SHARP MAXIMAL INEQUALITY FOR STOCHASTIC INTEGRALS

ADAM OSEKOWSKI

Abstract. Let $X = (X_t)_{t \geq 0}$ be a nonnegative supermartingale and $H = (H_t)_{t \geq 0}$ be a predictable process with values in $[-1,1]$. Let $Y$ denote the stochastic integral of $H$ with respect to $X$. The paper contains the proof of the sharp inequality

$$\sup_{t \geq 0} ||Y_t||_1 \leq \beta_0 ||\sup_{t \geq 0} X_t||_1,$$

where $\beta_0 = 2 + (3e)^{-1} = 2.1226\ldots$ A discrete-time version of this inequality is also established.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, which is filtered by a nondecreasing right-continuous family $(\mathcal{F}_t)_{t \geq 0}$ of sub-$\sigma$-fields of $\mathcal{F}$. Assume that $\mathcal{F}_0$ contains all the events of probability 0. Suppose $X = (X_t)_{t \geq 0}$ is an adapted real-valued right-continuous semimartingale with left limits. Let $Y$ be the Itô integral of $H$ with respect to $X$,

$$Y_t = H_0 X_0 + \int_{(0,t]} H_s dX_s, \quad t \geq 0,$$

where $H$ is a predictable process with values in $[-1,1]$. Let $||Y||_1 = \sup_{t \geq 0} ||Y_t||_1$ and $X^* = \sup_{t \geq 0} |X_t|$.

The objective of this paper is to compare the first moments of $Y$ and $X^*$. In [4], Burkholder introduced a method of proving related maximal inequalities for martingales and obtained the following sharp estimate.

**Theorem 1.1.** If $X$ is a martingale and $Y$ is as above, then we have

$$||Y||_1 \leq \gamma ||X^*||_1,$$

where $\gamma = 2.536\ldots$ is the unique solution of the equation

$$\gamma - 3 = - \exp\left(\frac{1-\gamma}{2}\right).$$

The constant is the best possible.

Using Burkholder’s techniques, we find the best constant in the inequality (1.1) in case $X$ is a nonnegative supermartingale. The main result of the paper is the following.

2000 Mathematics Subject Classification. Primary: 60H05. Secondary: 60G42.

Key words and phrases. Martingale, supermartingale, martingale transform, norm inequality, stochastic integral, maximal inequality.

Partially supported by MEiN Grant 1 PO3A 012 29.
Theorem 1.2. Suppose \( X \) is a nonnegative supermartingale and \( Y \) is as above. Then the inequality
\[
||Y||_1 \leq \beta_0||X^*||_1
\]
holds true with \( \beta_0 = 2 + (3e)^{-1} = 2.1226 \ldots \). The constant is the best possible. It is already the best possible if \( X \) is assumed to be a nonnegative martingale.

As usual, the inequality for stochastic integrals is accompanied by its discrete-time version. Suppose \((\Omega, \mathcal{F}, P)\) is a probability space, equipped with filtration \((\mathcal{F}_n)_{n \geq 0}\). Let \( f = (f_n)_{n \geq 0} \) be an adapted nonnegative supermartingale and \( g = (g_n)_{n \geq 0} \) be its transform by a predictable sequence \( v = (v_n)_{n \geq 0} \) bounded in absolute value by 1. That is,
\[
f_n = \sum_{k=0}^{n} df_k, \quad g_n = \sum_{k=0}^{n} v_k df_k, \quad n = 0, 1, 2, \ldots
\]
By predictability of \( v \) we mean that \( v_0 \) is \( \mathcal{F}_0 \)-measurable and for any \( k \geq 1 \), \( v_k \) is measurable with respect to \( \mathcal{F}_{k-1} \). Let \( f_n^* = \max_{k \leq n} f_k \) and \( f^* = \sup_{k} f_k \).

A discrete-time version of Theorem 1.2 can be stated as follows.

Theorem 1.3. Let \( f, g, \beta_0 \) be as above. Then we have
\[
||g||_1 \leq \beta_0||f^*||_1,
\]
and the constant \( \beta_0 \) is the best possible. It is already the best possible if \( f \) is assumed to be a nonnegative martingale.

The paper is organized as follows. In the next section we describe the Burkholder’s method. Section 3 is devoted to the proofs of the maximal inequalities. In the last section we complete the proofs of Theorem 1.2 and Theorem 1.3 by showing that the constant \( \beta_0 \) cannot be replaced by a smaller one.

2. The upper class of functions

Throughout this section we deal with the discrete-time setting. We start with some reductions. Standard approximation arguments (see page 350 of [4]) show that it is enough to prove Theorem 1.3 under an additional assumption that the supermartingale \( f \) is simple, i.e. for any \( n \) the variable \( f_n \) takes only a finite number of values and there is \( N \) such that \( f_N = f_{N+1} = f_{N+2} = \ldots \) with probability 1. Then, clearly, every transform \( g \) of \( f \) is also simple and the pointwise limits \( f_\infty, g_\infty \) exist. Furthermore, with no loss of generality, we may restrict ourselves to the special transforms \( g \) (called \( \pm 1 \) transforms), namely, those with all \( v_n \) being deterministic and taking values in \( \{-1, 1\} \): see Lemma A.1 on page 60 in [3] and observe \((f^*)_* = f_*^\prime \) on page 61. Finally, note that in order to prove inequality (1.2), it suffices to show that for any \( f, g \) as above and any integer \( n \) we have
\[
E|g_n| \leq \beta_0 E f_n^*.
\]
To describe Burkholder’s method, let us consider the following general problem, first in the martingale setting: let \( D = [0, \infty) \times \mathbb{R} \times [0, \infty) \) and \( V : D \to \mathbb{R} \) be any Borel function satisfying \( V(x, y, z) = V(x, y, x \vee z) \). Suppose we want to prove the inequality
\[
EV(f_n, g_n, f_n^*) \leq 0
\]
for all nonnegative integers \( n \) and all pairs \((f, g)\), where \( f \) is a simple nonnegative martingale and \( g \) is its \( \pm 1 \) transform.

The key idea is to study the family \( \mathcal{U} \) of all functions \( U : D \to \mathbb{R} \) satisfying the following three properties.

\[
U(x, y, z) = U(x, y, x \lor z) \quad \text{if} \quad (x, y, z) \in D,
\]

\[
V(x, y, z) \leq U(x, y, z) \quad \text{if} \quad (x, y, z) \in D
\]

and, furthermore, if \((x, y, z) \in D, \varepsilon \in \{-1, 1\}, \alpha \in (0, 1)\) and \( t_1, t_2 \geq -x \) with \( \alpha t_1 + (1 - \alpha) t_2 = 0 \), then

\[
\alpha U(x + t_1, y + \varepsilon t_1, z) + (1 - \alpha) U(x + t_2, y + \varepsilon t_2, z) \leq U(x, y, z).
\]

The interplay between the class \( \mathcal{U} \) and the maximal inequality (2.1) is described in the theorem below. It is a simple modification of Theorems 2.2 and 2.3 in [4] (see also Section 11 in [2] and Theorem 2.1 in [3]) to the case of nonnegative supermartingales. We omit the proof, as it requires only some minor changes.

**Theorem 2.1.** The inequality (2.1) holds for all \( n \) and all pairs \((f, g)\) as above if and only if the class \( \mathcal{U} \) is nonempty. Furthermore, if \( \mathcal{U} \) is nonempty, then there exists the least element in \( \mathcal{U} \), given by

\[
U^0(x, y, z) = \sup \{ \mathbb{E} V(f_\infty, g_\infty, f^* \lor z) \}.
\]

Here the supremum runs over all the pairs \((f, g)\), where \( f \) is a simple nonnegative martingale, \( \mathbb{P}(\{f_0, g_0\} = (x, y)) = 1 \) and \( dg_k = \pm df_k \) almost surely for all \( k \geq 1 \).

In case \( f \) is assumed to be a nonnegative supermartingale, we can proceed in a similar manner. For a given \( V \), consider the inequality (2.1). Suppose we want it to be valid for any \( n \), any nonnegative supermartingale \( f \) and any \( \pm 1 \) transform \( g \).

Let \( \mathcal{U}' \) be a subclass of \( \mathcal{U} \) containing those functions, which satisfy

\[
U(x, y, z) \geq U(x - \delta, y \pm \delta, z) \quad \text{if} \quad (x, y, z) \in D, \delta \in [0, x].
\]

The analogue of Theorem 2.1 is as follows (the straightforward proof is omitted).

**Theorem 2.2.** The inequality (2.1) holds for all \( n \) and all pairs \((f, g)\) as above if and only if the class \( \mathcal{U}' \) is nonempty.

Now we turn to (1.2) and assume from now on, that the function \( V \) is given by

\[
V(x, y, z) = V(x, y, x \lor z) = y - \beta(x \lor z),
\]

where \( \beta > 0 \) is a fixed number. The inequality (2.1) reads

\[
\mathbb{E} |g_n| \leq \beta \mathbb{E} f_n^*.
\]

Denote by \( \mathcal{U}(\beta), \mathcal{U}'(\beta) \) the classes \( \mathcal{U}, \mathcal{U}' \) corresponding to this choice of \( V \).

The rest of this section is devoted to the last part of Theorem 1.3. Let \( \beta(\text{possup}) \) (resp. \( \beta(\text{posmar}) \)) be the smallest constant \( \beta \) in the inequality (2.7), when \( f \) is assumed to run over the class of all nonnegative supermartingales (resp. nonnegative martingales).

**Theorem 2.3.** We have \( \beta(\text{posmar}) = \beta(\text{possup}) \).
Proof. We only need the inequality \( \beta = \beta(\text{posmar}) \geq \beta(\text{possup}) \), as the reverse one is trivial. By Theorem 2.2, it suffices to prove that the class \( \mathcal{U}(\beta) \) is nonempty. Theorem 2.1 guarantees the existence of the minimal element \( U^0 \) of the class \( \mathcal{U}(\beta) \), given by (2.5). By definition we get the following properties of \( U^0 \).

(2.8) \[ U^0(x, y, z) = U^0(x, -y, z), \]
(2.9) \[ U^0(1, -1, 1) = U^0(1, 1, 1) \leq 0, \]
(2.10) \[ U^0(\alpha x, \alpha y, \alpha z) = \alpha U^0(x, y, z) \quad \text{for any } \alpha > 0. \]

The equality (2.8) is clear, (2.9) follows from the fact that for any pair \((f, g)\) as in Theorem 2.1, starting from \((1, 1)\) or from \((1, -1)\), we have that \( g \) is a \( \pm 1 \) transform of \( f \) and therefore, by (2.7), we have \( EV(f_n, g_n, f^*_n) \leq 0 \) for any \( n \). For (2.10), we use the fact that \( V \) is homogeneous.

We will prove that the function \( U : D \to \mathbb{R} \) given by

(2.11) \[ U(x, y, z) = U^0(x, y, z) - U^0(1, 1, 1)x \]

belongs to \( \mathcal{U} \). The conditions (2.2), (2.3) and (2.4) hold true for \( U \), since they are satisfied for \( U^0 \) and, by (2.9), we have \( U \geq U^0 \). It remains to prove (2.6). Note that \( U \) satisfies \( U(x, y, z) = U(x, -y, z) \), \( U(1, -1, 1) = U(1, 1, 1) = 0 \) and is homogeneous. Fix \( y \in \mathbb{R} \), \( 0 \leq x \leq z \), \( \varepsilon \in \{-1, 1\} \) and let \( \delta \in (0, x] \), \( t > z - x \). Use (2.4) with \( t_1 = -\delta \), \( t_2 = t \) and \( \alpha = t/(t + \delta) \) to obtain

\[
\frac{t}{t + \delta} U(x - \delta, y - \varepsilon \delta, z) + \frac{\delta}{t + \delta} U(x + t, y + \varepsilon t, z) \leq U(x, y, z).
\]

By homogeneity of \( U \), this gives

(2.12) \[ \frac{t}{t + \delta} U(x - \delta, y - \varepsilon \delta, z) + \frac{\delta(x + t)}{t + \delta} U(1, y + \varepsilon t, 1) \leq U(x, y, z). \]

Now we let \( t \to \infty \); the inequality (2.6) will follow if we show that

(2.13) \[ \lim_{s \to 1} U(1, s, 1) \geq U(1, 1, 1) = 0. \]

For \( s > 1 \), use (2.4) with \( x = z = 1 \), \( y = s \), \( \varepsilon = -1 \), \( t_1 = -1 \), \( t_2 = (s - 1)/2 \) and get

\[
U(1, s, 1) \geq \frac{s - 1}{s + 1} U(0, s + 1, 1) + \frac{2}{s + 1} U\left(\frac{s + 1}{2}, \frac{s + 1}{2}, \frac{s + 1}{2}\right) \geq \frac{s - 1}{s + 1}(s + 1 - \beta),
\]

the latter inequality being a consequence of (2.3) and the homogeneity of \( U \). For \( 0 < s < 1 \), apply (2.12) to \( x = z = 1 \), \( y = s \), \( \varepsilon = -1 \), \( \delta = (1 - s)/2 \) and \( t = 2s/(1 - s) \) (so that \( (y + \varepsilon t)/(x + t) = -s \)) to obtain

\[
U(1, s, 1) \geq \frac{2}{s + 1} U\left(\frac{1 + s}{2}, \frac{1 + s}{2}, 1\right).
\]

Now we use the fact that, by (2.4), the function \( s \mapsto U(s, s, 1) \) is concave and therefore continuous. This completes the proof of (2.13) and, in consequence, we have \( U \in \mathcal{U}(\beta_0) \), so this class is nonempty. All that is left is to use Theorem 2.2. \( \square \)

Thus, to establish the inequality (1.2), we need to find an element \( U \) in \( \mathcal{U}(\beta_0) \). This will be done in the next section.
3. The proofs of the inequalities (1.1) and (1.2)

Here we construct the special function $U$ corresponding to the maximal inequality (1.2). This is the main section of the paper.

Let $S$ denote the strip $[-1, 1] \times \mathbb{R}$. Consider the following subsets of $S$.

$$D_1 = \{(x, y) : 0 \leq x < \frac{2}{3}, x + y \geq \frac{2}{3}\},$$

$$D_2 = \{(x, y) : \frac{2}{3} \leq x \leq 1, x - y \leq \frac{2}{3}\},$$

$$D_3 = \{(x, y) : 0 \leq x < \frac{2}{3}, y \geq 0, x + y \leq \frac{2}{3}\},$$

$$D_4 = \{(x, y) : \frac{2}{3} < x \leq 1, y \geq 0, x - y > \frac{2}{3}\}.$$

Let the function $u$ be defined on $S$ by the condition $u(x, y) = u(|x|, |y|)$ and

$$u(x, y) = \begin{cases} 
    y - \beta_0 + x \left\{ \exp\left[-\frac{3}{2}(y + x - \frac{3}{2})\right] + 1 \right\}, & (x, y) \in D_1, \\
    y - \beta_0 + \left( \frac{3}{4} - x \right) \exp\left[-\frac{3}{2}(y - x + \frac{3}{2})\right] + x, & (x, y) \in D_2, \\
    y - \beta_0 - x \log\left[ \frac{3}{2} (x + y) \right] + 2x, & (x, y) \in D_3, \\
    -\beta_0 - \frac{1}{4} (2 - 2x - y) (3 - 3x + 3y)^{1/2} + \frac{14}{9}, & (x, y) \in D_4.
\end{cases}$$

A function defined on the strip $S$ is said to be diagonally concave if it is concave on the intersection of $S$ with any line of slope $1$ or $-1$. The proof of the following statement is just a matter of elementary calculations.

**Lemma 3.1.** For any real number $y$ we have

(3.1) $u(0, y) = |y| - \beta_0$, $u(1, y) \geq |y| - \beta_0$,

(3.2) $u$ is diagonally concave,

(3.3) $u(1, \cdot)$ is convex,

(3.4) $u(1 - \delta, y \pm \delta) \leq u(1, y)$ for any $\delta \in [0, 1]$,

(3.5) $u(1, 1) = 0$.

Define $U : D \to \mathbb{R}$ by

$$U(x, y, z) = (x \vee z) u\left( \frac{x}{x \vee z}, \frac{y}{x \vee z} \right).$$

We have the following statement.

**Lemma 3.2.** The function $U$ belongs to $U(\beta_0)$.

This fact can be proved exactly in the same manner as Lemma 3.1 in [4]. We omit the details. Now we are ready to prove the maximal inequalities.

**Proof of the inequality (1.2):** It is an immediate consequence of Theorem 2.1, Theorem 2.3 and Lemma 3.2. □

**Proof of inequality (1.1):** This follows by approximation argument. See Section 16 of [2], where it is shown how the result of Bichteler [1] can be used to deduce the estimates for stochastic integrals from their discrete-time versions. □
4. Sharpness

Clearly, we need only to focus on the sharpness of (1.2), since it immediately implies that $\beta_0$ is also the best possible in (1.1).

Let $\beta = \beta(\text{posmar})$. By Theorem 2.3, we need to prove $\beta \geq \beta_0$. This can be done by constructing an appropriate example. However, we take a different approach.

By Theorem 2.1, the class $U(\beta)$ is nonempty, we can consider its minimal element $U_0$ and, as we have already proved, the function $U$ given by (2.11) belongs to $U'(\beta)$. Define $u : S \to \mathbb{R}$ by

\begin{equation}
(4.1) \quad u(x, y) = U(x, y, 1).
\end{equation}

The conditions (2.3), (2.4) and (2.6) imply that

\begin{equation}
(4.2) \quad u(x, y) \geq |y| - \beta,
\end{equation}

\begin{equation}
(4.3) \quad u \text{ is diagonally concave},
\end{equation}

\begin{equation}
(4.4) \quad u(x, y) \geq u(x - \delta, y \pm \delta) \text{ for } \delta \in [0, x]
\end{equation}

and, moreover, we have

\begin{equation}
(4.5) \quad u(1, 1) = U(1, 1, 1) = 0.
\end{equation}

Furthermore, note that for any $y$, by definition of $U_0$,

\begin{equation}
(4.6) \quad u(0, y) = U_0(0, y, 1) = |y| - \beta,
\end{equation}

since the only nonnegative martingale starting from 0 is the constant one.

We will show that the existence of $u$ satisfying the properties (4.2) – (4.6) implies $\beta \geq \beta_0$. This will be done in several steps. Set $B(x) = u(1, x + 1/3)$ and $C(x) = u(2/3, x)$.

**Step 1.** By properties (4.3) and (4.6), we have

\begin{equation}
\begin{aligned}
&u\left(\frac{2}{3} + \delta, 2k\delta + \delta\right) \geq (1 - 3\delta)C(2k\delta) + 3\delta B(2k\delta), \\
&C((2k + 2)\delta) \geq \frac{2}{2 + 3\delta} u\left(\frac{2}{3} + \delta, 2k\delta + \delta\right) + \frac{3\delta}{2 + 3\delta} (2k\delta + 2\delta + \frac{2}{3} - \beta),
\end{aligned}
\end{equation}

from which we deduce that

\begin{equation}
(4.7) \quad C((2k + 2)\delta) \geq \frac{2(1 - 3\delta)}{2 + 3\delta} C(2k\delta) + \frac{6\delta}{2 + 3\delta} B(2k\delta) + \frac{3\delta}{2 + 3\delta} (2k\delta + 2\delta + \frac{2}{3} - \beta).
\end{equation}

Furthermore, (4.3) and (4.4) yield

\begin{equation}
B(2k\delta) \geq u(1 - \delta, 2k\delta + \delta + \frac{1}{3}) \geq (1 - 3\delta)B((2k + 2)\delta) + 3\delta C((2k + 2)\delta).
\end{equation}

Multiply this inequality throughout by $\alpha > 0$ and add it to (4.7). We obtain

\begin{equation}
C((2k + 2)\delta)(1 - 3\alpha\delta) - \alpha(1 - 3\delta)B((2k + 2)\delta)
\end{equation}

\begin{equation}
\geq \frac{2(1 - 3\delta)}{2 + 3\delta} C(2k\delta) - \left(\alpha - \frac{6\delta}{2 + 3\delta}\right)B(2k\delta) + \frac{3\delta}{2 + 3\delta} ((2k + 2)\delta + \frac{2}{3} - \beta),
\end{equation}

or, equivalently, after substitution

\begin{equation}
(4.8) \quad \overline{B}(t) = B(t) - t - \frac{2}{3} + \beta, \quad \overline{C}(t) = C(t) - t - \frac{2}{3} + \beta,
\end{equation}
we get
\[
\mathcal{C}((2k + 2)\delta) - \frac{\alpha(1 - 3\delta)}{1 - 3\alpha\delta} \mathcal{B}((2k + 2)\delta)
\geq \frac{2(1 - 3\delta)}{(2 + 3\delta)(1 - 3\alpha\delta)} \left[ \mathcal{C}(2k\delta) - \frac{2\alpha + 3\alpha\delta - 6\delta}{2(1 - 3\delta)} \mathcal{B}(2k\delta) \right] + \frac{2\delta}{1 - 3\alpha\delta} (\alpha - \frac{2}{2 + 3\delta}).
\]
\[(4.9)\]

**Step 2.** Now we will use the inequality \((4.9)\) several times. The choice
\[
\alpha = \frac{5 + \sqrt{9 - 24\delta}}{2(2 + 3\delta)}
\]
gives
\[
\alpha(1 - 3\delta) = \frac{2\alpha + 3\alpha\delta - 6\delta}{2(1 - 3\delta)}
\]
and using \((4.9)\) for \(k = 1, k = 2, \ldots, l\) yields
\[
\mathcal{C}(2k\delta) - \frac{\alpha(1 - 3\delta)}{1 - 3\alpha\delta} \mathcal{B}(2k\delta)
\geq \left[ \frac{2(1 - 3\delta)}{(2 + 3\delta)(1 - 3\alpha\delta)} \right]^{k-1} \left[ \mathcal{C}(2\delta) - \frac{\alpha(1 - 3\delta)}{1 - 3\alpha\delta} \mathcal{B}(2\delta) \right] + \eta,
\]
\[(4.10)\]
where
\[
\eta = \frac{2\delta}{1 - 3\alpha\delta} (\alpha - \frac{2}{2 + 3\delta}) \sum_{r=0}^{k-1} \left[ \frac{2(1 - 3\delta)}{(2 + 3\delta)(1 - 3\alpha\delta)} \right]^r
= \frac{2(2\alpha + 3\alpha\delta - 2)}{-9 + 6\alpha + 9\alpha\delta} \left\{ \left[ \frac{2(1 - 3\delta)}{(2 + 3\delta)(1 - 3\alpha\delta)} \right]^{k-1} - 1 \right\}.
\]

Now fix \(K > L \geq 0\) with \(L/K\) rational. Then we may find arbitrarily large integers \(k\) and \(l\) such that \(K = 2k\delta\) and \(L = 2l\delta\) for some \(\delta > 0\). Letting \(k, l \to \infty\), we have \(\delta \to 0, \alpha \to 2^{\pm 1}\) and \((4.10)\) leads to
\[
\mathcal{C}(K) - \alpha \mathcal{B}(K) + \frac{4(\alpha - 1)}{-9 + 6\alpha} \geq \exp \left( \frac{(K - L)\left(-9 + 6\alpha \right)}{4} \right) \left[ \mathcal{C}(L) - \alpha \mathcal{B}(L) + \frac{4(\alpha - 1)}{-9 + 6\alpha} \right].
\]

Now we come back to the original functions \(B, C\). For \(\alpha = 2\), the inequality above takes form
\[
(4.11) \quad C(K) + K + 2 - \beta - 2B(K) \geq \exp \left( \frac{3}{4} (K - L) \right) [C(L) + L + 2 - \beta - 2B(L)],
\]
while for \(\alpha = 1/2\), we get
\[
(4.12) \quad 2C(K) - K + \beta - B(K) \geq \exp \left( \frac{-3}{2} (K - L) \right) [2C(L) - L + \beta - B(L)].
\]

**Step 3.** This is the final part. By \((4.2)\) and \((4.4)\), we have \(B(K) \geq K + \frac{1}{2} - \beta\) and \(B(K) \geq C(K)\). Plugging these estimates into \((4.11)\) we get that for any \(L,
\[
(4.13) \quad C(L) + L + 2 - \beta - 2B(L) \leq 0.
\]
Furthermore, the conditions \((4.3)\) and \((4.6)\) yield
\[
(4.14) \quad C(0) \geq \frac{2}{3} B(0) + \frac{1}{3} u(0, \frac{2}{3}) = \frac{2}{3} B(0) + \frac{1}{3} \left( \frac{2}{3} - \beta \right).
\]
Combining (4.14) with (4.13) applied to \( L = 0 \) gives

\[
0 \geq C(0) + 2 - \beta - 2B(0) \geq -\frac{4}{3}B(0) - \frac{4}{3}\beta + \frac{20}{9},
\]

which implies

\[
\beta + B(0) \geq \frac{5}{3}. \tag{4.15}
\]

The inequality (4.13), applied to \( L = 2/3 \), gives

\[
C(\frac{2}{3}) \leq \beta - \frac{8}{3}, \tag{4.16}
\]

since \( B(2/3) = 0 \), due to (4.5). Now use (4.12) for \( K = 2/3 \) and \( L = 0 \) to obtain

\[
2C(\frac{2}{3}) - \frac{2}{3} + \beta \geq \frac{1}{e}(2C(0) + \beta - B(0)).
\]

Combining this estimate with (4.14), (4.15) and (4.16) yields

\[
3\beta - 6 \geq \frac{1}{e}\left(\frac{5}{9} + \frac{4}{9}\right) = \frac{1}{e},
\]

or \( \beta \geq 2 + (3e)^{-1} \). This completes the proof of the sharpness of the inequality (1.2).

**Acknowledgement:** The results were obtained while the author was visiting Université de Franche-Comté in Besançon, France.

**References**


Department of Mathematics, Informatics and Mechanics, Warsaw University, Banacha 2, 02-097 Warsaw, Poland

Current address: Laboratoire de Mathematiques, Université de Franche-Comté, Rue de Gray 16, Besançon 25030 Cedex, France

E-mail address: ados@minuw.edu.pl