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SHARP MAXIMAL INEQUALITY FOR STOCHASTIC INTEGRALS

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ABSTRACT. Let $X = (X_t)_{t\geq 0}$ be a nonnegative supermartingale and $H = (H_t)_{t\geq 0}$ be a predictable process with values in $[-1,1]$. Let Y denote the stochastic integral of H with respect to X. The paper contains the proof of the sharp inequality

$$\sup_{t\geq 0} ||Y_t||_1 \leq \beta_0 || \sup_{t\geq 0} X_t||_1,$$

where $\beta_0 = 2 + (3e)^{-1} = 2,1226\ldots$. A discrete-time version of this inequality is also established.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, which is filtered by a nondecreasing right-continuous family $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-fields of $\mathcal{F}$. Assume that $\mathcal{F}_0$ contains all the events of probability 0. Suppose $X = (X_t)_{t\geq 0}$ is an adapted real-valued right-continuous semimartingale with left limits. Let Y be the Itô integral of H with respect to X,

$$Y_t = H_0 X_0 + \int_{(0,t]} H_s dX_s, \quad t \geq 0,$$

where H is a predictable process with values in $[-1,1]$. Let $||Y||_1 = \sup_{t\geq 0} ||Y_t||_1$ and $X^* = \sup_{t\geq 0} |X_t|$.

The objective of this paper is to compare the first moments of Y and X^*. In [4], Burkholder introduced a method of proving related maximal inequalities for martingales and obtained the following sharp estimate.

Theorem 1.1. *If X is a martingale and Y is as above, then we have*

$$||Y||_1 \leq \gamma ||X^*||_1,$$

where $\gamma = 2,536\ldots$ is the unique solution of the equation

$$\gamma - 3 = -\exp\left(\frac{1-\gamma}{2}\right).$$

The constant is the best possible.

Using Burkholder's techniques, we find the best constant in the inequality (1.1) in case X is a nonnegative supermartingale. The main result of the paper is the following.

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Theorem 1.2. *Suppose X is a nonnegative supermartingale and Y is as above. Then the inequality*

$$||Y||_1 \leq \beta_0 ||X^*||_1 \tag{1.1}$$

holds true with $\beta_0 = 2 + (3e)^{-1} = 2,1226\ldots$. The constant is the best possible. It is already the best possible if X is assumed to be a nonnegative martingale.

As usual, the inequality for stochastic integrals is accompanied by its discrete-time version. Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, equipped with filtration $(\mathcal{F}_n)_{n\geq 0}$. Let $f = (f_n)_{n\geq 0}$ be an adapted nonnegative supermartingale and $g = (g_n)_{n\geq 0}$ be its transform by a predictable sequence $v = (v_n)_{n\geq 0}$ bounded in absolute value by 1. That is,

$$f_n = \sum_{k=0}^{n} df_k, \qquad g_n = \sum_{k=0}^{n} v_k df_k, \qquad n = 0, 1, 2, \ldots.$$

By predictability of v we mean that v_0 is $\mathcal{F}_0$-measurable and for any $k \geq 1$, v_k is measurable with respect to $\mathcal{F}_{k-1}$. Let $f_n^* = \max_{k\leq n} f_k$ and $f^* = \sup_k f_k$.

A discrete-time version of Theorem 1.2 can be stated as follows.

Theorem 1.3. *Let f, g, β_0 be as above. Then we have*

$$||g||_1 \leq \beta_0 ||f^*||_1, \tag{1.2}$$

and the constant β_0 is the best possible. It is already the best possible if f is assumed to be a nonnegative martingale.

The paper is organized as follows. In the next section we describe the Burkholder's method. Section 3 is devoted to the proofs of the maximal inequalities. In the last section we complete the proofs of Theorem 1.2 and Theorem 1.3 by showing that the constant β_0 can not be replaced by a smaller one.

2. The upper class of functions

Throughout this section we deal with the discrete-time setting. We start with some reductions. Standard approximation arguments (see page 350 of [4]) show that it is enough to prove Theorem 1.3 under an additional assumption that the supermartingale f is simple, i.e. for any n the variable f_n takes only a finite number of values and there is N such that $f_N = f_{N+1} = f_{N+2} = \ldots$ with probability 1. Then, clearly, every transform g of f is also simple and the pointwise limits f_∞, g_∞ exist. Furthermore, with no loss of generality, we may restrict ourselves to the special transforms g (called ± 1 transforms), namely, those with all v_n being deterministic and taking values in $\{-1, 1\}$: see Lemma A.1 on page 60 in [3] and observe $(F^j)^* = f^*$ on page 61. Finally, note that in order to prove inequality (1.2), it suffices to show that for any f, g as above and any integer n we have

$$\mathbb{E}|g_n| \leq \beta_0 \mathbb{E} f_n^*.$$

To describe Burkholder's method, let us consider the following general problem, first in the martingale setting: let $D = [0, \infty) \times \mathbb{R} \times [0, \infty)$ and $V : D \to \mathbb{R}$ be any Borel function satisfying $V(x, y, z) = V(x, y, x \vee z)$. Suppose we want to prove the inequality

$$\mathbb{E} V(f_n, g_n, f_n^*) \leq 0 \tag{2.1}$$

for all nonnegative integers n and all pairs (f, g), where f is a simple nonnegative martingale and g is its ± 1 transform.

The key idea is to study the family $\mathcal{U}$ of all functions $U : D \to \mathbb{R}$ satisfying the following three properties.

$$U(x, y, z) = U(x, y, x \vee z) \quad \text{if } (x, y, z) \in D, \tag{2.2}$$

$$V(x, y, z) \leq U(x, y, z) \quad \text{if } (x, y, z) \in D \tag{2.3}$$

and, furthermore, if $(x, y, z) \in D$, $\varepsilon \in \{-1, 1\}$, $\alpha \in (0, 1)$ and $t_1, \ t_2 \geq -x$ with $\alpha t_1 + (1 - \alpha)t_2 = 0$, then

$$\alpha U(x + t_1, y + \varepsilon t_1, z) + (1 - \alpha)U(x + t_2, y + \varepsilon t_2, z) \leq U(x, y, z). \tag{2.4}$$

The interplay between the class $\mathcal{U}$ and the maximal inequality (2.1) is described in the theorem below. It is a simple modification of Theorems 2.2 and 2.3 in [4] (see also Section 11 in [2] and Theorem 2.1 in [3]) to the case of nonnegative supermartingales. We omit the proof, as it requires only some minor changes.

Theorem 2.1. *The inequality (2.1) holds for all n and all pairs (f, g) as above if and only if the class $\mathcal{U}$ is nonempty. Furthermore, if $\mathcal{U}$ is nonempty, then there exists the least element in $\mathcal{U}$, given by*

$$U^0(x, y, z) = \sup\{\mathbb{E}V(f_\infty, g_\infty, f^* \vee z)\}. \tag{2.5}$$

Here the supremum runs over all the pairs (f, g), where f is a simple nonnegative martingale, $\mathbb{P}((f_0, g_0) = (x, y)) = 1$ and $dg_k = \pm df_k$ almost surely for all $k \geq 1$.

In case f is assumed to be a nonnegative supermartingale, we can proceed in a similar manner. For a given V, consider the inequality (2.1). Suppose we want it to be valid for any n, any nonnegative supermartingale f and any ± 1 transform g. Let $\mathcal{U}'$ be a subclass of $\mathcal{U}$ containing those functions, which satisfy

$$U(x, y, z) \geq U(x - \delta, y \pm \delta, z) \quad \text{if } (x, y, z) \in D, \ \delta \in [0, x]. \tag{2.6}$$

The analogue of Theorem 2.1 is as follows (the straightforward proof is omitted).

Theorem 2.2. *The inequality (2.1) holds for all n and all pairs (f, g) as above if and only if the class $\mathcal{U}'$ is nonempty.*

Now we turn to (1.2) and assume from now on, that the function V is given by

$$V(x, y, z) = V(x, y, x \vee z) = y - \beta(x \vee z),$$

where $\beta > 0$ is a fixed number. The inequality (2.1) reads

$$\mathbb{E}|g_n| \leq \beta \mathbb{E} f_n^*. \tag{2.7}$$

Denote by $\mathcal{U}(\beta)$, $\mathcal{U}'(\beta)$ the classes $\mathcal{U}, \mathcal{U}'$ corresponding to this choice of V.

The rest of this section is devoted to the last part of Theorem 1.3. Let $\beta(possup)$ (resp. $\beta(posmar)$) be the smallest constant β in the inequality (2.7), when f is assumed to run over the class of all nonnegative supermartingales (resp. nonnegative martingales).

Theorem 2.3. *We have $\beta(posmar) = \beta(possup)$.*

Proof. We only need the inequality $\beta = \beta(posmar) \geq \beta(possup)$, as the reverse one is trivial. By Theorem 2.2, it suffices to prove that the class $\mathcal{U}'(\beta)$ is nonempty. Theorem 2.1 guarantees the existence of the minimal element U^0 of the class $\mathcal{U}(\beta)$, given by (2.5). By definition we get the following properties of U^0.

$$U^0(x,y,z) = U^0(x,-y,z), \tag{2.8}$$

$$U^0(1,-1,1) = U^0(1,1,1) \leq 0, \tag{2.9}$$

$$U^0(\alpha x, \alpha y, \alpha z) = \alpha U^0(x,y,z) \quad \text{for any } \alpha > 0. \tag{2.10}$$

The equality (2.8) is clear, (2.9) follows from the fact that for any pair (f,g) as in Theorem 2.1, starting from $(1,1)$ or from $(1,-1)$, we have that g is a ± 1 transform of f and therefore, by (2.7), we have $\mathbb{E}V(f_n, g_n, f_n^*) \leq 0$ for any n. For (2.10), we use the fact that V is homogeneous.

We will prove that the function $U : D \to \mathbb{R}$ given by

$$U(x,y,z) = U^0(x,y,z) - U^0(1,1,1)x \tag{2.11}$$

belongs to $\mathcal{U}'$. The conditions (2.2), (2.3) and (2.4) hold true for U, since they are satisfied for U^0 and, by (2.9), we have $U \geq U^0$. It remains to prove (2.6). Note that U satisfies $U(x,y,z) = U(x,-y,z)$, $U(1,-1,1) = U(1,1,1) = 0$ and is homogeneous. Fix $y \in \mathbb{R}$, $0 \leq x \leq z$, $\varepsilon \in \{-1,1\}$ and let $\delta \in (0,x]$, $t > z - x$. Use (2.4) with $t_1 = -\delta$, $t_2 = t$ and $\alpha = t/(t+\delta)$ to obtain

$$\frac{t}{t+\delta}U(x-\delta, y-\varepsilon\delta, z) + \frac{\delta}{t+\delta}U(x+t, y+\varepsilon t, z) \leq U(x,y,z).$$

By homogeneity of U, this gives

$$\frac{t}{t+\delta}U(x-\delta, y-\varepsilon\delta, z) + \frac{\delta(x+t)}{t+\delta}U\big(1, \frac{y+\varepsilon t}{x+t}, 1) \leq U(x,y,z). \tag{2.12}$$

Now we let $t \to \infty$; the inequality (2.6) will follow if we show that

$$\liminf_{s \to 1} U(1,s,1) \geq U(1,1,1) = 0. \tag{2.13}$$

For $s > 1$, use (2.4) with $x = z = 1$, $y = s$, $\varepsilon = -1$, $t_1 = -1$, $t_2 = (s-1)/2$ and get

$$U(1,s,1) \geq \frac{s-1}{s+1}U(0,s+1,1) + \frac{2}{s+1}U(\frac{s+1}{2}, \frac{s+1}{2}, \frac{s+1}{2}) \geq \frac{s-1}{s+1}(s+1-\beta),$$

the latter inequality being a consequence of (2.3) and the homogeneity of U. For $0 < s < 1$, apply (2.12) to $x = z = 1$, $y = s$, $\varepsilon = -1$, $\delta = (1-s)/2$ and $t = 2s/(1-s)$ (so that $(y+\varepsilon t)/(x+t) = -s$) to obtain

$$U(1,s,1) \geq \frac{2}{s+1}U(\frac{1+s}{2}, \frac{1+s}{2}, 1).$$

Now we use the fact that, by (2.4), the function $s \mapsto U(s,s,1)$ is concave and therefore continuous. This completes the proof of (2.13) and, in consequence, we have $U \in \mathcal{U}'(\beta)$, so this class is nonempty. All that is left is to use Theorem 2.2. □

Thus, to establish the inequality (1.2), we need to find an element U in $\mathcal{U}(\beta_0)$. This will be done in the next section.

3. The proofs of the inequalities (1.1) and (1.2)

Here we construct the special function U corresponding to the maximal inequality (1.2). This is the main section of the paper.

Let S denote the strip $[-1,1]\times\mathbb{R}$. Consider the following subsets of S.

$$D_1=\{(x,y):0\le x<\frac{2}{3},\ x+y\ge\frac{2}{3}\},$$
$$D_2=\{(x,y):\frac{2}{3}\le x\le 1,\ x-y\le\frac{2}{3}\},$$
$$D_3=\{(x,y):0\le x<\frac{2}{3},\ y\ge 0,\ x+y\le\frac{2}{3}\},$$
$$D_4=\{(x,y):\frac{2}{3}< x\le 1,\ y\ge 0,\ x-y>\frac{2}{3}\}.$$

Let the function u be defined on S by the condition $u(x,y)=u(|x|,|y|)$ and

$$u(x,y)=\begin{cases} y-\beta_0+x\{\exp[-\frac{3}{2}(y+x-\frac{2}{3})]+1\}, & (x,y)\in D_1,\\ y-\beta_0+(\frac{4}{3}-x)\exp[-\frac{3}{2}(y-x+\frac{2}{3})]+x, & (x,y)\in D_2,\\ y-\beta_0-x\log[\frac{3}{2}(x+y)]+2x, & (x,y)\in D_3,\\ -\beta_0-\frac{1}{3}(2-2x-y)(3-3x+3y)^{1/2}+\frac{14}{9}, & (x,y)\in D_4.\end{cases}$$

A function defined on the strip S is said to be *diagonally concave* if it is concave on the intersection of S with any line of slope 1 or -1. The proof of the following statement is just a matter of elementary calculations.

Lemma 3.1. *For any real number y we have*

$$u(0,y)=|y|-\beta_0,\quad u(1,y)\ge|y|-\beta_0, \tag{3.1}$$

$$u \text{ is diagonally concave}, \tag{3.2}$$

$$u(1,\cdot) \text{ is convex}, \tag{3.3}$$

$$u(1-\delta,y\pm\delta)\le u(1,y) \text{ for any } \delta\in[0,1], \tag{3.4}$$

$$u(1,1)=0. \tag{3.5}$$

Define $U:D\to\mathbb{R}$ by

$$U(x,y,z)=(x\vee z)u\Big(\frac{x}{x\vee z},\frac{y}{x\vee z}\Big).$$

We have the following statement.

Lemma 3.2. *The function U belongs to $\mathcal{U}(\beta_0)$.*

This fact can be proved exactly in the same manner as Lemma 3.1 in [4]. We omit the details. Now we are ready to prove the maximal inequalities.

Proof of the inequality (1.2): It is an immediate consequence of Theorem 2.1, Theorem 2.3 and Lemma 3.2. □

Proof of inequality (1.1): This follows by approximation argument. See Section 16 of [2], where it is shown how the result of Bichteler [1] can be used to deduce the estimates for stochastic integrals from their discrete-time versions. □.

4. Sharpness

Clearly, we need only to focus on the sharpness of (1.2), since it immediately implies that β_0 is also the best possible in (1.1).

Let $\beta = \beta(posmar)$. By Theorem 2.3, we need to prove $\beta \geq \beta_0$. This can be done by constructing an appropriate example. However, we take a different approach.

By Theorem 2.1, the class $\mathcal{U}(\beta)$ is nonempty, we can consider its minimal element U^0 and, as we have already proved, the function U given by (2.11) belongs to $\mathcal{U}'(\beta)$. Define $u : S \to \mathbb{R}$ by

$$u(x,y) = U(x,y,1). \tag{4.1}$$

The conditions (2.3), (2.4) and (2.6) imply that

$$u(x,y) \geq |y| - \beta, \tag{4.2}$$

$$u \text{ is diagonally concave}, \tag{4.3}$$

$$u(x,y) \geq u(x-\delta, y \pm \delta) \quad \text{for } \delta \in [0,x] \tag{4.4}$$

and, moreover, we have

$$u(1,1) = U(1,1,1) = 0. \tag{4.5}$$

Furthermore, note that for any y, by definition of U^0,

$$u(0,y) = U^0(0,y,1) = |y| - \beta, \tag{4.6}$$

since the only nonnegative martingale starting from 0 is the constant one.

We will show that the existence of u satisfying the properties (4.2) – (4.6) implies $\beta \geq \beta_0$. This will be done in several steps. Set $B(x) = u(1, x+1/3)$ and $C(x) = u\big(2/3, x\big)$.

Step 1. By properties (4.3) and (4.6), we have

$$u(\frac{2}{3}+\delta, 2k\delta+\delta) \geq (1-3\delta)C(2k\delta) + 3\delta B(2k\delta),$$

$$C((2k+2)\delta) \geq \frac{2}{2+3\delta} u(\frac{2}{3}+\delta, 2k\delta+\delta) + \frac{3\delta}{2+3\delta}(2k\delta+2\delta+\frac{2}{3}-\beta),$$

from which we deduce that

$$C((2k+2)\delta) \geq \frac{2(1-3\delta)}{2+3\delta}C(2k\delta) + \frac{6\delta}{2+3\delta}B(2k\delta) + \frac{3\delta}{2+3\delta}(2k\delta+2\delta+\frac{2}{3}-\beta). \tag{4.7}$$

Furthermore, (4.3) and (4.4) yield

$$B(2k\delta) \geq u(1-\delta, 2k\delta+\delta+\frac{1}{3}) \geq (1-3\delta)B((2k+2)\delta) + 3\delta C((2k+2)\delta).$$

Multiply this inequality throughout by $\alpha > 0$ and add it to (4.7). We obtain

$$\begin{aligned} &C((2k+2)\delta)(1-3\alpha\delta) - \alpha(1-3\delta)B((2k+2)\delta) \\ &\quad \geq \frac{2(1-3\delta)}{2+3\delta}C(2k\delta) - \big(\alpha - \frac{6\delta}{2+3\delta}\big)B(2k\delta) + \frac{3\delta}{2+3\delta}((2k+2)\delta+\frac{2}{3}-\beta), \end{aligned}$$

or, equivalently, after substitution

$$\overline{B}(t) = B(t) - t - \frac{2}{3} + \beta, \quad \overline{C}(t) = C(t) - t - \frac{2}{3} + \beta, \tag{4.8}$$

we get

$$
\begin{aligned}
&\overline{C}((2k+2)\delta) - \frac{\alpha(1-3\delta)}{1-3\alpha\delta}\overline{B}((2k+2)\delta) \\
&\quad \geq \frac{2(1-3\delta)}{(2+3\delta)(1-3\alpha\delta)}\Big[\overline{C}(2k\delta) - \frac{2\alpha+3\alpha\delta-6\delta}{2(1-3\delta)}\overline{B}(2k\delta)\Big] \\
&\qquad + \frac{2\delta}{1-3\alpha\delta}(\alpha - \frac{2}{2+3\delta}).
\end{aligned} \tag{4.9}
$$

Step 2. Now we will use the inequality (4.9) several times. The choice

$$\alpha = \frac{5 \pm \sqrt{9-24\delta}}{2(2+3\delta)}$$

gives

$$\frac{\alpha(1-3\delta)}{1-3\alpha\delta} = \frac{2\alpha+3\alpha\delta-6\delta}{2(1-3\delta)}$$

and using (4.9) for $k-1$, $k-2$, ..., l yields

$$
\begin{aligned}
&\overline{C}(2k\delta) - \frac{\alpha(1-3\delta)}{1-3\alpha\delta}\overline{B}(2k\delta) \\
&\quad \geq \Big[\frac{2(1-3\delta)}{(2+3\delta)(1-3\alpha\delta)}\Big]^{k-l}\Big[\overline{C}(2l\delta) - \frac{\alpha(1-3\delta)}{1-3\alpha\delta}\overline{B}(2l\delta)\Big] + \eta,
\end{aligned} \tag{4.10}
$$

where

$$
\begin{aligned}
\eta &= \frac{2\delta}{1-3\alpha\delta}(\alpha - \frac{2}{2+3\delta})\sum_{r=0}^{k-l-1}\Big[\frac{2(1-3\delta)}{(2+3\delta)(1-3\alpha\delta)}\Big]^r \\
&= \frac{2(2\alpha+3\delta\alpha-2)}{-9+6\alpha+9\alpha\delta}\Big\{\Big[\frac{2(1-3\delta)}{(2+3\delta)(1-3\alpha\delta)}\Big]^{k-l} - 1\Big\}.
\end{aligned}
$$

Now fix $K > L \geq 0$ with L/K rational. Then we may find arbitrarily large integers k and l such that $K = 2k\delta$ and $L = 2l\delta$ for some $\delta > 0$. Letting $k,\ l \to \infty$, we have $\delta \to 0$, $\alpha \to 2^{\pm 1}$ and (4.10) leads to

$$\overline{C}(K) - \alpha\overline{B}(K) + \frac{4(\alpha-1)}{-9+6\alpha} \geq \exp\big(\frac{(K-L)(-9+6\alpha)}{4}\big)\big[\overline{C}(L) - \alpha\overline{B}(L) + \frac{4(\alpha-1)}{-9+6\alpha}\big].$$

Now we come back to the original functions B, C. For $\alpha = 2$, the inequality above takes form

$$C(K) + K + 2 - \beta - 2B(K) \geq \exp(\frac{3}{4}(K-L))[C(L) + L + 2 - \beta - 2B(L)], \tag{4.11}$$

while for $\alpha = 1/2$, we get

$$2C(K) - K + \beta - B(K) \geq \exp(-\frac{3}{2}(K-L))[2C(L) - L + \beta - B(L)]. \tag{4.12}$$

Step 3. This is the final part. By (4.2) and (4.4), we have $B(K) \geq K + \frac{1}{3} - \beta$ and $B(K) \geq C(K)$. Plugging these estimates into (4.11) we get that for any L,

$$C(L) + L + 2 - \beta - 2B(L) \leq 0. \tag{4.13}$$

Furthermore, the conditions (4.3) and (4.6) yield

$$C(0) \geq \frac{2}{3}B(0) + \frac{1}{3}u(0, -\frac{2}{3}) = \frac{2}{3}B(0) + \frac{1}{3}(\frac{2}{3} - \beta). \tag{4.14}$$

Combining (4.14) with (4.13) applied to $L = 0$ gives

$$0 \geq C(0) + 2 - \beta - 2B(0) \geq -\frac{4}{3}B(0) - \frac{4}{3}\beta + \frac{20}{9},$$

which implies

$$\beta + B(0) \geq \frac{5}{3}. \tag{4.15}$$

The inequality (4.13), applied to $L = 2/3$, gives

$$C(\frac{2}{3}) \leq \beta - \frac{8}{3}, \tag{4.16}$$

since $B(2/3) = 0$, due to (4.5). Now use (4.12) for $K = 2/3$ and $L = 0$ to obtain

$$2C(\frac{2}{3}) - \frac{2}{3} + \beta \geq \frac{1}{e}(2C(0) + \beta - B(0)).$$

Combining this estimate with (4.14), (4.15) and (4.16) yields

$$3\beta - 6 \geq \frac{1}{e}(\frac{5}{9} + \frac{4}{9}) = \frac{1}{e},$$

or $\beta \geq 2 + (3e)^{-1}$. This completes the proof of the sharpness of the inequality (1.2).

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