WEAK TYPE INEQUALITIES FOR CONDITIONALLY SYMMETRIC MARTINGALES

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Abstract. Let f be a conditionally symmetric martingale and let $S(f)$ be its square function. The paper contains the proof of the sharp estimate

$$||f||_{p,\infty} \leq C_p ||S(f)||_p, \qquad 1 \leq p \leq 2,$$

where

$$C_p^p = \frac{2^{1-p/2}\pi^{p-3/2}\Gamma((p+1)/2)}{\Gamma(p+1)} \frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots}{1 - \frac{1}{3^{p+1}} + \frac{1}{5^{p+1}} - \frac{1}{7^{p+1}} + \dots}.$$

In addition, it is shown that the constant C_p is the best possible even for the class of dyadic martingales.

1. Introduction

Square function inequalities appear in many areas of mathematics, for example in harmonic analysis, potential theory and both classical and noncommutative probability, where they play an important role: see e.g. [4], [6], [13], [14], ... It is therefore of interest to establish sharp versions of such estimates. The primary objective of this paper is to determine the best constants in some weak-type estimates for the martingale square function under the assumption of conditional symmetry.

We start with introducing the background and notation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, filtered by $(\mathcal{F}_n)_{n\geq 0}$, a nondecreasing family of sub-σ-fields of $\mathcal{F}$. Let $f = (f_n)_{n\geq 0}$ be an adapted martingale taking values in a separable Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. Then $df = (df_n)_{n\geq 0}$, the difference sequence of f, is given by $df_0 = f_0$ and $df_n = f_n - f_{n-1}$. We define the *square function* of f by

$$S(f) = \left(\sum_{k=0}^{\infty} |df_k|^2 \right)^{1/2}.$$

We will also use the notation $S_n(f) = \left(\sum_{k=0}^n |df_k|^2\right)^{1/2}$ for $n = 0, 1, 2, \dots$ and write $||f||_p = \sup_n ||f_n||_p$, $||f||_{p,\infty} = \sup_n \sup_{\lambda>0} \lambda(\mathbb{P}(|f_n| \geq \lambda))^{1/p}$ for the strong and weak p-th moment of f.

A martingale f is *conditionally symmetric*, if for any $n \geq 1$ the conditional distributions of df_n and $-df_n$ given $\mathcal{F}_{n-1}$ coincide. For example, this is the case if f is a dyadic martingale. To recall what it means, let $(h_n)_{n\geq 0}$ be the system of Haar functions on $[0, 1]$. Then f is dyadic if for some $a_0, a_1, a_2, \dots \in \mathcal{H}$ we have $f_n = \sum_{k=0}^n a_k h_k$ for $n \geq 0$.

The problem of comparing the sizes of f and $S(f)$ goes back to the works of Khintchine [7], Littlewood [8], Marcinkiewicz [9], Marcinkiewicz and Zygmund [10]

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and Paley [12] (the concept of a martingale was not used there; the results concerned the partial sums of Rademacher and Haar series). Consider the inequality

$$a_p||S(f)||_p \leq ||f||_p \leq A_p||S(f)||_p, \tag{1.1}$$

to be valid for all conditionally symmetric martingales f. As shown by Burkholder in [3], for any $1 < p < \infty$ there are finite universal a_p and A_p such that the double inequality above holds. It follows from the results of Burkholder and Gundy [5] that the right inequality above holds also for $0 < p \leq 1$ with some absolute A_p. What about the optimal values of a_p and A_p? Let ν_p be the smallest positive zero of the confluent hypergeometric function and let μ_p be the largest positive zero of the parabolic cylinder function of parameter p. Wang [15] showed that $a_p = \nu_p$ for $p \geq 2$, $A_p = \nu_p$ for $0 < p \leq 2$ and $A_p = \mu_p$ for $p \geq 3$ are the best choices, even if we restrict ourselves in (1.1) to dyadic martingales. For the remaining values of parameter p, the optimal constants are not known. When $p = 1$, the left inequality in (1.1) does not hold with any universal $a_1 < \infty$. However, as shown by Bollobás [2] and the author [11], we have the sharp weak type inequality

$$||S(f)||_{1,\infty} \leq \left(\exp(-1/2) + \int_0^1 \exp(-s^2/2)\mathrm{d}s\right)||f||_1 = 1.4622\ldots||f||_1.$$

The purpose of this paper is to present a sharp comparison of the weak p-th norm of f with the strong p-th norm of $S(f)$, $1 \leq p \leq 2$. We will prove the following.

Theorem 1.1. *For any conditionally symmetric martingale f we have*

$$||f||_{p,\infty} \leq C_p||S(f)||_p, \qquad 1 \leq p \leq 2, \tag{1.2}$$

where

$$C_p^p = \frac{2^{1-p/2}\pi^{p-3/2}\Gamma((p+1)/2)}{\Gamma(p+1)}\frac{1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots}{1-\frac{1}{3^{p+1}}+\frac{1}{5^{p+1}}-\frac{1}{7^{p+1}}+\ldots}.$$

The constant C_p is the best possible, even for the class of real dyadic martingales.

The proof is based on Burkholder's technique: in the next section we introduce a special function and study its properties, which will be exploited in Section 3, where we establish Theorem 1.1.

2. A special function and its properties

Let $\mathbb{H} = \mathbb{R} \times (0,\infty)$, $S = \mathbb{R} \times (-1,1)$ and $S^+ = (0,\infty) \times (-1,1)$. Introduce a harmonic function $\mathcal{A} = \mathcal{A}_p : \mathbb{H} \to \mathbb{R}$, given by the Poisson integral

$$\mathcal{A}(\alpha,\beta) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\beta\left|\frac{2}{\pi}\log|t|\right|^p}{(\alpha-t)^2+\beta^2}\mathrm{d}t.$$

It is easy to see that the function $\mathcal{A}$ satisfies

$$\lim_{(\alpha,\beta)\to(z,0)}\mathcal{A}(\alpha,\beta) = \left(\frac{2}{\pi}\right)^p|\log|z||^p, \qquad z \neq 0. \tag{2.1}$$

Consider a conformal mapping φ given by $\varphi(z) = ie^{\pi z/2}$, or, in the real coordinates,

$$\varphi(x,y) = \left(-e^{\pi x/2}\sin\left(\frac{\pi}{2}y\right), e^{\pi x/2}\cos\left(\frac{\pi}{2}y\right)\right), \qquad (x,y) \in \mathbb{R}^2.$$

It can be easily verified that φ maps S onto $\mathbb{H}$. Introduce a function $A = A_p$ defined on the strip S by $A(x,y) = \mathcal{A}(\varphi(x,y))$. The function A is harmonic on S, since it is a real part of an analytic function. By (2.1), we can extend A to the continuous function on the closure $\overline{S}$ of S by $A(x,\pm 1) = |x|^p$. One easily checks that

$$A(x,y) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\cos\left(\frac{\pi}{2}y\right)\left|\frac{2}{\pi}\log|s| + x\right|^p}{(s-\sin(\frac{\pi}{2}y))^2 + \cos^2(\frac{\pi}{2}y)}\mathrm{d}s \tag{2.2}$$

for $|y| < 1$. Substituting $s := 1/s$ and $s := -s$ above, we see that A satisfies

$$A(x,y) = A(-x,y) = A(x,-y) \qquad \text{for } (x,y) \in S. \tag{2.3}$$

Finally, let $U = U_p : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be given by $U(x,y) = x^p$ for $|y| > 1$ and

$$U(x,y) = c_p \int_{\mathbb{R}} A(ux,y)\exp(-u^2/2)\mathrm{d}u$$

otherwise; here $c_p = \left(\int_{\mathbb{R}} |u|^p \exp(-u^2/2)\mathrm{d}u\right)^{-1} = \left(2^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)\right)^{-1}$. We easily check that U is continuous. By the symmetry condition (2.3), we have, for $|y| \leq 1$,

$$U(x,y) = 2c_p \int_0^\infty A(ux,y)\exp(-u^2/2)\mathrm{d}u. \tag{2.4}$$

Let us study the propertes of the function U which will be needed later.

Lemma 2.1. *We have $U(0,0) = C_p^{-p}$.*

Proof. This is straightforward: since $\pi^2/8 = \sum_{k=0}^\infty (2k+1)^{-2}$, we have

$$\begin{aligned}
U(0,0) = c_p\sqrt{2\pi}A(0,0) &= \frac{2^{-p/2}}{\Gamma\left(\frac{p+1}{2}\right)\sqrt{\pi}} \int_{-\infty}^\infty \frac{\left|\frac{2}{\pi}\log|s|\right|^p}{s^2+1}\mathrm{d}s \\
&= \frac{2^{1+p/2}}{\pi^{p+1/2}\Gamma\left(\frac{p+1}{2}\right)} \int_0^\infty \frac{|\log s|^p}{s^2+1}\mathrm{d}s \\
&= \frac{2^{1+p/2}}{\pi^{p+1/2}\Gamma\left(\frac{p+1}{2}\right)} \int_{-\infty}^\infty \frac{|s|^p e^s}{e^{2s}+1}\mathrm{d}s \\
&= \frac{2^{2+p/2}}{\pi^{p+1/2}\Gamma\left(\frac{p+1}{2}\right)} \int_0^\infty s^p e^{-s}\sum_{k=0}^\infty (-e^{-2s})^k \mathrm{d}s \\
&= \frac{2^{2+p/2}\Gamma(p+1)}{\pi^{p+1/2}\Gamma\left(\frac{p+1}{2}\right)} \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^p} = C_p^{-p}.
\end{aligned}$$

□

Lemma 2.2. *(i) The function U satisfies the differential equation*

$$U_x(x,y) + xU_{yy}(x,y) = 0 \qquad \text{on } S^+. \tag{2.5}$$

(ii) The function U is superharmonic on S^+.

Proof. By Fubini's theorem, we may take the derivatives inside the integral while computing the partial derivatives of U on S^+.

(i) Since A is harmonic, we have, for $(x,y) \in S^+$,

$$xU_{yy}(x,y) = x\int_{\mathbb{R}} A_{yy}(ux,y)\exp(-u^2/2)\mathrm{d}u = -\int_{\mathbb{R}} xA_{xx}(ux,y)\exp(-u^2/2)\mathrm{d}u$$

and the claim follows from the integration by parts: the above is equal to

$$-\int_{\mathbb{R}} A_x(ux,y)u\exp(-u^2/2)\mathrm{d}u = -U_x(x,y).$$

(ii) By the previous part, the assertion can be rewritten in the form

$$xU_{xx}(x,y) - U_x(x,y) \leq 0 \qquad \text{on } S^+.$$

Since $U_x(0+,y)=0$, we will be done if we show that $U_{xxx}\leq 0$ or, by (2.4), $A_{xxx}\leq 0$ on S^+. To this end, fix $x>0$, $\varepsilon\in(0,x)$ and introduce the function

$$f_\varepsilon(h) = 2|h|^{p-2}h - |h-\varepsilon|^{p-2}(h-\varepsilon) - |h+\varepsilon|^{p-2}(h+\varepsilon), \qquad h\in\mathbb{R}.$$

One easily verifies that

$$f_\varepsilon \text{ is odd and } f_\varepsilon\geq 0 \text{ on } [0,\infty). \tag{2.6}$$

We write

$$2A_x(x,y) - A_x(x-\varepsilon,y) - A_x(x+\varepsilon,y) = \frac{p}{\pi}\int_{-\infty}^{\infty} \frac{f_\varepsilon\left(x+\frac{2}{\pi}\log|s|\right)\cos(\frac{\pi}{2}y)}{(s-\sin(\frac{\pi}{2}y))^2+\cos^2(\frac{\pi}{2}y)}\mathrm{d}s,$$

split the integral into two, over the nonpositive and nonnegative halfline, and, finally, substitute $s=\pm e^r$. As the result, we get

$$2A_x(x,y) - A_x(x-\varepsilon,y) - A_x(x+\varepsilon,y) = \frac{p}{\pi}\int_{-\infty}^{\infty} f_\varepsilon\left(x+\frac{2}{\pi}r\right) g^y(r)\mathrm{d}r, \tag{2.7}$$

where the function g^y is given by

$$g^y(r) = \frac{\cos(\frac{\pi}{2}y)e^r}{(e^r-\sin(\frac{\pi}{2}y))^2+\cos^2(\frac{\pi}{2}y)} + \frac{\cos(\frac{\pi}{2}y)e^r}{(e^r+\sin(\frac{\pi}{2}y))^2+\cos^2(\frac{\pi}{2}y)}.$$

Note that g^y is even and nonincreasing on $[0,\infty)$: indeed, for $r>0$,

$$(g^y)'(r) = \frac{\cos(\frac{\pi}{2}y)e^r(1-e^r)}{[(e^r-\sin(\frac{\pi}{2}y))^2+\cos^2(\frac{\pi}{2}y)]^2} + \frac{\cos(\frac{\pi}{2}y)e^r(1-e^r)}{[(e^r+\sin(\frac{\pi}{2}y))^2+\cos^2(\frac{\pi}{2}y)]^2} \leq 0.$$

Thus, by (2.6), the integral in (2.7) is nonnegative and, since $\varepsilon\in(0,x)$ was arbitrary, the function A_x is concave on $(0,\infty)$. □

Lemma 2.3. *(i) For any $(x,y)\in[0,\infty)\times\mathbb{R}$ we have*

$$x^p \leq U(x,y) \leq x^p + U(0,0)1_{\{|y|<1\}}. \tag{2.8}$$

(ii) We have $U_x(x,y)\geq 0$ on $(0,\infty)\times\mathbb{R}$ and $U_y(x,y)\leq 0$ on $(0,\infty)\times((0,\infty)\setminus\{1\})$.

Proof. We may assume that $|y|<1$, since otherwise the claim is obvious, both in (i) and (ii). The lower bound in (2.8) follows from Jensen's inequality: we have

$$\begin{aligned} A(x,y) &= \int_{-\infty}^{\infty} \left|\frac{2}{\pi}\log|s| + x\right|^p \cdot \frac{1}{\pi}\frac{\cos(\frac{\pi}{2}y)}{(s-\sin(\frac{\pi}{2}y))^2+\cos^2(\frac{\pi}{2}y)}\mathrm{d}s \\ &\geq \left|\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos(\frac{\pi}{2}y)\left(\frac{2}{\pi}\log|s|+x\right)}{(s-\sin(\frac{\pi}{2}y))^2+\cos^2(\frac{\pi}{2}y)}\mathrm{d}s\right|^p = x^p \end{aligned}$$

(to see the latter equality, make the substitution $s:=1/s$) and it suffices to apply (2.4). Now we turn to (ii). Similar argument gives that the function $A(\cdot,y)$ is convex and hence the same is valid for U. Therefore, by part (ii) of Lemma 2.2 we have $U_{yy}\leq 0$ and hence by the first part of that lemma, $U_x\geq 0$. In addition, $U_y\leq 0$ for $y>0$, since the function $U(x,\cdot)$ is even (which follows from (2.3) and (2.4)). Thus we have shown (ii); moreover, we see that it suffices to establish the upper bound in (2.8) for $y = 0$. Using an elemetary inequality $|a+b|^p + |a-b|^p \leq 2|a|^p + 2|b|^p$ for a, $b \in \mathbb{R}$ yields

$$\begin{aligned} 2A(x,0) = A(x,0) + A(-x,0) &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{\left|\frac{2}{\pi}\log|s| + x\right|^p + \left|\frac{2}{\pi}\log|s| - x\right|^p}{s^2+1}\mathrm{d}s \\ &\leq \frac{1}{\pi}\int_{\mathbb{R}} \frac{2\left|\frac{2}{\pi}\log|s|\right|^p + 2|x|^p}{s^2+1}\mathrm{d}s \\ &= 2A(0,0) + |x|^p. \end{aligned}$$

It suffices to use (2.4) to get the claim. □

Lemma 2.4. *For any a, $b \in \mathcal{H}$ we have $|a^* + b^*| \leq |a+b|$.*

Proof. If both $|a|$, $|b|$ do not exceed 1, the claim is obvious. If $|a| > 1 \geq |b|$,

$$|a+b|^2 - |a^*+b^*|^2 = |a|^2 - 1 + 2\langle a,b\rangle(1-|a|^{-1}) \geq |a|^2 - 1 - 2|a|(1-|a|^{-1}) \geq 0$$

and similarly for $|b| > 1 \geq |a|$. Finally, if $|a| > 1$ and $|b| > 1$, then

$$|a+b|^2 - |a^*+b^*|^2 = |a|^2 + |b|^2 + 2\langle a,b\rangle(1-(|a||b|)^{-1}) - 2 \geq |a|^2 + |b|^2 - 2|a||b| \geq 0,$$

as desired. □

Lemma 2.5. *For any $(x,y) \in [0,\infty) \times \mathcal{H}$ and any $d \in \mathcal{H}$ we have*

$$2U(x,|y|) \leq U\big((x^2+|d|^2)^{1/2}, |y+d|\big) + U\big((x^2+|d|^2)^{1/2}, |y-d|\big). \tag{2.9}$$

Proof. It is convenient to split the proof into three parts.

Case 1: $|y| \geq 1$. Then the estimate is trivial: indeed, by the previous lemma,

$$2U(x,|y|) \leq 2(x^2+|d|^2)^{p/2} \leq U((x^2+|d|^2)^{1/2}, |y+d|) + U((x^2+|d|^2)^{1/2}, |y-d|).$$

Case 2: $|y| < 1$, $|y \pm d| \leq 1$. For $t \in [0,1]$, let $\psi(t) = U\big(x_+^t, |y_+^t|\big) + U\big(x_+^t, |y_-^t|\big)$, where $x_+^t = (x^2 + t^2|d|^2)^{1/2}$ and $y_\pm^t = y \pm td$. We have that $\psi'(t)/|d|$ equals

$$\frac{t|d|}{x_+^t}\left[U_x\big(x_+^t, |y_+^t|\big) + U_x\big(x_+^t, |y_-^t|\big)\right] + \langle U_y\big(x_+^t, |y_+^t|\big)(y_+^t)' - U_y\big(x_+^t, |y_-^t|\big)(y_-^t)', d'\rangle$$

(when $|y+td| = 0$, the differentiation is allowed since $U_y(x,0) = 0$). We will prove that this is nonnegative, which will clearly yield the claim. It suffices to show that

$$\frac{|y_+^t - y_-^t|}{2x_+^t}\left[U_x\big(x_+^t, |y_+^t|\big) + U_x\big(x_+^t, |y_-^t|\big)\right] \geq \left|U_y\big(x_+^t, |y_+^t|\big)(y_+^t)' - U_y\big(x_+^t, |y_-^t|\big)(y_-^t)'\right|.$$

To this end, note that if we square both sides, the estimate becomes $A \leq B \cdot \langle (y_+^t)', (y_-^t)'\rangle$, where A and B depend only on $|y_+^t|$ and $|y_-^t|$. Thus it suffices to prove it for $(y_+^t)' = \pm(y_-^t)'$. When $(y_+^t)'$ and $(y_-^t)'$ are equal, we use (2.5) and the fact that $U_y \leq 0$ on $(0,\infty)\times(0,1)$ (by Lemma 2.8) and see that the inequality reads

$$-\frac{\big||y_+^t| - |y_-^t|\big|}{2}\left[U_{yy}\big(x_+^t, |y_+^t|\big) + U_{yy}\big(x_+^t, |y_-^t|\big)\right] \geq \left|\int_{|y_-^t|}^{|y_+^t|} U_{yy}(x_+^t, s)\mathrm{d}s\right|.$$

This follows from the fact that U_{yy} is nonpositive and concave: by Lemma 2.2, we have $x^2 U_{yyyy}(x,y) = U_{xx}(x,y) + U_{yy}(x,y) \leq 0$ for $(x,y) \in S^+$. The case $(y_+^t)' = -(y_-^t)'$ is dealt with in the same manner.

Case 3: $|y| < 1$, $d > 1 - |y|$. This can be reduced to the previous case. Set $y_+ = (y+d)^*$, $y_- = (y-d)^*$ and $\tilde{y} = (y_+ + y_-)/2$, $\tilde{d} = (y_+ - y_-)/2$, $\tilde{x} = (x^2 + |d|^2 - \tilde{d}^2)^{1/2}$.

By Lemma 2.4, $|\tilde{y}| \leq |y|$ and $|\tilde{d}| \leq |d|$, so $\tilde{x} \geq x$. Using Lemma 2.3 and the fact that $\tilde{y}$, $\tilde{d}$ satisfy the assumptions of Case 2, we may write

$$2U(x,y) \leq 2U(\tilde{x},\tilde{y}) \leq U((\tilde{x}^2+\tilde{d}^2)^{1/2}, y_+) + U((\tilde{x}^2+\tilde{d}^2)^{1/2}, y_-)$$

and the latter sum is precisely the right hand side of (2.9). □

Remark 2.1. *The choice $x=0$, $y=0$ in* (2.9) *gives $U(0,0) \leq U(|d|,d)$ for all d.*

3. Proof of Theorem 1.1

Proof of (1.2). We may assume that $S(f) \in L^p$, since otherwise there is nothing to prove. By homogeneity, it suffices to show that for any $n \geq 0$,

$$\mathbb{P}(|f_n| \geq 1) \leq C_p^p \mathbb{E} S_n^p(f).$$

The key ingredient of the proof of this estimate is the fact that

$$\text{the process } (U(S_n(f), f_n))_{n\geq 0} \text{ is an } (\mathcal{F}_n)\text{-submartingale.} \tag{3.1}$$

Indeed, for $n \geq 0$ the variable $U(S_n(f), f_n)$ is integrable by the condition $S(f) \in L^p$ and (2.8). Moreover, by the conditional symmetry, $2\mathbb{E}(U(S_{n+1}(f), f_{n+1})|\mathcal{F}_n)$ equals

$$\mathbb{E}\left[U((S_n^2(f)+df_{n+1}^2)^{1/2}, f_n+df_{n+1}) + U((S_n^2(f)+df_{n+1}^2)^{1/2}, f_n-df_{n+1})\middle|\mathcal{F}_n\right].$$

This is not smaller than $2U(S_n(f), f_n)$: apply (2.9) with $x = S_n(f)$, $y = f_n$ and $d = df_{n+1}$. Thus (3.1) follows and, using (2.8) and Remark 2.1, we get

$$U(0,0) \leq \mathbb{E}U(S_0(f), f_0) \leq \mathbb{E}U(S_n(f), f_n) \leq \mathbb{E}S_n^p(f) + U(0,0)\mathbb{P}(|f_n|<1).$$

This completes the proof, by virtue of Lemma 2.1. □

Sharpness. Suppose that γ_p is the optimal constant in (1.2) for real-valued dyadic martingales. Arguing as in [11], this yields a corresponding weak type inequality

$$\mathbb{P}(|B_\tau| \geq 1) \leq \gamma_p^p \mathbb{E}\tau^{p/2}, \tag{3.2}$$

where B is a standard Brownian motion and τ is any stopping time of B. On the other hand, let $\eta = \inf\{t : |B_t| = 1\}$ and consider the process $(U(\sqrt{\eta\wedge t}, B_{\eta\wedge t}))_{t\geq 0}$. By (2.5) and Itô's formula, it is a martingale with expectation equal to $U(0,0)$. By (2.8) and exponential integrability of η, this martingale converges in L^1 to $\eta^{p/2}$, which, by Lemma 2.1, yields $C_p^{-p} = \mathbb{E}\eta^{p/2}$ and, consequently, $1 = \mathbb{P}(|B_\eta| \geq 1) = C_p^p \mathbb{E}\eta^{p/2}$. By (3.2), this implies $\gamma_p \geq C_p$ and completes the proof. □

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