ON Φ-INEQUALITIES FOR BOUNDED SUBMARTINGALES AND SUBHARMONIC FUNCTIONS

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Abstract. Let $f = (f_n)$ be a submartingale such that $||f||_\infty \leq 1$ and $g = (g_n)$ be a martingale, adapted to the same filtration, satisfying

$$|dg_n| \leq |df_n|, \quad n = 0, 1, 2, \ldots.$$

The paper contains the proof of the sharp inequality

$$\sup_n \mathbb{E}\Phi(|g_n|) \leq \Phi(1)$$

for a class of convex increasing functions Φ on $[0, \infty)$, satisfying certain growth condition. As an application, we show a continuous-time version for stochastic integrals and a related estimate for smooth functions on Euclidean domain.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space filtered by $(\mathcal{F}_n)_{n\geq 0}$, a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$. Let $f = (f_n)_{n\geq 0}$, $g = (g_n)_{n\geq 0}$ denote adapted real-valued integrable processes, such that f is a submartingale and g is differentially subordinate to f: we have, almost surely,

$$|dg_n| \leq |df_n|, \qquad n = 0, 1, 2, \ldots. \tag{1.1}$$

Here $df = (df_n)_{n\geq 0}$ and $dg = (dg_n)_{n\geq 0}$ are the difference sequences of f and g, defined by

$$df_0 = f_0,\ df_n = f_n - f_{n-1},\ \ dg_0 = g_0,\ dg_n = g_n - g_{n-1}, \qquad n = 1, 2, \ldots.$$

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Throughout the paper, Φ is an increasing convex function on $[0,\infty)$. The purpose of the paper is to study sharp estimates for $\sup_n \mathbb{E}\Phi(|g_n|)$ under some additional assumptions on boundedness of f.

Let us start with discussing some related inequalities from the literature. In the three results below, Φ is assumed to satisfy the following conditions: the integral $\int_0^\infty \Phi(t)e^{-t}dt$ is finite and Φ is twice differentiable on $(0,\infty)$ with a strictly convex first derivative satisfying $\Phi'(0+) = 0$ (for example, one can take $\Phi(t) = t^p$, $p > 2$, or $\Phi(t) = e^{at} - 1 - at$ for $a \in (0,1)$). Burkholder [2] proved that if f and g are martingales satisfying (1.1) and $||f||_\infty \leq 1$, then we have a sharp bound

$$\sup_n \mathbb{E}\Phi(|g_n|) < \frac{1}{2}\int_0^\infty \Phi(t)e^{-t}dt.$$

Then, in [4], Burkholder extended this inequality to the submartingale case. Assume f is a nonnegative submartingale bounded from above by 1 and g is differentially subordinate to f and conditionally differentially subordinate to f, that is, almost surely,

$$|\mathbb{E}(dg_{n+1}|\mathcal{F}_n)| \leq \mathbb{E}(df_{n+1}|\mathcal{F}_n), \quad n = 0,\, 1,\, 2,\, \ldots.$$

Then we have a sharp estimate

$$\sup_n \mathbb{E}\Phi\big(\frac{|g_n|}{2}\big) < \frac{2}{3}\int_0^\infty \Phi(t)e^{-t}dt.$$

A further generalization was given by Kim and Kim in [5]. Let α be a number belonging to the interval $[0,1]$. Suppose f is a nonnegative submartingale bounded from above by 1 and g is differentially subordinate to f and α-conditionally differentially subordinate to f, which means

$$|\mathbb{E}(dg_{n+1}|\mathcal{F}_n)| \leq \alpha\mathbb{E}(df_{n+1}|\mathcal{F}_n), \quad n = 0,\, 1,\, 2,\, \ldots$$

with probability 1. Then

$$\mathbb{E}\Phi\big(\frac{|g_n|}{1+\alpha}\big) < \frac{1+\alpha}{2+\alpha}\int_0^\infty \Phi(t)e^{-t}dt. \tag{1.2}$$

In the paper we prove a related result for other class of functions Φ. From now on, we assume that Φ is three times differentiable on $(1,\infty)$ and twice differentiable on $(0,\infty)$, with

$$\Phi''(t)t \leq \Phi'(t) \quad \text{for} \quad t \in (0,1] \quad \text{and} \quad \Phi'''(t) \leq 0 \ \ \text{for} \ \ t > 1. \tag{1.3}$$

Note that for any Φ as above, Φ' is concave on $(1,\infty)$ and hence this new class of functions is disjoint from the one considered previously. For example, one can take $\Phi(t) = t^p$, $p \in [1,2]$.

The main result of the paper can be stated as follows.

Theorem 1.1. *Assume f is a nonnegative submartingale satisfying $||f||_\infty \leq 1$ and g is a martingale which is differentially subordinate to f. Then for any Φ satisfying (1.3) we have*

$$\sup_n \mathbb{E}\Phi(|g_n|) \leq \Phi(1). \tag{1.4}$$

The inequality is sharp. It is already sharp if f is assumed to be a nonnegative martingale bounded by 1*.*

Suppose $v = (v_n)$ is a real predictable sequence, that is, v_0 is measurable with respect to $\mathcal{F}_0$ and for any $n \geq 1$, v_n is measurable with respect to $\mathcal{F}_{n-1}$. The process g is a transform of f by v, if for any nonnegative integer n we have $dg_n = v_n df_n$. The theorem above implies the following estimate for martingale transforms.

Corollary 1.2. *Let f be a nonnegative martingale such that $||f||_\infty \leq 1$ and let g be a transform of f by a predictable sequence v with $||v||_\infty \leq 1$. Then for any Φ satisfying (1.3) we have*

$$\sup_n \mathbb{E}\Phi(|g_n|) \leq \Phi(1) \tag{1.5}$$

and the inequality is sharp.

There is a continuous-time version of the corollary above. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, filtered by a nondecreasing family $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-algebras of $\mathcal{F}$, with $\mathcal{F}_0$ containing all the events of probability 0. Assume $X = (X_t)_{t\geq 0}$ is an adapted right-continuous martingale with left limits, satisfying $\mathbb{P}(0 \leq X_t \leq 1) = 1$ for all t and let $H = (H_t)_{t\geq 0}$ be a predictable process taking values in the interval $[-1, 1]$. Let $Y = (Y_t)_{t\geq 0}$ be an Itô stochastic integral of H with respect to X:

$$Y_t = H_0 X_0 + \int_{(0,t]} H_s dX_s.$$

Theorem 1.3. *For X, Y as above and Φ satisfying (1.3), we have*

$$\sup_t \mathbb{E}\Phi(|Y_t|) \leq \Phi(1) \tag{1.6}$$

and the bound on the right is the best possible.

Theorems 1.1 and 1.3 are accompanied by a version for smooth functions. Let n be a fixed positive integer and D be an open connected subset of $\mathbb{R}^n$. Fix $\xi \in D$ and consider a twice continuously differentiable subharmonic function $v : D \to [0, 1]$ and a harmonic function $w : D \to \mathbb{R}$ such that

$$|w(\xi)| \leq v(\xi) \tag{1.7}$$

and

$$|\nabla w| \leq |\nabla v|. \tag{1.8}$$

These two inequalities are the analogue of differential subordination in this setting. Let D_0 be a subset of D such that $\overline{D_0} \subset D$ and $\xi \in D_0$. Let $\mu^{\xi}_{D_0}$ be a harmonic measure on ∂D_0 with respect to ξ.

Theorem 1.4. *For v, w as above and Φ satisfying (1.3) we have*

$$\sup \int_{\partial D_0} \mathbb{E}\Phi(|w(x)|) d\mu^{\xi}_{D_0}(x) \leq \Phi(1), \tag{1.9}$$

where the supremum is taken over all the sets D_0 as above. The bound on the right is the best possible.

For the proof of the inequality (1.4), we use the method invented by Burkholder. It reduces the problem of showing a (sub-)martingale inequality to a problem of constructing a certain special function (see [2] and [3] for details). Such a function is presented in the next section and we study its properties there. The proof of Theorem 1.1 is postponed to the last section, which also contains the proofs of the remaining results of this paper.

2. The special function

Let S denote the strip $[0,1]\times\mathbb{R}$ and let us consider the following subsets of S,

$$D_1=\{(x,y)\in S: |y|\geq x\},\quad D_2=\{(x,y)\in S:|y|<x\}.$$

Define the function $u:S\to\mathbb{R}$ by the formulas

$$u(x,y)=(1-x)e^{x-|y|}\Big[\int_0^{-x+|y|}\Phi(s+1)e^s ds+\Phi(1)\Big]+x\Phi(-x+|y|+1)$$

if $(x,y)\in D_1$ and

$$u(x,y)=\Phi(1)(1-x)+(1-x)\int_0^{x-|y|}\frac{\Phi(1-s)}{(1-s)^2}ds+|y|\frac{\Phi(-x+|y|+1)}{-x+|y|+1}$$

if $(x,y)\in D_2$.

Let us study the properties of u. This is done in the two lemmas below.

Lemma 2.1. *(i) The function u and its first and second order partial derivatives of u are continuous on $(0,1)\times\mathbb{R}$. Furthermore, u_x and u_y can be extended to continuous and locally bounded functions on the whole strip S (these extensions are still denoted by u_x, u_y, respectively).*

(ii) We have

$$u_x\leq 0. \tag{2.1}$$

(iii) We have

$$\Phi(|y|) \leq u(x,y). \tag{2.2}$$

Furthermore, if $|y| \leq x$*, then*

$$u(x,y) \leq \Phi(1). \tag{2.3}$$

Proof. Let us start with computing the partial derivatives. We have

$$u_x(x,y) = -xe^{x-|y|}\Big[\int_0^{-x+|y|}\Phi(s+1)e^s ds + \Phi(1)\Big] \\ - x\Phi'(-x+|y|+1) + x\Phi(-x+|y|+1)$$

if (x,y) belongs to D_1^o, the interior of D_1, and

$$u_x(x,y) = -\Phi(1) - \int_0^{x-|y|}\frac{\Phi(1-s)}{(1-s)^2}ds + \frac{\Phi(-x+|y|+1)}{-x+|y|+1} - |y|\frac{\Phi'(-x+|y|+1)}{-x+|y|+1}$$

if $(x,y) \in D_2^o$. Furthermore,

$$u_y(x,y) = \operatorname{sgn} y \cdot \Big\{ -(1-x)e^{x-|y|}\Big[\int_0^{-x+|y|}\Phi(s+1)e^s ds + \Phi(1)\Big] \\ + (1-x)\Phi(-x+|y|+1) + x\Phi'(-x+|y|+1)\Big\}$$

if $(x,y) \in D_1^o$ and

$$u_y(x,y) = y\frac{\Phi'(-x+|y|+1)}{-x+|y|+1}$$

if $(x,y) \in D_2^o$.

For the second order partial derivatives, we have

$$u_{xx}(x,y) = -(1+x)e^{x-|y|}\Big[\int_0^{-x+|y|}\Phi(s+1)e^s ds + \Phi(1)\Big] \\ + (1+x)\Phi(-x+|y|+1) - (1+x)\Phi'(-x+|y|+1) + x\Phi''(-x+|y|+1)$$

if $(x,y) \in D_1^o$ and

$$u_{xx}(x,y) = (x-2|y|-1)\frac{\Phi'(-x+|y|+1)}{(-x+|y|+1)^2} + |y|\frac{\Phi''(-x+|y|+1)}{-x+|y|+1}$$

if $(x,y)\in D_2^o$. Moreover,

$$u_{xy}(x,y)=x\,\mathrm{sgn}\,y\cdot\Big\{e^{x-|y|}\Big[\int_0^{-x+|y|}\Phi(s+1)e^s ds+\Phi(1)\Big]-\Phi(-x+|y|+1)$$
$$-\Phi''(-x+|y|+1)+\Phi'(-x+|y|+1)\Big\}$$

if $(x,y)\in D_1^o$ and

$$u_{xy}(x,y)=-y\frac{\Phi''(-x+|y|+1)(-x+|y|+1)-\Phi'(-x+|y|+1)}{(-x+|y|+1)^2}$$

if $(x,y)\in D_2^o$. Finally,

$$u_{yy}(x,y)=(1-x)e^{x-|y|}\Big[\int_0^{-x+|y|}\Phi(s+1)e^s ds+\Phi(1)\Big]$$
$$-(1-x)\Phi(-x+|y|+1)+(1-x)\Phi'(-x+|y|+1)+x\Phi''(-x+|y|+1)$$

if $(x,y)\in D_1^o$ and

$$u_{yy}(x,y)=(1-x)\frac{\Phi'(-x+|y|+1)}{(-x+|y|+1)^2}+|y|\frac{\Phi''(-x+|y|+1)}{-x+|y|+1}$$

if $(x,y)\in D_2^o$.

Now we turn to the properties (i)–(iii).

(i) Straightforward.

(ii) If (x,y) belongs to D_1, then the inequality $u_x\leq 0$, after substitution $t=-x+|y|+1\geq 1$, is equivalent to

$$\int_1^t\Phi(s)e^s ds+\Phi(1)e-[\Phi(t)-\Phi'(t)]e^t\geq 0.$$

In fact we will prove a stronger statement, namely, for $t\geq 1$,

$$\int_1^t\Phi(s)e^s ds+\Phi(1)e-[\Phi(t)-\Phi'(t)+\Phi''(t)]e^t\geq 0. \tag{2.4}$$

For $t=1$ the inequality takes form $\Phi'(1)\geq\Phi''(1)$, which follows by (1.3). It suffices to note that the left hand side, as a function of t, has a derivative $-e^t\Phi'''(t)$, which is nonnegative by (1.3). Thus (2.1) is established for $(x,y)\in D_1$. For $(x,y)\in D_2$, we again use the substitution $t=-x+|y|+1\in(|y|,1)$, which transforms (2.1) into

$$-\Phi(1)-\int_t^1\frac{\Phi(s)}{s^2}ds+\frac{\Phi(s)}{s}-\frac{|y|}{s}\Phi'(s)\leq 0.$$

For $t=1$ the inequality above reads $-|y|\Phi'(1)\leq 0$, a true statement. Furthermore, the left hand side, as a function of t, has a derivative

$$\frac{t+|y|}{t^2}\Big[\Phi'(t)-t\frac{|y|}{t+|y|}\Phi''(t)\Big]\geq\frac{t+|y|}{t^2}\Big[\Phi'(t)-t\Phi''(t)\Big]\geq 0,$$

the latter inequality being a consequence of (1.3). This completes the proof of (2.1).

(iii) By (2.1), we have $u(x,y)\geq u(1,y)=\Phi(|y|)$, which is (2.2). To prove (2.3), note that for $|y|\leq x$, by (2.1), $u(x,y)\leq u(|y|,y)=\Phi(1)$. □

Lemma 2.2. *Let x, y, h, k be real numbers satisfying x, $x+h\in[0,1]$ and $|k|\leq|h|$. Then*

$$u(x+h,y+k)\leq u(x,y)+u_x(x,y)h+u_y(x,y)k. \tag{2.5}$$

Proof. By continuity, we may assume $x\notin\{0,1\}$. Consider a function $G=G_{x,y,h,k}$ defined on the set $\{t:x+th\in[0,1]\}$ by formula $G(t)=u(x+th,y+tk)$. The inequality (2.5) is equivalent to $G(1)\leq G(0)+G'(0)$ and will follow once we have proved that G is concave. Since $G''_{x,y,h,k}(t)=G''_{x+th,y+tk,h,k}(0)$, it suffices to prove $G''(0)\leq 0$.

Suppose $(x,y)\in D_1$. We have $G''(0)=A+B$, where

$$\begin{aligned}A=(k^2-h^2)\Big\{e^{x-|y|}\Big[\int_0^{-x+|y|}\Phi(s+1)e^s ds+\Phi(1)\Big]\\ -\Phi(-x+|y|+1)+\Phi'(-x+|y|+1)\Big\}\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}B=-x(h-k\,\mathrm{sgn}\,y)^2\Big\{e^{x-|y|}\Big[\int_0^{-x+|y|}\Phi(s+1)e^s ds+\Phi(1)\Big]\\ -\Phi(-x+|y|+1)+\Phi'(-x+|y|+1)-\Phi''(-x+|y|+1)\Big\}.\end{aligned} \tag{2.7}$$

We will show that A and B are nonpositive. To this end, it suffices to show that

$$e^{x-|y|}\Big[\int_0^{-x+|y|}\Phi(s+1)e^s ds+\Phi(1)\Big]$$
$$-\Phi(-x+|y|+1)+\Phi'(-x+|y|+1)-\Phi''(-x+|y|+1)\geq 0,$$

which is (2.4) after substitution $t=-x+|y|+1\geq 1$.

Suppose that $(x,y)\in D_2$. We have $G''(0)=A+B$, where

$$A=(k^2-h^2)\frac{\Phi'(-x+|y|+1)}{-x+|y|+1}\leq 0 \tag{2.8}$$

and

$$B=\frac{y(h-k\,\mathrm{sgn}\,y)^2}{(-x+|y|+1)^2}[\Phi''(-x+|y|+1)(-x+|y|+1)-\Phi'(-x+|y|+1)]\leq 0. \tag{2.9}$$

The latter inequality follows from (1.3) and $-x+|y|+1\in(0,1)$.

The proof is complete. □

3. The proofs of the theorems

In this section we provide the proofs of the results announced in the introduction.

Proof of Theorem 1.1: Clearly, it suffices to show that for any nonnegative integer n we have

$$\mathbb{E}\Phi(|g_n|)\leq\Phi(1). \tag{3.1}$$

Let k be any nonnegative integer. The inequality (2.5) and the differential subordination imply that, almost surely,

$$u(f_{k+1},g_{k+1})\leq u(f_k,g_k)+u_x(f_k,g_k)df_{k+1}+u_y(f_k,g_k)dg_{k+1}.$$

Furthermore, again by differential subordination, we have $|dg_l|\leq 1$, $l=0,\,1,\,\ldots,\,k$ almost surely, and therefore $\mathbb{P}(|g_k|\leq k)=1$. Now applying Lemma 2.1 (i) yields the integrability of both sides of the above inequality. Taking the conditional expectation with respect to $\mathcal{F}_k$, we obtain

$$\mathbb{E}[u(f_{k+1},g_{k+1})|\mathcal{F}_k] \leq u(f_k,g_k) + u_x(f_k,g_k)\mathbb{E}(df_{k+1}|\mathcal{F}_k) \leq u(f_k,g_k),$$

due to (2.1) and the submartingale condition $\mathbb{E}(df_{k+1}|\mathcal{F}_k) \geq 0$. Now take expectation to get $\mathbb{E}u(f_{k+1},g_{k+1}) \leq \mathbb{E}u(f_k,g_k)$. Combining this with (2.2), (2.3) and $|g_0| \leq f_0$ (which is due to differential subordination), we arrive at

$$\mathbb{E}\Phi(|g_n|) \leq \mathbb{E}u(f_n,g_n) \leq Eu(f_0,g_0) \leq \Phi(1).$$

This completes the proof of (3.1) and, in consequence, also the proof of (1.4). To prove the sharpness of this estimate, take constant processes

$$f_n = g_n \equiv 1, \qquad n = 0, 1, 2, \ldots. \quad \Box \tag{3.2}$$

Proof of Corollary 1.2: If f is a martingale, then so is its transform g. Furthermore, the condition $||v||_\infty \leq 1$ implies g is differentially subordinate to f. Thus (1.5) follows from (1.4). To prove that the estimate is sharp, note that in the example (3.2), g is a transform of f by a deterministic sequence $v = (1,1,\ldots)$. $\Box$

Proof of Theorem 1.3: The inequality (1.6) follows from (1.5) by approximation argument. See Section 16 of [2], where it is shown how similar inequalities for stochastic integrals are implied by their discrete-time analogues and the result of Bichteler [1]. To prove the sharpness, take $X_t = H_t \equiv 1$; then $Y_t \equiv 1$ and we have equality in (1.6). $\Box$

Proof of Theorem 1.4: Let $W : D \to \mathbb{R}$ be given by $W(x) = u(v(x), w(x))$. We will show that W is superharmonic, which will yield (1.9): by (1.7), (2.1), (2.2) and (2.3), we will have, for any D_0 as in the statement,

$$\int_{\partial D_0} \Phi(w(x))d\mu^\xi_{D_0}(x) \leq \int_{\partial D_0} W(x)d\mu^\xi_{D_0}(x) \leq W(\xi) \leq \Phi(1).$$

We have $\Delta W = \Delta_1 + \Delta_2$, where

$$\Delta_1 = u_x(v,w)\Delta v + u_y(v,w)\Delta w$$

and

$$\Delta_2 = \sum_{i=1}^n \left[u_{xx}(v,w)v_i^2 + 2u_{xy}(v,w)v_iw_i + u_{yy}(v,w)w_i^2\right].$$

Note that Δ_1 is nonpositive due to (2.1) and the conditions $\Delta v \geq 0$, $\Delta u = 0$. To complete the proof of (1.9) it is enough to show that Δ_2 is also nonpositive. Fix $x \in D$. For any $1 \leq i \leq n$, the function G_i given by $G_i = G_{v(x),w(x),v_i(x),w_i(x)}$ is concave and we have

$$\Delta_2(x) = \sum_{i=1}^n G_i''(0) = \sum_{i=1}^n (A_i + B_i) \leq \sum_{i=1}^n A_i.$$

Here $A_i = A(G_i)$, $B_i = B(G_i)$ are given by (2.6), (2.7) or (2.8), (2.9), depending on whether $(v(x), w(x))$ belongs to D_1 or D_2. Therefore Δ_2 equals

$$\begin{aligned}(|\nabla w(x)|^2 - |\nabla v(x)|^2)\Big\{e^{v(x)-|w(x)|}\Big[\int_0^{-v(x)+|w(x)|}\Phi(s+1)e^s ds + \Phi(1)\Big]\\ - \Phi(-v(x)+|w(x)|+1) + \Phi'(-v(x)+|w(x)|+1)\Big\} \leq 0\end{aligned}$$

if $(v(x), w(x)) \in D_1$ and

$$(|\nabla w(x)|^2 - |\nabla v(x)|^2)\frac{\Phi'(-v(x)+|w(x)|+1)}{-v(x)+|w(x)|+1} \leq 0$$

if $(v(x), w(x)) \in D_2$. Thus $\Delta W \leq 0$ and W is superharmonic. To see that that inequality (1.9) is sharp, consider constant functions $v = w \equiv 1$. $\square$

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