SHARP $L^1(\ell^q)$ ESTIMATE FOR A SEQUENCE AND ITS PREDICTABLE PROJECTION

ADAM OSEKOWSKI

Abstract. Let $f = (f_n)_{n\geq 0}$ be a sequence of integrable Banach-space valued random variables and $g = (g_n)_{n\geq 0}$ denote its predictable projection. We prove that, for $1 \leq q < \infty$,

$$E \left( \sum_{n=0}^{\infty} |g_n|^q \right)^{1/q} \leq 2^{(q-1)/q} E \left( \sum_{n=0}^{\infty} |f_n|^q \right)^{1/q},$$

and that the constant $2^{(q-1)/q}$ is the best possible. The proof rests on the construction of a certain special function enjoying appropriate majorization and concavity.

1. Introduction

Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a fixed probability space, equipped with a filtration $(\mathcal{F}_n)_{n \geq 0}$, a nondecreasing sequence of sub-$\sigma$-algebras of $\mathcal{F}$. Let $f = (f_n)_{n \geq 0}$ be an adapted sequence of integrable real-valued random variables and let $g = (g_n)_{n \geq 0}$ stand for the predictable projection of the sequence $f$, that is, $g_0 = f_0$ and $g_n = E(f_n | \mathcal{F}_{n-1})$, for $n = 1, 2, \ldots$. For $1 \leq p, q \leq \infty$, we introduce the mixed norms

$$||f||_{p,q} = ||f||_{L^p(\ell^q)} = \left[ E \left( \sum_{n=1}^{\infty} |f_n|^q \right)^{p/q} \right]^{1/p},$$

with the usual convention if $p$ or $q$ is infinite. The problem of comparing the $(p,q)$-norms of $f$ and $g$ was first studied by Stein [10]. He showed that for $1 < p < \infty$ and $1 \leq q \leq \infty$ there is an absolute constant $C_{p,q} < \infty$ (not depending on $f$, $g$, the probability space or filtration) such that

$$||g||_{p,q} \leq C_{p,q} ||f||_{p,q}. \quad (1.1)$$

Actually, Stein focused on the case $q = 2$ only, but his proof works for other values of $q$ as well. It is also worth to mention here that the result is true for the sequences $f$ which are not necessarily adapted.

The inequality (1.1) has been studied by many authors. Johnson, Maurey, Schechtman and Tzafriri [6] established an extension with $L^p$ replaced by a rearrangement invariant space $X$ with Boyd indices satisfying $0 < \beta_X \leq \alpha_X < 1$. Then Bourgain [1] showed the estimate for $p = 1$ and $q = 2$ with $C_{1,2} = 3$; later Lépingle and Yor (see [7]) managed to decrease this constant to 2. See also the recent note by Qiu [9] for the noncommutative counterpart of this result. Delbaen and Schachermayer needed in [3] the version of (1.1) with $p = 2$ and $q = \infty$, and

2000 Mathematics Subject Classification. Primary: 60G42. Secondary: 46E30.

Key words and phrases. Predictable projection, best constants.
in [4] they proved the estimate for $1 \leq p \leq q \leq \infty$ with $C_{p,q} = 2$ (in fact the proof yields $C_{p,q} = 2^{1/p}$). The constant $2^{1/p}$, for $p \in \{1, \infty\}$ and $q = \infty$, turns out to be the best possible. The author identified in [8] the value of $C_{p,\infty}$ for $1 \leq p < \infty$: it is equal to $1 + (p - 1)^{p-1}/p^p$. The case $q = 1$ is strictly related to estimates for martingale conditional square function. It was studied by Burkholder [2], Garsia in [5] and Wang [11].

The contribution of this paper is to provide the optimal values of the constants $C_{1,q}$ for $1 \leq q < \infty$. Here is our main result.

**Theorem 1.1.** For any $1 \leq q < \infty$ we have

$$\|g\|_{1,q} \leq 2^{(q-1)/q} \|f\|_{1,q}. \quad (1.2)$$

The inequality is sharp.

Of course, this result remains valid if we allow the sequence $f$ to take values in a separable Banach space $B$ (with an appropriate modification of $\|f\|_{p,q}$: we need to replace $|f_n|$ with $\|f_n\|_B$, the norm of $f_n$ in $B$). This follows at once from the fact that the passage from $(f_n)_{n \geq 0}$ to $(\|f_n\|_B)_{n \geq 0}$ does not change the $(p,q)$-norm of the sequence and does not decrease the $(p,q)$-norm of the projection.

The proof of the inequality (1.2) will rest on the existence of a certain special function. Our approach is novel and can be regarded as a version of the so-called Bellman function method (or Burkholder’s method), a powerful technique which has been successfully applied in numerous problems in probability and analysis. We do not know whether this approach can be used to identify the optimal constants $C_{p,q}$ in the full range of exponents.

2. **Proof of Theorem 1.1**

2.1. **A special function and its properties.** Let $1 \leq q < \infty$ be fixed. Introduce a function $u : [0, \infty) \times [0, \infty) \to \mathbb{R}$, given by the formula

$$u(x,y) = \begin{cases} 
-(x^q - y^q)^{1/q} & \text{if } x \geq 2^{1/q}y, \\
 y - 2^{(q-1)/q}x & \text{if } x < 2^{1/q}y.
\end{cases}$$

It is easy to see that first-order partial derivatives of $u$ exist and are continuous on $(0, \infty) \times (0, \infty)$. The next three lemmas are devoted to certain pointwise estimates for $u$.

**Lemma 2.1.** For any $x$, $y \geq 0$ we have the majorization

$$u(x,y) \geq y - 2^{(q-1)/q}x. \quad (2.1)$$

**Proof.** Clearly, it is enough to show the claim for $x \geq 2^{1/q}y$. By homogeneity, we may assume that $y = 1$; then the bound can be rewritten in the equivalent form

$$F(x) := 1 - 2^{(q-1)/q}x + (x^q - 1)^{1/q} \leq 0$$

for $x \geq 2^{1/q}$. However, one easily checks that $F(2^{1/q}) = F'(2^{1/q}+) = 0$ (where $F'(2^{1/q}+)$ denotes the right-hand derivative of $F$ at the point $2^{1/q}$). In addition, a direct differentiation shows that $F$ is concave on $[2^{1/q}, \infty)$. This proves the claim. \hfill \Box

**Lemma 2.2.** Let $x$, $y \geq 0$. Then the function $\psi(s) = u((x^q + s)^{1/q}, (y^q + s)^{1/q})$ is nonincreasing on $[0, \infty)$. 


Proof. We will consider two cases.

The case $x \leq 2^{1/q} y$. Then for each $s$ we have

\[ (x^q + s)^{1/q} \leq 2^{1/q} (y^q + s)^{1/q} \]  

(this is evident: it suffices to rise both sides to the power $q$ and do some trivial manipulations) and hence $\psi(s) = (y^q + s)^{1/q} - 2^{1/q}(x^q + s)^{1/q}$. So,

\[ \psi'(s) = q^{-1} \left[(y^q + s)^{1/q-1} - 2^{1/q}(x^q + s)^{1/q-1}\right] \leq 0, \]

where the latter bound is again due to (2.2).

The case $x > 2^{1/q} y$. For such $x$ and $y$, we have $(x^q + s)^{1/q} > 2^{1/q}(y^q + s)^{1/q}$ when $s < x^q - 2y^q$ and $(x^q + s)^{1/q} \leq 2^{1/q}(y^q + s)^{1/q}$ for $s \geq x^q - 2y^q$. Consequently,

\[ \psi(s) = \begin{cases} -(x^q - y^q)^{1/q} & \text{if } s < x^q - 2y^q, \\ (y^q + s)^{1/q} - 2^{1/q}(x^q + s)^{1/q} & \text{if } s \geq x^q - 2y^q. \end{cases} \]

So, $\psi$ is constant on $[0, x^q - 2y^q]$ and nonincreasing on $[x^q - 2y^q, \infty)$, where the latter fact is proved word-by-word as in the preceding case. \hfill \Box

The main property of $u$ is described by the following concavity-type condition.

Lemma 2.3. For any $x$, $y \geq 0$ and any simple random variable $d \geq 0$, we have

\[ \mathbb{E} u \left( (x^q + d^q)^{1/q}, (y^q + (\mathbb{E} d)^q)^{1/q} \right) \leq u(x, y). \]

Proof. Consider the $C^1$ function

\[ \xi(s) = u \left( (x^q + s^q)^{1/q}, (y^q + (\mathbb{E} d)^q)^{1/q} \right), \quad s \geq 0, \]

and let $\zeta$ stand for the smallest concave majorant of $\xi$. Clearly, the left-hand side of (2.3) can be bounded from above by $\zeta(\mathbb{E} d)$, and in what follows, we will find the explicit formulas for $\zeta$, depending on the relations between $x$, $y$, and $\mathbb{E} d$. We will need the following elementary fact: if $A \geq 0$, then the function $s \mapsto (s^q + A)^{1/q}$ is convex on $[0, \infty)$; if $A < 0$, then the function $s \mapsto (s^q + A)^{1/q}$ is concave on $[(-A)^{1/q}, \infty)$.

For the sake of convenience and clarity, we consider two cases separately.

The case $x^q \geq y^q + (\mathbb{E} d)^q$. For such $x$, $y$ and $d$, we have

\[ \xi(s) = \begin{cases} -(x^q - y^q - (\mathbb{E} d)^q + s^q)^{1/q} & \text{if } s^q > 2y^q + 2(\mathbb{E} d)^q - x^q, \\ (y^q + (\mathbb{E} d)^q)^{1/q} - 2^{1/q}(x^q + s^q)^{1/q} & \text{if } s^q \leq 2y^q + 2(\mathbb{E} d)^q - x^q. \end{cases} \]

Using the above fact concerning the convexity/concavity of the function $s \mapsto (s^q + A)^{1/q}$, we see that the function $\xi$ is concave and hence $\zeta = \xi$. Consequently, the left-hand side of (2.3) does not exceed $u \left( (x^q + (\mathbb{E} d)^q)^{1/q}, (y^q + (\mathbb{E} d)^q)^{1/q} \right)$, which, by Lemma 2.2, is not bigger than $u(x, y)$.

The case $x^q < y^q + (\mathbb{E} d)^q$. For these $x$, $y$ and $d$, the function $\xi$ has the same explicit formula as above, but its behavior is slightly different. Namely, it is concave on $[0, (2y^q + 2(\mathbb{E} d)^q - x^q)^{1/q}]$ and convex on $[(2y^q + 2(\mathbb{E} d)^q - x^q)^{1/q}, \infty)$. To find the least concave majorant $\zeta$, observe that $\lim_{s \to \infty} \xi'(s) = -1$; furthermore, a little
calculation shows that \( x \leq (2y^q + 2(\mathbb{E}d)^q - x^q)^{1/q} \) and \( \zeta'(x) = -1 \). Consequently, \( \zeta \) is given by

\[
\zeta(s) = \begin{cases} 
(y^q + (\mathbb{E}d)^q)^{1/q} - 2(q-1)/q(x^q + s^q)^{1/q} & \text{if } s \leq x, \\
(y^q + (\mathbb{E}d)^q)^{1/q} - x - s & \text{if } s > x.
\end{cases}
\]

Therefore, if \( \mathbb{E}d \leq x \), then the left-hand side of (2.3) does not exceed

\[
(y^q + (\mathbb{E}d)^q)^{1/q} - 2(q-1)/q(x^q + (\mathbb{E}d)^q)^{1/q} = u \left( (x^q + \mathbb{E}d)^q, (y^q + \mathbb{E}d)^q \right),
\]

which is not larger than \( u(x, y) \) by means of Lemma 2.2. On the other hand, if \( \mathbb{E}d > x \), then

\[
\mathbb{E}u \left( (x^q + d)^q, (y^q + \mathbb{E}d)^q \right) \leq (y^q + \mathbb{E}d)^q - x - \mathbb{E}d.
\]

However, the right-hand side increases if we decrease \( \mathbb{E}d \) to \( x \); consequently,

\[
(y^q + (\mathbb{E}d)^q)^{1/q} - x - \mathbb{E}d \leq (y^q + x^q)^{1/q} - 2x = u((x^q + x^q)^{1/q}, (y^q + x^q)^{1/q}),
\]

which is not bigger than \( u(x, y) \) by Lemma 2.2. This completes the proof. \( \square \)

**Remark 2.1.** The lemma above implies that for any \( s \geq 0 \) we have

\[(2.4) \quad u(s, s) \leq u(0, 0) = 0.\]

To see this, it suffices to apply (2.3) with \( x = y = 0 \) and \( d \equiv s \).

### 2.2. Proof of Theorem 1.1

We split the reasoning into two parts.

**Proof of (1.2).** For any sequence \( a = (a_n)_{n \geq 0} \) and any \( q \geq 1 \), define the operator

\[
S_n(a) = S_n^{(q)}(a) = \left( \sum_{k=0}^{n} |a_k|^q \right)^{1/q}.
\]

Fix \( f, g \) as in the statement; with no loss of generality, we may and do assume that \( f \) is nonnegative, passing from \( (f_n)_{n \geq 0} \) to \( (|f_n|)_{n \geq 0} \), if necessary. The crucial property is that for such \( f \) and \( g \), the sequence \( \{u(S_n(f), S_n(g))\}_{n \geq 0} \) is a supermartingale. To see this, pick an arbitrary \( n \geq 0 \) and write

\[
u(S_{n+1}(f), S_{n+1}(g)) = u \left( (S_n^q(f) + f_{n+1}^q)^{1/q}, (S_n^q(g) + g_{n+1}^q)^{1/q} \right).
\]

It suffices to apply the inequality (2.3), conditionally with respect to \( \mathcal{F}_n \), with \( x = S_n(f), \ y = S_n(g) \) and \( d = f_{n+1} \), to establish the aforementioned supermartingale property. Hence, by the majorization (2.1) and then (2.4), we may write

\[
\mathbb{E}S_n(g) - 2(q-1)/q\mathbb{E}S_n(f) \leq \mathbb{E}u(S_n(f), S_n(g)) \leq \mathbb{E}u(S_0(f), S_0(g)) = \mathbb{E}u(f_0, f_0) \leq 0.
\]

Thus, the claim follows by letting \( n \to \infty \) and applying Lebesgue’s monotone convergence theorem. \( \square \)

**Sharpness.** Assume that the underlying probability space is the interval \([0, 1]\) equipped with its Borel subsets and Lebesgue measure. We will construct a family of appropriate examples. Let \( r \) be an arbitrary number belonging to \((0, 1)\) and let \( a < (1-r)^{-1} \) be a fixed number. If we take \( r \) sufficiently close to 1 and \( a \) sufficiently
Therefore, we get
\[
f_n(\omega) = \begin{cases} 
a^n & \text{if } \omega \in [0, (1-r)^n), \\
a^{n-1} & \text{if } \omega \in [(1-r)^n, (1-r)^{n-1}), \\
0 & \text{otherwise.}
\end{cases}
\]

Let \((\mathcal{F}_n)_{n \geq 0}\) be the filtration generated by \((f_n)_{n \geq 0}\). Then \(g_0 = f_0 \equiv 1\) and, directly from the above formula, we have that if \(\omega \in [0, (1-r)^n)\), then
\[
g_n(\omega) = \frac{a^n(1-r)^n + a^{n-1}(1-r)^{n-1} - (1-r)^n}{(1-r)^{n-1}} = a^{n-1}(r + (1-r)a).
\]
Consequently,
\[
\|g\|_{L^1(\mathbb{F})} \geq \mathbb{E}\left( \sum_{n=1}^{\infty} g_n^1(1-r)^n,(1-r)^{n-1})\right)^{1/q}
\]
\[
= \mathbb{E}\sum_{n=1}^{\infty} g_n(1-r)^n,(1-r)^{n-1})
\]
\[
= r(r + (1-r)a) \sum_{n=1}^{\infty} a^{n-1}(1-r)^{n-1} = \frac{r(r + (1-r)a)}{1 - a(1-r)}.
\]
On the other hand, for \(\omega \in [(1-r)^n, (1-r)^{n-1})\) we have
\[
\|f\|_{L^1(\mathbb{F})}(\omega) = \left(1 + a^q + a^{2q} + \ldots + a^{(n-1)q} + a^{(n-1)q}\right)^{1/q} \leq \left(2a^{(n-2)q} + 2a^{(n-1)q}\right)^{1/q}.
\]
The latter bound is evident for \(n = 1\), while for \(n \geq 2\) we note that
\[
1 + a^q + a^{2q} + \ldots + a^{(n-2)q} = a^{(n-1)q} - \frac{1}{a^q - 1} \leq \frac{a^{(n-1)q} + a^{(n-2)q}(a^q - 2)}{a^q - 1} = 2a^{(n-2)q}.
\]
Therefore, we get
\[
\|f\|_{L^1(\mathbb{F})} \leq r \sum_{n=1}^{\infty} (1-r)^{n-1}(2a^{(n-2)q} + 2a^{(n-1)q})^{1/q} = \frac{r(2a^{q-2} + 2)^{1/q}}{1 - a(1-r)},
\]
which implies that
\[
\frac{\|g\|_{L^1(\mathbb{F})}}{\|f\|_{L^1(\mathbb{F})}} \geq \frac{r + (1-r)a}{(2a^{q-2} + 2)^{1/q}}.
\]
However, if we take \(r\) sufficiently close to 1 and then \(a\) sufficiently close to \((1-r)^{-1}\) (which in particular means that \(a\) is huge), then the ratio on the right can be made arbitrarily close to \(2^{1-1/q}\). This shows that the constant in (1.2) is indeed the best possible.

\[\square\]

Acknowledgments

The research was supported by the NCN grant DEC-2012/05/B/ST1/00412.
References


Department of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland

E-mail address: ados@mimuw.edu.pl