Hill and Kertz [9]: if $X_1, X_2, \ldots$ are arbitrarily dependent and take values in $[0,1]$, then we have the sharp bound

$$M \leq V - V \log V. \tag{1.1}$$

There is a very natural problem concerning the L^p-version of this result, where p is a fixed bumber between 1 and infinity. For example, consider the following interesting question. Suppose that $X_1, X_2, \ldots$ are nonnegative random variables satisfying $\sup_n ||X_n||_p \leq 1$. What is the analogue of (1.1)? Unfortunately, as we will see in Section 5 below, there is no non-trivial prophet inequality in this setting. More precisely, for any $K > 0$ one can construct a sequence X bounded in L^p which satisfies $V = 1$ and $M \geq K$.

We will work under the more restrictive assumption

$$X_1,\, X_2,\, \ldots \text{ are nonnegative and satisfy } \sup_{\tau \in \mathcal{T}} ||X_\tau||_p^p \leq t, \tag{1.2}$$

where t is a given positive number. The main result of the paper is the following.

Theorem 1.1. *Let $t > 0$, $1 < p < \infty$ be fixed and suppose that $X_1, X_2, \ldots$ satisfy* (1.2)*. Then*

$$M \leq V + \frac{V}{p-1} \log\left(\frac{te}{V^p}\right) \tag{1.3}$$

and the inequality is sharp: the bound on the right cannot be replaced by a smaller number.

Note that this statement generalizes the inequality (1.1) of Hill and Kertz: it suffices to take $t = 1$ and let p go to ∞ to recover the bound. On the other hand, the expression on the right of (1.3) explodes as $p \downarrow 1$, which indicates that there is no prophet inequality in the limit case $p = 1$.

A few words about the proof. Our approach is based on the following two-step procedure: first we show that it suffices to establish (1.3) under the additional assumption that X is a nonnegative supermartingale; second, we prove that in the supermartingale setting, the validity of (1.3) is equivalent to the existence of a certain special function which enjoys appropriate majorization and convexity properties. In the literature this equivalence is often referred to as Burkholder's method or Bellman function method, and it has turned out to be extremely efficient in numerous problems in probability and analysis: consult e.g. [5], [17], [18], [23] and references therein.

We have organized the paper as follows. In the next section we reduce the problem to the supermartingale setting. Section 3 contains the description of Burkholder's method (or rather its variant which is needed in the study of (1.3) for supermartingales). In Section 4 we apply the method and provide the proof of Theorem 1.1. In the final part of the paper we show that there are no interesting prophet inequalities in the case when the variables X_1, X_2, ... are only assumed to be bounded in L^p.

2. A reduction

The first step in the analysis of the inequality (1.3) is to relate it to a certain inequality for nonnegative supermartingales. Throughout, we use the notation $X^* = \sup_{n\geq 1} X_n$ and $X^*_m = \sup_{1\leq n\leq m} X_n$ for the maximal and the truncated maximal function of X. Recall that $V = \sup_{\tau\in\mathcal{T}} \mathbb{E}X_\tau$.

Lemma 2.1. *Suppose that $X = (X_n)_{n\geq 1}$ is an arbitrary sequence of dependent random variables satisfying $X_1 \equiv 0$ and (1.2). Then there is a supermartingale $Y = (Y_n)_{n\geq 1}$ adapted to the filtration of X, which satisfies*

$$Y_n \geq X_n \text{ almost surely for all } n, \tag{2.1}$$

$$\mathbb{P}(Y_1 = V) = 1 \tag{2.2}$$

and

$$\sup_{\tau\in\mathcal{T}} ||Y_\tau||_p^p \leq t. \tag{2.3}$$

Note that the additional assumption $X_1 \equiv 0$ is not restrictive at all: we can always replace the initial sequence X_1, X_2, ... with 0, X_1, X_2, ..., and the prophet inequality remains the same. In the proof of the above lemma we will need the notion of essential supremum, the well-known object in the optimal stopping theory. Let us briefly recall its definition, for details and properties we refer the reader to the monographs of Peskir and Shiryaev [19] and Shiryaev [20].

Definition 2.1. Let $(Z_\alpha)_{\alpha\in I}$ be a family of random variables. Then there is a countable subset J of I such that the random variable $Z^* = \sup_{\alpha\in J} Z_\alpha$ satisfies the following two properties:

(i) $\mathbb{P}(Z_\alpha \leq Z^*) = 1$ for each $\alpha \in I$,
(ii) if $\tilde{Z}$ is another random variable satisfying (i) in the place of Z^*, then $\mathbb{P}(Z^* \leq \tilde{Z}) = 1$.

The random variable Z^* is called the *essential supremum* of $(Z_\alpha)_{\alpha\in I}$.

Proof of Lemma 2.1. We will use some basic facts from the optimal stopping theory, see e.g. Chapter I in Peskir and Shiryaev [19]. Let Y be the Snell envelope of X, i.e., the smallest adapted supermartingale majorizing the sequence $(X_n)_{n\geq 1}$. It is a well-known fact that for each n the variable Y_n is given by the formula

$$Y_n = \operatorname{ess\,sup}\Big\{\mathbb{E}(X_\sigma|\mathcal{F}_n)\,:\,\sigma\in\mathcal{T}_n\Big\},$$

where $\mathcal{T}_n$ denotes the class of all finite adapted stopping times not smaller than n. Thus, (2.1) is given for free. To show (2.2), note that Y_1 is a constant random variable, since the σ-algebra $\mathcal{F}_1=\sigma(X_1)$ is trivial. Thus the bound $Y_1\geq V$ follows directly from the above formula for Y_1 and the definition of an essential supremum. On the other hand, for any finite stopping time σ we have $V\geq \mathbb{E}X_\sigma=\mathbb{E}(X_\sigma|\mathcal{F}_1)$ almost surely, which implies $\mathbb{P}(V\geq Y_1)=1$, again from the definition of an essential supremum. This gives (2.2). Finally, to prove (2.3), fix a positive integer N and let

$$Y_n^N = \operatorname{ess\,sup}\Big\{\mathbb{E}(X_\sigma|\mathcal{F}_n)\,:\,\sigma\in\mathcal{T}_n^N\Big\},\quad n\leq N,$$

where $\mathcal{T}_n^N$ is the class of all stopping times taking values in $[n,N]$. Then Y^N can be alternatively defined by the backward induction

$$Y_N^N=X_n,\quad Y_n^N=\max\big\{X_n,\mathbb{E}(Y_{n+1}^N|\mathcal{F}_n)\big\},\quad n=1,\,2,\,\ldots,\,N-1.$$

This implies that $Y_n^N=\mathbb{E}(X_{\tau_n^N}|\mathcal{F}_n)$, where the stopping time τ_n^N is given by

$$\tau_n^N=\inf\big\{k\in\{n,n+1,\ldots,N\}:Y_k^N=X_k\big\}.$$

Thus, if we fix an arbitrary $\tau\in\mathcal{T}$, then

$$Y_{\tau\wedge N}^N=\sum_{n=1}^N Y_n^N 1_{\{\tau\wedge N=n\}}=\sum_{n=1}^N 1_{\{\tau\wedge N=n\}}\mathbb{E}\big(X_{\tau_n^N}|\mathcal{F}_n\big),$$

which gives

$$||Y_{\tau\wedge N}^N||_p^p=\sum_{n=1}^N \big|\big|\mathbb{E}\big(1_{\{\tau\wedge N=n\}}X_{\tau_n^N}|\mathcal{F}_n\big)\big|\big|_p^p\leq\sum_{n=1}^N \big|\big|1_{\{\tau\wedge N=n\}}X_{\tau_n^N}\big|\big|_p^p=||X_\sigma||_p^p,$$

where $\sigma=\sum_{n=1}^N 1_{\{\sigma\wedge N=n\}}\tau_n^N$. We easily check that σ is a stopping time: for any $1\leq k\leq N$,

$$\{\sigma=k\}=\bigcup_{n=1}^N\big(\{\sigma\wedge N=n\}\cap\{\tau_n^N=k\}\big)=\bigcup_{n=1}^k\big(\{\sigma\wedge N=n\}\cap\{\tau_n^N=k\}\big)$$

belongs to $\mathcal{F}_k$. Therefore, the boundedness assumption (1.2) implies $||Y_{\tau\wedge N}^N||_p^p\leq t$. However, $Y_n^N\uparrow Y_n$ almost surely as $N\to\infty$: this is one of the fundamental facts from the optimal stopping theory (and describes the passage from finite to infinite horizon). Consequently, Fatou's lemma imples $||Y_\tau||_p^p\leq t$ and we are done. □

Therefore, it suffices to establish the inequality (1.3) under the additional assumption that the process X is a supermartingale and the variable X_1 is constant almost surely. By some standard approximation arguments, we may further restrict ourselves to the class of *simple* supermartingales; recall that the sequence $X=(X_n)_{n\geq 1}$ is called simple, if for each n the random variable takes only a finite number of values and there is a deterministic N such that $X_N=X_{N+1}=X_{N+2}=\ldots=X_\infty$ almost surely. We are ready to apply Burkholder's method, which is introduced in the next section.

3. Burkholder's method

Now we will describe the main tool which will be used to establish the inequality (1.3). Distinguish the set

$$D = \{(x, y, t) \in [0, \infty) \times [0, \infty) \times [0, \infty) : x^p \leq t\}$$

and, for each $(x, y, t) \in D$, let $\mathcal{S}(x, y, t)$ denote the class of all simple supermartingales $X = (X_n)_{n\geq 1}$ satisfying $X_1 \equiv x$ and $\sup_{\tau\in\mathcal{T}} ||X_\tau||_p^p \leq t$ (the class does not depend on y, but we keep the notation $\mathcal{S}(x, y, t)$ for the sake of convenience). Suppose that we are interested in the explicit formula for the function

$$\mathbb{B}(x, y, t) = \sup \Big\{ \mathbb{E}\big(X_n^* \vee y\big) \Big\}$$

where the supremum is taken over all positive integers n and all $X \in \mathcal{S}(x, y, t)$. The key idea in the study of this problem is to introduce the class $\mathcal{C}$ which consists of all functions $B : D \to \mathbb{R}$ satisfying

$$B(x, y, t) = B(x, x \vee y, t) \qquad \text{for any } (x, y, t) \in D, \tag{3.1}$$

$$B(x, y, t) \geq x \vee y \qquad \text{for any } (x, y, t) \in D, \tag{3.2}$$

and the following concavity-type property: if $\alpha \in [0, 1]$, $(x, y, t) \in D$ and $(x_\pm, y, t_\pm) \in D$ satisfy $x \leq y$, $\alpha x_- + (1 - \alpha)x_+ \leq x$ and $\alpha t_- + (1 - \alpha)t_+ \leq t$, then

$$B(x, y, t) \geq \alpha B(x_-, y, t_-) + (1 - \alpha)B(x_+, y, t_+). \tag{3.3}$$

We turn to the main result of this section. Recall that the probability space is the interval $[0, 1]$ with its Borel subsets and Lebesgue measure.

Theorem 3.1. *The function $\mathbb{B}$ is the least element of the class $\mathcal{C}$.*

Proof. It is convenient to split the reasoning into two parts.

Step 1. First we will show that $\mathbb{B}$ belongs to the class $\mathcal{C}$. The condition (3.1) is evident, since for any $n \geq 1$ and any $X \in \mathcal{S}(x, y, t)$ we have $X_n^* \vee y = X_n^* \vee X_1 \vee y = X_n^* \vee x \vee y$. The majorization (3.2) is also immediate: the constant process $X = (x, x, \ldots)$ belongs to the class $\mathcal{S}(x, y, t)$. The main difficulty lies in proving the concavity property (3.3). Fix the parameters α, x, t, $x_\pm$, $t_\pm$ as in the statement and pick arbitrary supermartingales $X^- \in \mathcal{S}(x_-, y, t_-)$, $X^+ \in \mathcal{S}(x_+, y, t_+)$. We splice these two processes into one sequence $X = (X_n)_{n\geq 1}$ by setting $X_1 \equiv 1$ and, for $n \geq 2$,

$$X_n(\omega) = \begin{cases} X^-_{n-1}(\omega/\alpha) & \text{if } \omega \in [0, \alpha], \\ X^+_{n-1}((\omega - \alpha)/(1 - \alpha)) & \text{if } \omega \in (\alpha, 1]. \end{cases}$$

Then X is a supermartingale (with respect to its natural filtration), because the proceses $X_\pm$ have this property and $\alpha x_- + (1 - \alpha)x_+ \leq x$. Furthermore, for any stopping time $\tau \in \mathcal{T}$ we have $||X_\tau||_p^p \leq t$. Indeed, we easily verify that the variables $\tau^\pm$, given by

$$\tau^-(\omega) = \tau(\alpha\omega), \qquad \tau^+ = \tau((1 - \alpha)\omega),$$

are stopping times of X^- and X^+. Therefore,

$$||X_\tau||_p^p = \alpha||X^-_{\tau^-}||_p^p + (1 - \alpha)||X^+_{\tau^+}||_p^p \leq \alpha t_- + (1 - \alpha)t_+ \leq t$$

and hence $X \in \mathcal{S}(x, y, t)$. Since $x \leq y$, we have $X_n^* \vee y = \sup_{2\leq k\leq n} X_k \vee y$ and thus

$$\mathbb{B}(x, y, t) \geq \mathbb{E}(X_n^* \vee y) = \alpha\mathbb{E}\big((X^-_{n-1})^* \vee y\big) + (1 - \alpha)\mathbb{E}\big((X^+_{n-1})^* \vee y\big).$$

Take the supremum over all n and $X^\pm$ as above to obtain the desired bound (3.3).

Step 2. Now suppose that B is an arbitrary element of $\mathcal{C}$; we must prove that $\mathbb{B} \le B$. To do this, rephrase the condition (3.3) as follows. Suppose that (X,T) is an arbitrary random variable with two-point distribution, such that $\mathbb{P}(X^p \le T) = 1$. Then for any $(x,y,t) \in D$ such that $x \le y$ and $\mathbb{E}X \le x$, $\mathbb{E}T \le t$ we have

$$B(x,y,t) \ge \mathbb{E}B(X,y,T). \tag{3.4}$$

Note that the set $\{(x,t) : x^p \le t\}$ is convex. Therefore, by straightforward induction, the above inequality extends to the case when (X,T) is an arbitrary simple random variable satisfying $X^p \le T$ with probability 1. Now, pick $X \in \mathcal{S}(x,y,t)$ and consider the sequence (X,Y,T), where $Y_n = X_n^* \vee y$ and $T_n = \sup_{\tau \in \mathcal{T}_n} \mathbb{E}\big(X_\tau^p | \mathcal{F}_n\big)$ (recall that $\mathcal{T}_n$ denotes the class of all adapted stopping times not smaller than n). Then the process $B(X,Y,T)$ is a supermartingale: to see this, fix $n \ge 1$ and apply (3.4) conditionally with respect to $\mathcal{F}_n$, with $\tilde{x} := X_n$, $\tilde{y} = Y_n$, $\tilde{t} = T_n$, $\tilde{X} = X_{n+1}$ and $\tilde{T} = T_{n+1}$. Let us verify the assumptions: the inequalities $\tilde{x}^p \le \tilde{t}$, $\tilde{x} \le \tilde{y}$ and $\tilde{X}^p \le \tilde{T}$ are evident; the inequalities $\mathbb{E}(\tilde{X}|\mathcal{F}_n) \le \tilde{x}$ and $\mathbb{E}(\tilde{T}|\mathcal{F}_n) \le \tilde{t}$ follow from the supermartingale property of X and T (T is a supermartingale, since it is the Snell envelope of the sequence $(X_n^p)_{n \ge 1}$). Thus, (3.4) yields

$$B(X_n,Y_n,T_n) \ge \mathbb{E}\Big[B\big(X_{n+1},Y_n,T_{n+1}\big)|\mathcal{F}_n\Big] = \mathbb{E}\Big[B(X_{n+1},Y_{n+1},T_{n+1})|\mathcal{F}_n\Big],$$

where in the last passage we have exploited (3.1). Combining this with (3.2) yields

$$\mathbb{E}(X_n^* \vee y) \le \mathbb{E}B(X_n,Y_n,T_n) \le \mathbb{E}B(X_1,Y_1,T_1) = B(x,y,\sup_{\tau \in \mathcal{T}} \mathbb{E}X_\tau^p) \le B(x,y,t).$$

Here in the last inequality we have used the fact that the function $t \mapsto B(x,y,t)$ is nondecreasing; this follows immediately from (3.3), applied to $x_+ = x_- = x$ and $t_+ = t_- < t$. Taking the supremum over all n and all $X \in \mathcal{S}(x,y,t)$, we obtain the bound $\mathbb{B}(x,y,t) \le B(x,y,t)$. This finishes the proof. □

4. Proof of Theorem 1.1

We will prove that the function $\mathbb{B}$ admits the following explicit formula:

$$\mathbb{B}(x,y,t) = \begin{cases} (x \vee y) + \dfrac{x}{p-1} \log\left(\dfrac{te}{x(x\vee y)^{p-1}}\right) & \text{if } y \le (t/x)^{1/(p-1)}, \\[2ex] (x \vee y) + t(x \vee y)^{1-p}/(p-1) & \text{if } y > (t/x)^{1/(p-1)}. \end{cases}$$

This will clearly yield the assertion of Theorem 1.1, which is nothing else but the explicit formula for $\mathbb{B}(V,V,t)$. Denote expression on the right above by $B(x,y,t)$.

4.1. **Proof of $\mathbb{B} \le B$.** In the light of Theorem 3.1, it suffices to verify that $B \in \mathcal{C}$. The conditions (3.1) and (3.2) are obvious, and the main problem is to establish (3.3). First, we easily check that the functions $x \mapsto B(x,y,t)$ and $t \mapsto B(x,y,t)$ are nondecreasing. Consequently, in the proof of (3.3) we may restrict ourselves to the case $x = \alpha x_- + (1-\alpha)x_+$ and $t = \alpha t_- + (1-\alpha)t_+$. Since the region $\{(x,t) : x^p \le t\}$ is convex, it is enough to prove the following. For any $h \in \mathbb{R}$ and any $(x,y,t) \in D$ satisfying $y \ge x$, the function

$$\varphi(s) = B(x+sh, y, t+s)$$

(defined for those s, for which $(x+sh,y,t+s) \in D$) is concave. We start from observing that φ is of class C^1; this follows immediately from the fact that B is of class C^1 and $B_y(x,x,t) = 0$ (the latter condition guarantees that the one-sided derivatives $\varphi'(s-)$ and $\varphi'(s+)$ will match for $x+sh=y$). To deal with the concavity of φ on the set $\{s : x+sh \leq y\}$, we must prove that the matrix

$$\begin{bmatrix} B_{xx}(x+sh,y,t+s) & B_{xt}(x+sh,y,t+s) \\ B_{xt}(x+sh,y,t+s) & B_{tt}(x+sh,y,t+s) \end{bmatrix}$$

(defined for $y \neq (t+s)/(x+sh)$) is nonpositive-definite. Substituting $\tilde{x} = x+sh$ and $\tilde{t} = t+s$ if necessary, we may assume that $s=0$. Now, if $y < (t/x)^{1/(p-1)}$, then the matrix equals

$$\begin{bmatrix} -\big((p-1)x\big)^{-1} & \big(p-1)t\big)^{-1} \\ \big(p-1)t\big)^{-1} & -x/((p-1)t^2) \end{bmatrix},$$

which clearly has the required property; if $y > (t/x)^{1/(p-1)}$, the situation is even simpler, since all the entries of the matrix are 0. Finally, it remains to show the concavity of φ on $\{s : t+s \geq x+sh > y\}$. On this set, we have $y < \big((t+s)/(x+sh)\big)^{1/(p-1)}$ and hence

$$\varphi(s) = x+sh + \frac{x+sh}{p-1}\log\left(\frac{(t+s)e}{(x+sh)^p}\right).$$

A direct differentiation yields

$$\varphi''(s) = -\frac{h^2}{x+sh} - \frac{(x-th)^2}{(p-1)(t+s)^2(x+sh)} \leq 0$$

and the claim follows.

4.2. **Proof of $\mathbb{B} \geq B$.** Now we will use the second half of Theorem 3.1, which states that $\mathbb{B} \in \mathcal{C}$. We will also exploit the following additional homogeneity property of $\mathbb{B}$: for any $(x,y,t) \in D$ and $\lambda > 0$ we have

$$\mathbb{B}(\lambda x, \lambda y, \lambda^p t) = \lambda \mathbb{B}(x,y,t). \tag{4.1}$$

This condition follows at once from the very definition of $\mathbb{B}$ and the fact that $X \in \mathcal{S}(x,y,t)$ if and only if $\lambda X \in \mathcal{S}(\lambda x, \lambda y, \lambda^p t)$.

For the sake of clarity, we have split the reasoning into a few parts.

Step 1. Let δ be a small positive number. Using (3.3), we can write

$$\mathbb{B}(1,1,1) \geq \left(1 - \frac{1}{(1+\delta)^p}\right)\mathbb{B}(0,1,0) + \frac{1}{(1+\delta)^p}\mathbb{B}(1+\delta,1,(1+\delta)^p).$$

By (3.1), we have $\mathbb{B}(1+\delta,1,(1+\delta)^p) = \mathbb{B}(1+\delta,1+\delta,(1+\delta)^p)$. Thus, using (3.2) and (4.1), the right-hand side is not smaller than

$$1 - \frac{1}{(1+\delta)^p} + \frac{1}{(1+\delta)^p}\mathbb{B}(1+\delta,1+\delta,(1+\delta)^p) = 1 - \frac{1}{(1+\delta)^p} + \frac{\mathbb{B}(1,1,1)}{(1+\delta)^{p-1}}.$$

Combining the above facts, we get

$$\mathbb{B}(1,1,1) \geq \frac{(1+\delta)^p - 1}{(1+\delta)\big((1+\delta)^{p-1}-1\big)},$$

so letting $\delta \to 0$ gives

$$\mathbb{B}(1,1,1) \geq \frac{p}{p-1}. \tag{4.2}$$

Step 2. Now we provide a lower bound for $\mathbb{B}(1,1,t)$, where t is larger than 1. We argue as in the previous step, applying (3.3) and combining it with (3.1), (3.2) and (4.1). Precisely, we fix a small positive δ and write

$$\begin{aligned}\mathbb{B}(1,1,t) &\geq \frac{\delta}{1+\delta}\mathbb{B}(0,1,0) + \frac{1}{1+\delta}\mathbb{B}\big(1+\delta,1,t(1+\delta)\big)\\ &\geq \frac{\delta}{1+\delta} + \mathbb{B}\left(1,1,\frac{t}{(1+\delta)^{p-1}}\right).\end{aligned}$$

By induction, this implies

$$\mathbb{B}(1,1,t) \geq \frac{n\delta}{1+\delta} + \mathbb{B}\left(1,1,\frac{t}{(1+\delta)^{n(p-1)}}\right),$$

if only $(1+\delta)^{n(p-1)} \leq t$. Now we fix a large positive integer n, put $\delta = t^{1/(n(p-1))}-1$ (so that $(1+\delta)^{n(p-1)} = t$) and let $n \to \infty$. Then the above bound gives

$$\mathbb{B}(1,1,t) \geq \mathbb{B}(1,1,1) + \frac{\log t}{p-1},$$

which combined with (4.2) yields

$$\mathbb{B}(1,1,t) \geq \frac{p}{p-1} + \frac{\log t}{p-1}. \tag{4.3}$$

Step 3. The next move in our analysis is to prove the estimate $\mathbb{B}(x,y,t) \geq B(x,y,t)$ for $y \leq (t/x)^{1/(p-1)}$. Of course, since both $\mathbb{B}$ and B satisfy (3.1), we may assume that $x \leq y$. We proceed as previously: first apply (3.3) to obtain

$$\mathbb{B}(x,y,t) \geq \frac{y-x}{y}\mathbb{B}(0,y,0) + \frac{x}{y}\mathbb{B}\left(y,y,\frac{ty}{x}\right)$$

(here we use the assumption $y \leq (t/x)^{1/(p-1)}$; if this inequality does not hold, the point $(y,y,ty/x)$ does not belong to D and $\mathbb{B}(y,y,ty/x)$ does not make sense). Next, using (3.2), (4.1) and finally (4.3), we get

$$\begin{aligned}\mathbb{B}(x,y,t) &\geq \frac{y-x}{y}\cdot y + \frac{x}{y}\cdot y\mathbb{B}\left(1,1,\frac{t}{xy^{p-1}}\right)\\ &\geq y-x+x\left(\frac{p}{p-1} + \frac{\log(t/(xy^{p-1}))}{p-1}\right) = B(x,y,t).\end{aligned}$$

Step 4. Now we will deal with $\mathbb{B}(1,y,1)$ for $y>1$. By (3.3), (3.2) and (4.1), we have, for small positive δ,

$$\begin{aligned}\mathbb{B}(1,y,1) &\geq (1-(1+\delta)^{-p})\mathbb{B}(0,y,0) + (1+\delta)^{-p}\mathbb{B}(1+\delta,y,(1+\delta)^p)\\ &\geq (1-(1+\delta)^{-p})y + (1+\delta)^{1-p}\mathbb{B}(1,y/(1+\delta),1).\end{aligned}$$

By induction, this implies

$$\begin{aligned}\mathbb{B}(1,y,1) &\geq (1+\delta)^{n(1-p)}\mathbb{B}(1,y/(1+\delta)^n,1) + y\big(1+(1+\delta)^{-p}\big)\sum_{k=0}^{n-1}(1+\delta)^{-kp}\\ &= (1+\delta)^{n(1-p)}\mathbb{B}(1,y/(1+\delta)^n,1) + y\left(1-(1+\delta)^{-np}\right),\end{aligned}$$

if only $y/(1+\delta)^n \geq 1$. Now put $\delta = y^{1/n}-1$ (so that $(1+\delta)^n$ and y are equal) and let $n\to\infty$. As the result, we obtain

$$\mathbb{B}(1,y,1) \geq y^{1-p}\mathbb{B}(1,1,1) + y(1-y^p),$$

or, in the light of (4.2),

$$\mathbb{B}(1,y,1)\geq y+\frac{y^{1-p}}{p-1}. \tag{4.4}$$

Step 5. This is the final part. Pick $(x,y,t)\in D$ such that $y>(t/x)^{1/(p-1)}$ and apply (3.3) and then (3.2) to get

$$\begin{aligned}\mathbb{B}(x,y,t)&\geq \alpha\mathbb{B}(0,y,0)+(1-\alpha)\mathbb{B}\left(\frac{x}{1-\alpha},y,\frac{t}{1-\alpha}\right)\\&\geq \alpha y+\mathbb{B}(x,y(1-\alpha),t(1-\alpha)^{p-1}),\end{aligned}$$

where $\alpha=1-(x^p/t)^{1/(p-1)}$. For this choice of α, we have $x^p=t(1-\alpha)^{p-1}$ and hence, by (4.1) and (4.4),

$$\begin{aligned}\mathbb{B}(x,y,t)&\geq \alpha y+x\mathbb{B}\left(1,\frac{y(1-\alpha)}{x},1\right)\\&=\left(1-\left(\frac{x^p}{t}\right)^{1/(p-1)}\right)y+x\left[\frac{1}{p-1}\left(\frac{x}{y}\right)^{p-1}\frac{t}{x^p}+\frac{y}{x}\left(\frac{x^p}{t}\right)^{1/(p-1)}\right]\\&=B(x,y,t).\end{aligned}$$

This completes the proof of the inequality $\mathbb{B}\geq B$ on the whole domain. Thus, Theorem 1.1 follows.

5. Lack of prophet inequalities for L^p bounded variables

In the final part of the paper we show that if the condition (1.2) is replaced by

$$X_1,\ X_2,\ \dots\ \text{are nonnegative and}\ \sup_n ||X_n||_p^p\leq 1, \tag{5.1}$$

then no non-trivial prophet inequality holds. To prove this, we will exploit the results of Section 4. Fix an arbitrary positive number K. We have $\mathbb{B}(1,1,t)\to\infty$ as $t\to\infty$, and hence there is a positive integer L and a finite nonnegative supermartingale $(Y_n)_{n=1}^N$ on $([0,1],\mathbb{B}([0,1]),|\cdot|)$ which satisfies

$$\sup_{\tau\in\mathcal{T}}||Y_\tau||_p^p\leq L \qquad\text{and}\qquad \mathbb{E}Y^*\geq \mathbb{E}Y_1+K. \tag{5.2}$$

Now we construct L "distinct" copies of Y which evolve on pairwise disjoint time intervals. Precisely, consider a sequence $X=(X_n)_{n=1}^{LN}$ defined as follows:

- If $\omega\in[0,1/L)$, then $X_n(\omega)=Y_n(L\omega)$ for $n=1,\ 2,\ \dots,\ N$ and $X_n(\omega)=0$ for other n.
- If $\omega\in[1/L,2/L)$, then $X_n(\omega)=Y_{n-N}(L\omega-1)$ for $n=N+1,\ N+2,\ \dots,\ 2N$ and $X_n(\omega)=0$ for other n.
- If $\omega\in[2/L,3/L)$, then $X_n(\omega)=Y_{n-2N}(L\omega-2)$ for $n=2N+1,\ 2N+2,\ \dots,\ 3N$ and $X_n(\omega)=0$ for other n.
- ...
- If $\omega\in[1-1/L,1)$, then $X_n(\omega)=Y_{n-(L-1)N}(L\omega-(L-1))$ for $n=(L-1)N+1,\ (L-1)N+2,\ \dots,\ LN$ and $X_n(\omega)=0$ for other n.

The variables $X_1,\ X_2,\ \dots,\ X_{LN}$ are nonnegative and enjoy the following properties. First, note that the conditional distribution of X_{mN+n} on $[m/L,(m+1)/L)$ conicides with the distribution of Y_n, so $||X_{mN+n}||_p^p=||Y_n||_p^p/L$ and thus

$$\sup_{1\leq n\leq LN}||X_n||_p^p=\sup_{1\leq n\leq N}||Y_n||_p^p/L\leq 1,$$

by the first inequality in (5.2). Next, if $\omega \in [m/L,(m+1)/L)$, we have $X^*(\omega) = Y^*(L\omega - m)$ and hence $\mathbb{E}X^* = \mathbb{E}Y^*$. Finally, we will prove that $\sup_{\tau\in\mathcal{T}} \mathbb{E}X_\tau = \mathbb{E}Y_1$. The inequality "$\geq$" follows by considering the stopping time given by $\tilde{\tau}(\omega) = mL+1$ for $\omega \in [m/L,(m+1)/L)$. To prove the reverse bound, note that while computing $\sup_{\tau\in\mathcal{T}} \mathbb{E}X_\tau$ we may restrict to stopping times which on each $[m/L,(m+1)/L)$ take values $\{mL+1, mL+2, \ldots, (m+1)L\}$. Indeed, if τ is an arbitrary stopping time, then $\bar{\tau} = (\tau \vee \tilde{\tau}) \wedge (\tilde{\tau} + L - 1)$ has this property and $\mathbb{E}X_\tau \leq \mathbb{E}X_{\bar{\tau}}$. Therefore, since Y is a supermartingale, we have

$$\mathbb{E}(X_\tau | [m/L,(m+1)/L)) \leq \mathbb{E}(X_{mL+1} | [m/L,(m+1)/L)) = \mathbb{E}Y_1$$

and hence $\mathbb{E}X_\tau \leq \mathbb{E}Y_1$.

Consequently, by the second estimate in (5.2), the sequence X satisfies

$$\mathbb{E}X^* \geq \sup_{\tau\in\mathcal{T}} \mathbb{E}X_\tau + K.$$

Since K was arbitrary, no universal prophet inequality holds under (5.1). This completes the proof.

References

[1] P. C. Allaart, *Pseudo-prophet inequalities in average-optimal stopping*, Sequential Anal. **22** (2003), pp. 233–239.

[2] P. C. Allaart, *Prophet inequalities for I.I.D. random variables with random arrival times*, Sequential Anal. **26** (2007), pp. 403–413.

[3] F. Boshuizen,*Multivariate prophet inequalities for negatively dependent random vectors*, Strategies for sequential search and selection in real time (Amherst, MA, 1990), pp. 183–190, Contemp. Math. **125**, Amer. Math. Soc., Providence, RI, 1992.

[4] F. Boshuizen, *Prophet compared to gambler: additive inequalities for transforms of sequences of random variables*, Statist. Probab. Lett. **29** (1996), pp. 23–32.

[5] D. L. Burkholder, *Explorations in martingale theory and its applications*, Ecole d'Ete de Probabilités de Saint-Flour XIX—1989, pp. 1–66, Lecture Notes in Math., **1464**, Springer, Berlin, 1991.

[6] T. P. Hill, *Prophet inequalities and order selection in optimal stopping problems*, Proc. Amer. Math. Soc. **88** (1983), pp. 131–137.

[7] T. P. Hill, *Prophet inequalities for averages of independent non-negative random variables*, Math. Z. **192** (1986), pp. 427–436.

[8] T. P. Hill and R. P. Kertz, *Additive comparisons of stop rule and supremum expectations of uniformly bounded independent random variables*, Proc. Amer. Math, Soc. **83** (1981), pp. 582–585.

[9] T. P. Hill and R. P. Kertz, *Stop rule inequalities for uniformly bounded sequences of random variables*, Trans. Amer. Math. Soc. **278** (1983), pp. 197–207

[10] T. P. Hill and R. P. Kertz, *A survey of prophet inequalities in optimal stopping theory*, Strategies for sequential search and selection in real time (Amherst, MA, 1990), pp. 191–207, Contemp. Math. **125**, Amer. Math. Soc., Providence, RI, 1992.

[11] R. P. Kennedy, *Optimal stopping of independent random variables and maximizing prophets*, Ann. Probab. **13** (1985), pp. 566–571.

[12] R. P. Kennedy, *Prophet-type inequalities for multi-choice optimal stopping*, Stoch. Proc. Appl. **24** (1987), pp. 77–88.

[13] R. P. Kertz, *Comparison of optimal value and constrained maxima expectations for independent random variables*, Adv. Appl. Prob. **18** (1986), pp. 311–340.

[14] U. Krengel and L. Sucheston, *Semiamarts and finite values*, Bull. Amer. Math. Soc. **83** (1977), pp. 745–747.

[15] U. Krengel and L. Sucheston, *On semiamarts, amarts, and processes with finite value*, Probability on Banach Spaces, Ed. by J. Kuelbs, Marcel Dekker, New York, 1978.

[16] U. Krengel and L. Sucheston, *Prophet compared to the gambler: an inequality for transforms of processes*, Ann. Probab. **15** (1987), pp. 1593–1599.

[17] F. Nazarov and S. Treil, *The hunt for Bellman function: applications to estimates of singular integral operators and to other classical problems in harmonic analysis*, Algebra i Analis 8 (1997), pp. 32–162.

[18] A. Osękowski, *Sharp martingale and semimartingale inequalities*, Monografie Matematyczne **72**, Birkhäuser, 2012.

[19] G. Peskir and A. N. Shiryaev, *Optimal stopping and free-boundary problems*, Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2006.

[20] A. N. Shiryaev, *Optimal stopping rules*, Translated from the 1976 Russian second edition by A. B. Aries. Reprint of the 1978 translation. Stochastic Modelling and Applied Probability, 8. Springer-Verlag, Berlin, 2008.

[21] T. Tanaka, *Prophet inequalities for two-parameter optimal stopping problems*, J. Inf. Optim. Sci. **28** (2007), pp. 159–172.

[22] T. Tanaka, *Lower semicontinuity property of multiparameter optimal stopping value and its application to multiparameter prophet inequalities*, J. Math. Anal. Appl. **359** (2009), pp. 240–251.

[23] V. Vasyunin and A. Volberg, *Monge-Ampére equation and Bellman optimization of Carleson embedding theorems*, Linear and complex analysis, pp. 195–238, Amer. Math. Soc. Transl. Ser. 2, 226, Amer. Math. Soc., Providence, RI, 2009.

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