SHARP MAXIMAL INEQUALITY FOR MARTINGALES AND STOCHASTIC INTEGRALS

ADAM OSĘKOWSKI

Abstract. Let $X = (X_t)_{t\geq 0}$ be a semimartingale and $H = (H_t)_{t\geq 0}$ be a predictable process taking values in $[-1, 1]$. Let Y denote the stochastic integral of H with respect to X. We show that if X is a martingale, then

$$||\sup_{t\geq 0} Y_t||_1 \leq \beta_0 ||\sup_{t\geq 0} |X_t|||_1,$$

where $\beta_0 = 2,0856\ldots$ is the best possible. Furthermore, if X is assumed to be a nonnegative supermartingale, then

$$||\sup_{t\geq 0} Y_t||_1 \leq \beta_0^+ ||\sup_{t\geq 0} X_t||_1,$$

where $\beta_0^+ = \frac{14}{9}$ is the best possible.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, which is filtered by a nondecreasing right-continuous family $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-fields of $\mathcal{F}$. Assume that $\mathcal{F}_0$ contains all the events of probability 0. Suppose $X = (X_t)_{t\geq 0}$ is an adapted real-valued right-continuous semimartingale with left limits. Let Y be the Itô integral of H with respect to X,

$$Y_t = H_0 X_0 + \int_{(0,t]} H_s dX_s, \quad t \geq 0,$$

where H is a predictable process with values in $[-1, 1]$. Let $||Y||_1 = \sup_{t\geq 0} ||Y_t||_1$ and $X^* = \sup_{t\geq 0} X_t$.

The main interest of this paper is in the comparison of the sizes of Y^* and $|X|^*$. Let us first describe two related results from the literature. In [4], Burkholder introduced a method of proving maximal inequalities for martingales and obtained the following sharp estimate.

Theorem 1.1. *If X is a martingale and Y is as above, then we have*

$$||Y||_1 \leq \gamma || \, |X|^* ||_1, \tag{1.1}$$

where $\gamma = 2,536\ldots$ is the unique solution of the equation

$$\gamma - 3 = -\exp\left(\frac{1-\gamma}{2}\right).$$

The constant is the best possible.

2000 *Mathematics Subject Classification.* Primary: 60G42. Secondary: 60G44.

Key words and phrases. Martingale, stochastic integral, maximal function.

Partially supported by MEiN Grant 1 PO3A 012 29 and Foundation for Polish Science.

It was then proved by the author in [5], that if X is positive, then the optimal constant γ in (1.1) equals $2 + (3e)^{-1} = 2,1226\ldots$.

We study here a related estimate with Y is replaced by its one-sided supremum:

$$||Y^*||_1 \leq \beta|| \, |X|^*||_1. \tag{1.2}$$

Let $\beta_0 = 2,0856\ldots$ be the positive solution to the equation

$$2\log\left(\frac{8}{3} - \beta_0\right) = 1 - \beta_0$$

and $\beta_0^+ = \frac{14}{9} = 1,555\ldots$. The main result of the paper can be stated as follows.

Theorem 1.2. *(i) If X is a martingale and Y is as above, then* (1.2) *holds with $\beta = \beta_0$ and the inequality is sharp.*

(ii) If X is a nonnegative supermartingale and Y is as above, then (1.2) *holds with $\beta = \beta_0^+$ and the constant is the best possible. It is already the best even if X is assumed to be a positive martingale.*

To prove this theorem, we establish first its discrete-time version. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, equipped with filtration $(\mathcal{F}_n)_{n\geq 0}$. Adding σ-field $\mathcal{F}_{-1}$ if necessary, we may assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Let $f = (f_n)_{n\geq 0}$ be an adapted sequence of integrable variables and $g = (g_n)_{n\geq 0}$ be its transform by a predictable sequence $v = (v_n)_{n\geq 0}$ bounded in absolute value by 1. That is, for any $n = 0, 1, 2, \ldots$ we have

$$f_n = \sum_{k=0}^{n} df_k \quad \text{and} \quad g_n = \sum_{k=0}^{n} v_k df_k.$$

By predictability of v we mean that v_0 is $\mathcal{F}_0$-measurable (and hence deterministic) and for any $k \geq 1$, v_k is measurable with respect to $\mathcal{F}_{k-1}$. In the special case when each v_k is deterministic and takes values in $\{-1, 1\}$ we will say that g is a ± 1 transform of f. Let $f_n^* = \max_{k\leq n} f_k$ and $f^* = \sup_k f_k$.

A discrete-time version of Theorem 1.2 is the following.

Theorem 1.3. *Let f, g, β_0, β_0^+ be as above.*

(i) If f is a martingale, then

$$||g^*||_1 \leq \beta_0|||f|^*||_1, \tag{1.3}$$

and the constant β_0 is the best possible.

(ii) If f is a nonnegative supermartingale, then

$$||g^*||_1 \leq \beta_0^+||f^*||_1, \tag{1.4}$$

and the constant β_0^+ is the best possible. It is already the best possible if f is assumed to be a nonnegative martingale.

A few words about the organization of the paper. The proof of Theorem 1.3 is based on Burkholder's technique, which reduces the problem of proving a martingale inequality to finding a certain special function. The description of this technique can be found in Section 2. Then, in the following two sections we provide the special functions corresponding to (1.3) and (1.4) and study their properties. In the last section we complete the proofs of Theorem 1.2 and Theorem 1.3 by showing that the constants β_0 and β_0^+ can not be replaced by smaller ones.

2. Burkholder's method

Throughout this section we deal with discrete-time setting. Let us start with some standard reductions. Assume f, g are as in the statement of Theorem 1.3. With no loss of generality we may assume that the process f is *simple*: for any integer n the random variable f_n takes only a finite number of values and there exists a number N such that $f_N = f_{N+1} = \dots$ with probability 1. Furthermore, it suffices to prove Theorem 1.3 for ± 1 transforms. To see this, let us consider the following extension of the Lemma A.1 from [4]. The proof is identical as in the original setting and hence it is omitted.

Lemma 2.1. *Let g be the transform of a martingale (resp., nonnegative martingale, nonnegative supermartingale) f by a real-valued predictable sequence v uniformly bounded in absolute value by 1. Then there exist martingales (resp., nonnegative martingales, nonnegative supermartingales) $F^j = (F^j_n)_{n\geq 0}$ and Borel measurable functions $\phi_j : [-1,1] \to \{-1,1\}$ such that, for $j \geq 1$ and $n \geq 0$,*

$$f_n = F^j_{2n+1}, \quad |f|^* = |F^j|^*,$$

$$g_n = \sum_{j=1}^{\infty} 2^{-j} \phi_j(v_0) G^j_{2n+1},$$

where G^j is the transform of F^j by $\varepsilon = (\varepsilon_k)_{k\geq 0}$ with $\varepsilon_k = (-1)^k$.

Suppose we have established Theorem 1.3 for ± 1 transforms and let β denote β_0 or β_0^+, depending on whether f is a martingale or nonnegative supermartingale. Lemma 2.1 gives us the processes F^j and the functions ϕ_j, $j \geq 1$. As v_0 is deterministic, for any $j \geq 1$ the sequence $\phi_j(v_0)G^j$ is a ± 1 transform of F^j and hence we may write

$$||g^*||_1 \leq \left|\left| \sum_{j=1}^{\infty} 2^{-j} \sup_n \left(\phi_j(v_0) G^j_{2n+1}\right) \right|\right|_1 \leq \sum_{j=1}^{\infty} 2^{-j} \left|\left| \left(\phi_j(v_0) G^j\right)^* \right|\right|_1$$

$$\leq \beta \sum_{j=0}^{\infty} 2^{-j} |||F^j|^*||_1 = \beta |||f|^*||_1.$$

The final reduction is that it suffices to prove that for any integer n we have

$$\mathbb{E}\left[g_n^* - \beta |f_n|^*\right] \leq 0. \tag{2.1}$$

To establish the above estimate, consider the following general problem, first in the martingale setting. Let $D = \mathbb{R} \times \mathbb{R} \times (0,\infty) \times (0,\infty)$ and $V : D \to \mathbb{R}$ be a Borel function. Suppose we want to prove the inequality

$$\mathbb{E} V(f_n, g_n, |f_n|^*, g_n^*) \leq 0 \tag{2.2}$$

for any integer n, any martingale f and g being its ± 1 transform.

The key idea is to study the family $\mathcal{U}$ of all functions $U : D \to \mathbb{R}$ satisfying the following three properties.

$$U(x,y,z,w) = U(x,y,|x| \vee z, y \vee w), \quad \text{if } (x,y,z,w) \in D, \tag{2.3}$$

$$V(x,y,z,w) \leq U(x,y,z,w), \quad \text{if } (x,y,z,w) \in D \tag{2.4}$$

and, furthermore,

$$\begin{aligned}\alpha U(x+t_1,y+\varepsilon t_1,z,w)+(1-\alpha)U(x+t_2,y+\varepsilon t_2,z,w)\leq U(x,y,z,w),\\ \text{for any } |x|\leq z,\ y\leq w,\ \varepsilon\in\{-1,1\},\ \alpha\in(0,1) \text{ and } t_1,\ t_2\\ \text{with } \alpha t_1+(1-\alpha)t_2=0.\end{aligned}\tag{2.5}$$

The relation between the class $\mathcal{U}$ and the estimate (2.2) is described in the following theorem. It is a simple modification of Theorems 2.2 and 2.3 in [4] (see also Section 11 in [2] and Theorem 2.1 in [3]). We omit the proof.

Theorem 2.2. *The inequality (2.2) holds for all n and all pairs (f,g) as above if and only if the class $\mathcal{U}$ is nonempty. Furthermore, if $\mathcal{U}$ is nonempty, then there exists the least element in $\mathcal{U}$, given by*

$$U_0(x,y,z,w)=\sup\{\mathbb{E}V(f_\infty,g_\infty,|f|^*\vee z,g^*\vee w)\}.\tag{2.6}$$

Here the supremum runs over all the pairs (f,g), where f is a simple martingale, $\mathbb{P}((f_0,g_0)=(x,y))=1$ and $dg_k=\pm df_k$ almost surely for all $k\geq 1$.

A similar statement is valid when we want the inequality (2.2) to hold for any nonnegative martingale f and its ±1 transform g. Let $D^+=[0,\infty)\times\mathbb{R}\times(0,\infty)\times(0,\infty)$ and let $\mathcal{U}^+$ denote the class of functions $U:D^+\to\mathbb{R}$ satisfying (2.3), (2.4) and (2.5) (with D replaced by D^+ and, in (2.5), an extra assumption $t_1,\ t_2\geq -x$).

Theorem 2.3. *The inequality (2.2) holds for all n and all pairs (f,g) as above if and only if the class $\mathcal{U}^+$ is nonempty. Furthermore, if $\mathcal{U}^+$ is nonempty, then there exists the least element in $\mathcal{U}^+$, given by*

$$U_0^+(x,y,z,w)=\sup\{\mathbb{E}V(f_\infty,g_\infty,|f|^*\vee z,g^*\vee w)\}.\tag{2.7}$$

Here the supremum runs over all the pairs (f,g), where f is a simple nonnegative martingale, $\mathbb{P}((f_0,g_0)=(x,y))=1$ and $dg_k=\pm df_k$ almost surely for all $k\geq 1$.

In the case when f is assumed to be a nonnegative supermartingale we need to impose another condition on the special functions. Let $\overline{\mathcal{U}^+}$ be a subclass of $\mathcal{U}^+$ such that if $U\in\overline{\mathcal{U}^+}$, then

$$U(x,y,z,w)\geq U(x-\delta,y\pm\delta,z,w),\quad \text{if } (x,y,z,w)\in D^+,\ \delta\in[0,x].\tag{2.8}$$

Here is the analogue of Theorems 2.2 and 2.4. Again, we omit the straightforward proof.

Theorem 2.4. *The inequality (2.2) holds for all n and all pairs (f,g) as above if and only if the class $\overline{\mathcal{U}^+}$ is nonempty.*

Let us now turn to (1.3) and assume, from now on, that the function V is given by

$$V(x,y,z,w)=V(x,y,|x|\vee z,y\vee w)=y\vee w-\beta(x\vee z),$$

where $\beta>0$ is a fixed number. Denote by $\mathcal{U}(\beta)$, $\mathcal{U}^+(\beta)$ and $\overline{\mathcal{U}^+}(\beta)$ the classes $\mathcal{U}$, $\mathcal{U}^+$ and $\overline{\mathcal{U}^+}$ corresponding to this choice of V. The purpose of the next two sections is to show that the classes $\mathcal{U}(\beta_0)$ and $\overline{\mathcal{U}^+}(\beta_0^+)$ are nonempty.

3. The special function: a general case

We start with the class $\mathcal{U}(\beta_0)$. Let us introduce an auxiliary parameter. The equation

$$2\log\left(2-\frac{2}{3a}\right)=\frac{a-2}{3a}, \qquad a>\frac{1}{3}, \tag{3.1}$$

has a unique solution $a=0.46986\ldots$, related to β_0 by the identity

$$\beta_0=\frac{2a+2}{3a}.$$

Let S denote the strip $[-1,1]\times(-\infty,0]$ and consider the following subsets of S.

$$\begin{aligned} D_1 &= \{(x,y):|x|+y>0\},\\ D_2 &= \{(x,y):0\geq |x|+y>1-\beta_0\},\\ D_3 &= \{(x,y):|x|+y\leq 1-\beta_0\}.\end{aligned}$$

Introduce the special function $u:S\to\mathbb{R}$ by

$$u(x,y)=\begin{cases} a(2|x|-y-2)(1-|x|-y)^{1/2}-3a|x|+y & \text{if } (x,y)\in D_1,\\ 3a(2-|x|)\exp(\frac{1}{2}(|x|+y))+(1-3a)y-8a & \text{if } (x,y)\in D_2,\\ \frac{9a^2}{4(3a-1)}(1-|x|)\exp(|x|+y)-\beta_0 & \text{if } (x,y)\in D_3.\end{cases}$$

A function defined on the strip S is said to be *diagonally concave* if it is concave on the intersection of S with any line of slope 1 or -1. We have the following fact.

Lemma 3.1. *The function u has the following properties.*

$$u(1,\cdot) \text{ is convex,} \tag{3.2}$$

$$u(1,y)\geq -\beta_0, \tag{3.3}$$

$$u(x,0)\geq -\beta_0, \tag{3.4}$$

$$u \text{ is diagonally concave.} \tag{3.5}$$

Proof. It is easy to check that u is of class C^1 in the interior of S. Now the condition (3.2) is apparent and hence so is (3.3). To see that (3.4) holds, note that

$$u(x,0)=-a(2(1-|x|)^{3/2}+3|x|)$$

attains its minimum $-2a>-\beta_0$ at $x=0$. Due to the symmetry, it suffices to check the diagonal concavity of u restricted to the set $(0,1)\times(-\infty,0)$. This is obvious on the lines of slope -1. On the remaining lines, fix $(x,y)\in(0,1)\times(-\infty,0)$ and introduce the function F by $F(t)=u(x+t,y+t)$ for t belonging to a certain open interval containing 0. Denoting by A^o the interior of a set A, we easily check that

$$F''(0)=\begin{cases} 3ay(1-x-y)^{-3/2} & \text{if } (x,y)\in D_1^o,\\ -3ax\exp(\frac{1}{2}(x+y)) & \text{if } (x,y)\in D_2^o,\\ -\frac{9a^2}{3a-1}x\exp(x+y) & \text{if } (x,y)\in D_3^o\end{cases}$$

is nonpositive. This completes the proof. $\square$

Define $U:\mathbb{R}\times\mathbb{R}\times(0,\infty)\times(0,\infty)\to\mathbb{R}$ by

$$U(x,y,z,w)=y\vee w+(|x|\vee z)u\left(\frac{x}{|x|\vee z},\frac{y-(y\vee w)}{|x|\vee z}\right). \tag{3.6}$$

We have

Lemma 3.2. *The function U belongs to $\mathcal{U}(\beta_0)$.*

Proof. The condition (2.3) follows from the definition of U. The inequality (2.4) is equivalent to $u \geq -\beta_0$ on the whole strip S, an estimate which follows directly from (3.3), (3.4) and (3.5). The main technical difficulty lies in proving (2.5). Let us start with some reductions. First, we may assume $\varepsilon = 1$, as $U(x,y,z,w) = U(-x,y,z,w)$. Secondly, by homogeneity, it is enough to show (2.5) for $z = 1$. Finally, we may set $w = 0$, since $U(x,y,z,w) = U(x,y-w,z,0) + w$. Now fix $(x,y) \in S$ and introduce the function $\Phi : \mathbb{R} \to \mathbb{R}$ by $\Phi(t) = U(x+t, y+t, 1, 0)$. The condition (2.5) will follow if we show that there exists a concave function Ψ on $\mathbb{R}$ such that $\Phi \leq \Psi$ and $\Phi(0) = \Psi(0)$. The existence will be a consequence of the properties (3.7) – (3.11) below.

$$\Phi \text{ is continuous,} \tag{3.7}$$

$$\Phi \text{ is concave on } [-1-x, 1-x], \tag{3.8}$$

$$\Phi \text{ is convex on } (-\infty, -1-x] \text{ and on } [1-x, \infty), \tag{3.9}$$

$$\lim_{t \to -\infty} \Phi'(t) \geq \lim_{t \downarrow -1-x} \Phi'(t), \tag{3.10}$$

$$\lim_{t \to \infty} \Phi'(t) \leq \lim_{t \uparrow 1-x} \Phi'(t). \tag{3.11}$$

The property (3.7) is straightforward to check. If $1 - x \leq -y$, then the condition (3.8) follows from (3.5). If $1 - x > -y$ then (3.5) gives the concavity only on $[-1-x, -y]$, but for $t \in (-y, 1-x)$ we have

$$\Phi(t) = y + t - a(2(1 - |x+t|)^{3/2} + 3|x+t|),$$

so $\Phi''(t) < 0$. In addition, one-sided derivatives of Φ match at $-y$ and we are done.

To show (3.9), fix $\alpha_1,\ \alpha_2 > 0$ satisfying $\alpha_1 + \alpha_2 = 1$, choose $t_1, t_2 \in (-\infty, -1-x]$ and let $t = \sum \alpha_k t_k$. We have

$$\begin{aligned}
\sum \alpha_k \Phi(t_k) &= \sum \alpha_k U(x+t_k, y+t_k, 1, 0) \\
&= \sum \alpha_k \left[(-x-t_k) u \left(-1, \frac{y+t_k}{-x-t_k} \right) \right] \\
&= -(x+t) \sum \frac{\alpha_k (x+t_k)}{x+t} u \left(1, \frac{y+t_k}{-x-t_k} \right)
\end{aligned}$$

By (3.2), this can be bounded from below by

$$-(x+t) u \left(1, \sum \frac{y+t_k}{-x-t_k} \cdot \frac{\alpha_k(x+t_k)}{x+t} \right) = -(x+t) u \left(1, -\frac{y+t}{x+t} \right) = \Phi(t).$$

Hence Φ is convex on $(-\infty, -1-x]$. If $1-x < -y$, then convexity on $[1-x, -y]$ can be established exactly in the same manner. Furthermore, for $t > \max\{1-x, -y\}$ we have

$$\Phi(t) = y - 3ax + (1-3a)t \tag{3.12}$$

and one-sided derivatives of Φ are equal at $\max\{1-x, -y\}$. Thus (3.9) follows.

To prove (3.10), note that the limit on the left equals $1+2a$, while the one on the right equals

$$\begin{aligned}
&3a-\frac{3}{2}a(-y+1+x)^{1/2}\leq 3a, && \text{if } -x+y\geq 0,\\
&\frac{3a}{2}\exp(\frac{1}{2}(y-x))\leq\frac{3a}{2}, && \text{if } 1-\beta_0\leq -x+y<0,\\
&\frac{9a^2}{4(3a-1)}\exp(y-x)\leq 3a-1, && \text{if } -x+y<1-\beta_0.
\end{aligned}$$

Finally, let us turn to (3.11). The limit on the left is equal to $1-3a$, due to (3.12). If $-x+y\geq -1+\beta_0$, then the right hand side is also $1-3a$; for $-x+y\leq -1+\beta_0$ the inequality (3.11) becomes

$$-\frac{9a^2}{2(3a-1)}\exp(2-x+y)\geq 1-3a,$$

which is a consequence of the fact that the left hand side is a nonincreasing function of y and both sides are equal for $-x+y=-1+\beta_0$. □

4. The special function for the case of positive supermartingales

Let S^+ denote the strip $[0,1]\times(-\infty,0]$ and let

$$\begin{aligned}
D_1 &= \left\{(x,y)\in S^+ : x-y>\tfrac{2}{3},\ x\leq\tfrac{2}{3}\right\},\\
D_2 &= \left\{(x,y)\in S^+ : x+y<\tfrac{2}{3},\ x>\tfrac{2}{3}\right\},\\
D_3 &= \left\{(x,y)\in S^+ : x+y\geq\tfrac{2}{3}\right\},\\
D_4 &= \left\{(x,y)\in S^+ : x-y\leq\tfrac{2}{3}\right\}.
\end{aligned}$$

Introduce the function $u^+ : S^+\to\mathbb{R}$ by

$$u^+(x,y)=\begin{cases} x\exp[\frac{3}{2}(-x+y)+1]-\beta_0^+, & \text{if } (x,y)\in D_1,\\ (\frac{4}{3}-x)\exp[\frac{3}{2}(x+y)-1]-\beta_0^+, & \text{if } (x,y)\in D_2,\\ -x+y-\frac{1}{\sqrt{3}}(1-x-y)^{1/2}(2-2x+y), & \text{if } (x,y)\in D_3,\\ x-x\log(\frac{3}{2}(x-y))-\beta_0^+, & \text{if } (x,y)\in D_4.\end{cases}$$

Here is the analogue of Lemma 3.1.

Lemma 4.1. *The function u^+ has the following properties.*

$$u^+(1,\cdot)\ \textit{is convex,} \tag{4.1}$$

$$u^+(1,y)\geq-\beta_0^+\ \textit{for}\ y\leq 0, \tag{4.2}$$

$$u^+(x,0)\geq-\beta_0^+\ \textit{for}\ x\in[0,1]. \tag{4.3}$$

$$u^+\ \textit{is diagonally concave.} \tag{4.4}$$

Proof. It is not difficult to check that u^+ has continuous partial derivatives in the interior of S^+. Now the properties (4.1) and (4.2) are easy to see. To show (4.3) observe that the function $u^+(\cdot,0)$ is concave on $[0,1]$ and $u^+(0,0)=-\beta_0^+<u^+(1,0)$. Finally, it is obvious that u^+ is concave along the lines of slope 1 on

$D_1 \cup D_4$, and along the lines of slope -1 on $D_2 \cup D_3$. For $x \in D_1^o \cup D_4^o$, let $F_-(t) = u(x+t, y-t)$ and derive that

$$F_-''(0) = \begin{cases} (9x-6)\exp[\frac{3}{2}(-x+y)+1], & \text{if } (x,y) \in D_1^o, \\ 4y(x-y)^{-2}, & \text{if } (x,y) \in D_2^o, \end{cases}$$

so $F_-''(0) < 0$. Similarly, for $x \in D_2^o \cup D_3^o$, introduce $F_+(t) = u(x+t, y+t)$ and check that

$$F_+''(0) = \begin{cases} (-9x+6)\exp[\frac{3}{2}(x+y)-1], & \text{if } (x,y) \in D_2^o, \\ \sqrt{3}y(1-x-y)^{-3/2}, & \text{if } (x,y) \in D_3^o, \end{cases}$$

which gives $F_+''(0) < 0$. This completes the proof. $\square$

Now we define the special function $U^+ : D^+ \to \mathbb{R}$ by the same formula as in (3.6), namely

$$U^+(x,y,z,w) = y \vee w + (|x| \vee z)u^+\left(\frac{x}{|x| \vee z}, \frac{y-(y\vee w)}{|x| \vee z}\right). \tag{4.5}$$

The following is the analogue of Lemma 3.2.

Lemma 4.2. *The function U^+ belongs to $\overline{\mathcal{U}^+}(\beta_0^+)$.*

Proof. The condition (2.3) is immediate, while (2.4) follows from (4.2), (4.3), (4.4) and the equality $u^+(0,y) = -\beta_0^+$. Again, the condition (2.5) is the most elaborate. As previously, we may assume $z = 1$ and $w = 0$. We fix $\varepsilon \in \{-1, 1\}$ $x \in [0,1]$, $y \in (-\infty, 0]$, introduce the function $\Phi(t) = U^+(x+t, y+\varepsilon t, 1, 0)$ (given for $t \geq -x$) and observe that (2.5) follows from existence of a concave function Ψ satisfying $\Psi \geq \Phi$ and $\Psi(0) = \Phi(0)$. Let us list the properties of Φ which imply the existence. The proof is left to the reader.

$$\Phi \text{ is continuous,} \tag{4.6}$$

$$\Phi \text{ is concave on } [-x, 1-x]. \tag{4.7}$$

$$\Phi \text{ is convex on } (1-x, \infty), \tag{4.8}$$

$$\lim_{t \uparrow 1-x} \Phi'(t) \geq \lim_{t \to \infty} \Phi'(t). \tag{4.9}$$

$\square$

Proof of Theorem 1.2. This follows from Theorem 1.3 and the approximation result by Bichteler [1]. For similar argumentation, see [2]. $\square$

5. Optimality of the constants

In this section we prove that the inequalities (1.3) and (1.4) are sharp.

The constant β_0 is optimal in (1.3). By Theorem 2.2, the class $\mathcal{U}(\beta_0)$ is nonempty and let U_0 denote its minimal element. This function enjoys the following properties.

$$U_0(tx, ty, tz, tw) = tU_0(x,y,z,w) \quad \text{for } t > 0,$$

and

$$U_0(x,y,z,w) = U_0(x, y+t, z, w+t) - t \quad \text{for } t > -w.$$

Introduce the functions A, $B : (-\infty, 0] \to \mathbb{R}$, $C : [0,1] \to \mathbb{R}$ by

$$A(y) = U_0(0,y,1,0), \quad B(y) = U_0(1,y,1,0) = U_0(-1,y,1,0), \quad C(x) = U_0(x,0,1,0).$$

Step 1. We start with the observation that for $x \in (0,1]$ and $\delta \in (0,x]$, the property (2.5) gives

$$\begin{aligned} C(x) &\geq \frac{2\delta}{1-x+2\delta}B(x-1)+\frac{1-x}{1-x+2\delta}(C(x-2\delta)+\delta) \\ &\geq \frac{2\delta}{1-x+2\delta}B(x-1-2\delta)+\frac{1-x}{1-x+2\delta}(C(x-2\delta)+\delta), \end{aligned}$$

where the latter inequality follows from the fact that B is nondecreasing (by the very definition). Furthermore,

$$B(x-1)\geq \delta+\delta B(0)+\frac{\delta}{1-x+2\delta}C(x-2\delta)+\frac{1-x+\delta}{1-x+2\delta}B(x-1-2\delta).$$

Equivalenty,

$$\begin{aligned} C(x)-C(x-2\delta) &\geq 2\delta\left[\frac{B(x-1-2\delta)}{1-x+2\delta}-\frac{C(x-2\delta)}{1-x+2\delta}\right]+\frac{2\delta(1-x)}{1-x+2\delta}, \\ 2B(x-1)-2B(x-1-2\delta) &\geq 2\delta\left[\frac{C(x-2\delta)}{1-x+2\delta}-\frac{B(x-1-2\delta)}{1-x+2\delta}\right]+2\delta(1+B(0)). \end{aligned}$$

Adding the two estimates above gives

$$C(x)+2B(x-1)-C(x-2\delta)-2B(x-1-2\delta)\geq 2\delta(2+B(0))-\frac{4\delta^2}{1-x+2\delta}. \tag{5.1}$$

Now fix an integer n, substitute $\delta = 1/(2n)$, $x = k/n$, $k = 1, 2, \ldots, n$ and sum these inequalities; we get

$$C(1)+2B(0)-C(0)-2B(-1)\geq 2+B(0)-\frac{1}{n^2}\sum_{k=1}^{n}\frac{1}{1-\frac{k-1}{n}}.$$

Passing to the limit $n\to\infty$ and using the equalities $C(1)=B(0)$, $C(0)=A(0)$ we arrive at

$$2B(0)-A(0)-2B(-1)\geq 2. \tag{5.2}$$

Step 2. Now we will show that

$$A(0)\geq B(-1)+1. \tag{5.3}$$

To do this, use the property (2.5) twice to obtain

$$\begin{aligned} A(0) &\geq \frac{\delta}{1+\delta}B(-1)+\frac{1}{1+\delta}(C(\delta)+\delta) \\ &\geq \frac{\delta}{1+\delta}B(-1)+\frac{1}{1+\delta}\big(\delta B(-1)+(1-\delta)(\delta+A(0))+\delta\big), \end{aligned}$$

or, equivalently, $A(0)\geq B(-1)+1-\frac{\delta}{2}$. As δ is arbitrary, (5.3) follows.

Step 3. The property (2.5), used twice, yields

$$\begin{aligned} A(y-2\delta) &\geq \frac{\delta}{1+\delta}B(y-2\delta-1)+\frac{1}{1+\delta}U_0(-\delta,y-\delta,1,0) \\ &\geq \frac{\delta}{1+\delta}B(y-2\delta-1)+\frac{\delta}{1+\delta}B(y-1)+\frac{1-\delta}{1+\delta}A(y) \end{aligned} \tag{5.4}$$

if $\delta < 1$ and $y \leq 0$. Moreover, if $y < 0$, $\delta \in (0,1)$ and $t > -y+1$, then

$$\begin{aligned} B(y-1) &\geq \frac{t}{t+\delta} U_0(1-\delta, y-1-\delta, 1, 0) + \frac{\delta}{t+\delta} U_0(1+t, y-1+t, 1, 0) \\ &= \frac{t}{t+\delta} U_0(1-\delta, y+1-\delta, 1, 0) + \frac{\delta(1+t)}{t+\delta} \left(\frac{y-1+t}{1+t} + U_0(1,0,1,0) \right), \end{aligned}$$

which gives, if one takes $t \to \infty$,

$$B(y-1) \geq U_0(1-\delta, y-1-\delta, 1, 0) + \delta(1 + B(0)). \tag{5.5}$$

Combining this estimate with the following consequence of (2.5):

$$U_0(1-\delta, y-1-\delta, 1, 0) \geq \delta A(y-2\delta) + (1-\delta) B(y-1-2\delta)$$

gives

$$B(y-1) \geq \delta A(y-2\delta) + (1-\delta) B(y-1-2\delta) + \delta(1 + B(0)). \tag{5.6}$$

Now multiply (5.4) throughout by $1+\delta$ and add it to (5.6) to obtain

$$A(y-2\delta) - B(y-1-2\delta) \geq (1-\delta)(A(y) - B(y-1)) + \delta(1 + B(0)),$$

which, by induction, leads to the estimate

$$A(-2n\delta) - B(-2n\delta - 1) - 1 - B(0) \geq (1-\delta)^n (A(0) - B(-1) - 1 - B(0)),$$

valid for any nonnegative integer n. Fix $y < 0$, $\delta = -y/(2n)$ and let $n \to \infty$ to obtain

$$A(y) - B(y-1) - 1 - B(0) \geq e^{y/2}(A(0) - B(-1) - 1 - B(0)) \geq -B(0)e^{y/2}, \tag{5.7}$$

where the latter estimate follows from (5.3).

Now we come back to (5.6) and write it in equivalent form

$$B(y-1) - B(y-1-2\delta) \geq \delta(A(y-2\delta) - B(y-1-2\delta)) + \delta(1 + B(0)).$$

By (5.7), we get

$$B(y-1) - B(y-1-2\delta) \geq \delta(-e^{y/2}B(0) + 2 + 2B(0)).$$

This gives, by induction,

$$\begin{aligned} B(-1) - B(-2n\delta - 1) &= \sum_{k=0}^{n} [B(-2k\delta - 1) - B(-2k\delta - 1 - 2\delta)] \\ &\geq n\delta(2 + 2B(0)) - \delta B(0) \frac{1 - e^{-n\delta}}{1 - e^{-\delta}}. \end{aligned}$$

Now fix $y < 0$, take $\delta = -y/(2n)$ and let $n \to \infty$ to obtain

$$B(-1) - B(y-1) \geq -y(1 + B(0)) - B(0)(1 - e^{y/2}). \tag{5.8}$$

Now, by (5.2) and (5.3),

$$B(-1) = \frac{1}{3}B(-1) + \frac{2}{3}B(-1) \leq \frac{1}{3}A(0) + \frac{2}{3}B(-1) + \frac{1}{3} \leq \frac{2}{3}B(0) - 1.$$

Furthermore, by the definition of B we have $B(y-1) \geq -\beta$. Plugging these estimates into (5.8) yields

$$\beta \geq -y(1 + B(0)) - B(0)(1 - e^{y/2}) + 1 - \frac{2}{3}B(0).$$

We must have $1 + B(0) \leq 0$, otherwise the estimate above would give $\beta_0 = \infty$, which, as we already know, is impossible. Take $y \in (-\infty, 0]$ satisfying

$$e^{y/2} = \frac{2}{B(0)} + 2.$$

We get

$$\beta \geq -2(1 + B(0)) \log\left(2 + \frac{2}{B(0)}\right) + 3 + \frac{1}{3}B(0)$$

and the right hand side, as a function of $B(0) \in (-\infty, -1]$, attains its minimum β_0 at $B(0) = -3a$ (where a is given by (3.1)). Hence $\beta \geq \beta_0$ and the proof is complete.

The constant β_0^+ *is optimal in* (1.4), *even in the case of nonnegative martingales.* Suppose for any nonnegative martingale f and its ± 1 transform g we have

$$||g^*||_1 \leq \beta ||f^*||_1.$$

Then the class $\mathcal{U}^+(\beta)$ is nonempty, so we may consider its minimal element U_0^+. From the very definition (and from the special form of V) it follows that

$$U_0^+(tx, ty, tz, tw) = tU_0^+(x, y, z, w) \quad \text{for } t > 0, \tag{5.9}$$

and

$$U_0^+(x, y, z, w) = U_0^+(x, y + t, z, w + t) - t \quad \text{for } t > -w.$$

Furthermore,

$$\text{the function } U_0^+(1, \cdot, 1, 0) \text{ is nondecreasing.} \tag{5.10}$$

It will be convenient to work with the functions

$$A(y) = U_0^+\big(\frac{2}{3}, y, 1, 0\big),\ B(y) = U_0^+(1, y, 1, 0),\ C(x) = U_0^+(x, 0, 1, 0).$$

As previously, we divide the proof into a few intermediate steps.

Step 1. First let us note that the arguments leading to (5.1) are valid for these functions and hence so is this estimate. For a fixed positive integer n, let us write (5.1) for $\delta = 1/(6n)$, $x = \frac{2}{3} + 2k\delta$, $k = 1, 2, \ldots, n$ and sum all these inequalities to obtain

$$C(1) + 2B(0) - C\big(\frac{2}{3}\big) - 2B\big(-\frac{1}{3}\big) \geq \frac{1}{3}(1 + B(0)) - \frac{1}{9n^2}\sum_{k=1}^{n} \frac{1}{\frac{1}{3} - \frac{k-1}{3n}}.$$

Now let $n \to \infty$ and use $C(1) = B(0)$ to get

$$3B(0) \geq C\big(\frac{2}{3}\big) + 2B\big(-\frac{1}{3}\big) + \frac{1}{3}(1 + B(0)). \tag{5.11}$$

Step 2. We will show that

$$C\big(\frac{2}{3}\big) \geq \frac{2}{3}B\big(-\frac{1}{3}\big) + \frac{4}{9} - \frac{\beta}{3}. \tag{5.12}$$

To this end, note that, using (2.5) twice, for $\delta < 1/3$,

$$\begin{aligned} C\big(\frac{2}{3}\big) &\geq \frac{3\delta}{1 + 3\delta}B\big(-\frac{1}{3}\big) + \frac{1}{1 + 3\delta}\left[\delta + C\big(\frac{2}{3} - \delta\big)\right] \\ &\geq \frac{3\delta}{1 + 3\delta}B\big(-\frac{1}{3}\big) + \frac{1}{1 + 3\delta}\left\{\delta + \frac{3\delta}{2}(-\beta) + \frac{2 - 3\delta}{2}\left[\delta + C\big(\frac{2}{3}\big)\right]\right\}. \end{aligned}$$

This is equivalent to

$$C\big(\frac{2}{3}\big) \geq \frac{2}{3}B\big(-\frac{1}{3}\big) + \frac{2}{9}\big(2 - \frac{3}{2}\delta\big) - \frac{\beta}{3}$$

and it suffices to let $\delta \to 0$.

Step 3. Using the property (2.5), we get, for $y < -1/3$ (see (5.5) and the arguments leading to it),

$$B(y) \geq U_0^+(1-\delta, y-\delta, 1, 0) + \delta(1 + B(0)).$$

Furthermore, again by (2.5),

$$U_0^+(1-\delta, y-\delta, 1, 0) \geq (1-3\delta)B(y-2\delta) + 3\delta A\big(y + \frac{1}{3} - 2\delta\big)$$

and hence

$$B(y) \geq (1-3\delta)B(y-2\delta) + 3\delta A\big(y + \frac{1}{3} - 2\delta\big) + \delta(1 + B(0)). \tag{5.13}$$

Moreover,

(5.14)

$$\begin{aligned} A\big(y + \frac{1}{3} - 2\delta\big) &\geq \frac{3\delta}{2+3\delta}U_0^+(0, y - \frac{1}{3} - 2\delta, 1, 0) + \frac{2}{2+3\delta}U_0^+(\frac{2}{3} + \delta, y + \frac{1}{3} - \delta, 1, 0) \\ &\geq \frac{3\delta}{2+3\delta}(-\beta) + \frac{2}{2+3\delta}\left[3\delta B(y) + (1-3\delta)A(y + \frac{1}{3})\right]. \end{aligned}$$

Step 4. Now we will combine (5.13) and (5.14) and use them several times. Multiply (5.14) by $\gamma > 0$ (to be specified later) and add it to (5.13). We obtain

$$\begin{aligned} &B(y) \cdot \left(1 - \frac{6\gamma\delta}{2+3\delta}\right) - A(y + \frac{1}{3}) \cdot \frac{(2-6\delta)\gamma}{2+3\delta} \\ &\geq B(y-2\delta) \cdot (1-3\delta) - A(y + \frac{1}{3} - 2\delta) \cdot (\gamma - 3\delta) + \delta\left(1 + B(0) - \frac{3\beta\gamma}{2+3\delta}\right) \\ &\geq B(y-2\delta) \cdot (1-3\delta) - A(y + \frac{1}{3} - 2\delta) \cdot (\gamma - 3\delta) + \delta\left(1 + B(0) - \frac{3\beta\gamma}{2}\right). \end{aligned}$$

Now the choice $\gamma = (5 - \sqrt{9 - 24\delta})/4$ allows to write the inequality above in the form

$$F(y) \geq Q_\delta F(y - 2\delta) + \delta\left(1 + B(0) - \frac{3\beta\gamma}{2}\right), \tag{5.15}$$

where

$$F(y) = B(y) \cdot \left(1 - \frac{6\gamma\delta}{2+3\delta}\right) - A(y + \frac{1}{3}) \cdot \frac{(2-6\delta)\gamma}{2+3\delta}$$

and

$$Q_\delta = \frac{1-3\delta}{1 - \frac{6\gamma\delta}{2+3\delta}}.$$

The inequality (5.15), by induction, leads to

$$F(-1/3) \geq Q_\delta^n F(-1/3 + 2n\delta) + \delta\left(1 + B(0) - \frac{3\beta\gamma}{2}\right) \cdot \frac{Q_\delta^n - 1}{Q_\delta - 1}.$$

Now fix $Y < -1/3$, take $\delta = (Y + 1/3)/(2n)$ and let $n \to \infty$. Then

$$\gamma \to \frac{1}{2}, \quad Q_\delta^n \to \exp\left(\frac{3}{4}(Y + \frac{1}{3})\right)$$

and we arrive at

$$B(-\frac{1}{3}) - \frac{1}{2}A(0) \geq \exp\left(\frac{3}{4}(Y + \frac{1}{3})\right)(B(Y) - \frac{1}{2}A(Y + \frac{1}{3}))$$
$$+ \frac{2}{3}\left(1 + B(0) - \frac{3\beta}{4}\right)\left[\exp\left(\frac{3}{4}(Y + \frac{1}{3})\right) - 1\right].$$

Now we have $B(Y) \geq -\beta$ and $A(Y + \frac{1}{3}) \leq A(0)$. Hence, letting $Y \to -\infty$ yields

$$F(-1/3) \geq -\frac{2}{3}\left(1 + B(0) - \frac{3\beta}{4}\right). \tag{5.16}$$

Now combine (5.11), (5.12) and (5.16) to obtain $\beta \geq 14/9$. The proof is complete.

References

[1] K. Bichteler, *Stochastic integration and L^p-theory of semimartingales*, Ann. Probab. 9 (1980), pp. 49–89.

[2] D. L. Burkholder, *Boundary value problems and sharp inequalities for martingale transforms*, Ann. Probab. **12** (1984), pp. 647–702.

[3] ———, *Explorations in martingale theory and its applications*, Ecole d'Ete de Probabilits de Saint-Flour XIX—1989, pp. 1–66, Lecture Notes in Math., 1464, Springer, Berlin, 1991.

[4] ———, *Sharp norm comparison of martingale maximal functions and stochastic integrals,* Proceedings of the Norbert Wiener Centenary Congress, 1994 (East Lansing, MI, 1994), pp. 343–358, Proc. Sympos. Appl. Math., 52, Amer. Math. Soc., Providence, RI, 1997.

[5] A. Osękowski, *Sharp maximal inequality for stochastic integrals*, Proc. Amer. Math. Soc. **136** (2008), 2951–2958.

Department of Mathematics, Informatics and Mechanics, Warsaw University, Banacha 2, 02-097 Warsaw, Poland

E-mail address: ados@mimuw.edu.pl