SHARP MOMENT INEQUALITIES FOR DIFFERENTIALLY SUBORDINATED MARTINGALES

ADAM OSĘKOWSKI

Abstract. We determine the optimal constants $C_{p,q}$ in the moment inequalities

$$||g||_p \leq C_{p,q}||f||_q, \quad 1 \leq p < q < \infty,$$

where $f = (f_n)$, $g = (g_n)$ are two martingales, adapted to the same filtration, satisfying

$$|dg_n| \leq |df_n|, \qquad n = 0, 1, 2, \ldots,$$

with probability 1. Furthermore, we establish related sharp estimates

$$||g||_1 \leq \sup_n \mathbb{E}\Phi(|f_n|) + L(\Phi),$$

where Φ is an increasing convex function satisfying certain growth conditions and $L(\Phi)$ depends only on Φ.

1. Introduction

The purpose of this paper is to provide new interesting sharp estimates for martingales under the assumption of differential subordination. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, filtered by a nondecreasing family $(\mathcal{F}_n)$ of sub-σ-fields of $\mathcal{F}$. Throughout, $f = (f_n)$, $g = (g_n)$ will stand for martingales adapted to this filtration, taking values in a certain separable Hilbert space $\mathcal{H}$. Their difference sequences $df = (df_n)$, $dg = (dg_n)$ are defined by the equations

$$f_n = \sum_{k=0}^{n} df_k, \qquad g_n = \sum_{k=0}^{n} dg_k, \quad n = 0, 1, 2, \ldots.$$

We assume that g is *differentially subordinate* to f, that is, for any $n = 0, 1, 2, \ldots$,

$$\mathbb{P}(|dg_n| \leq |df_n|) = 1.$$

For example, this is valid for martingale transforms. Suppose that we have $dg_n = v_n df_n$ for $n = 0, 1, 2, \ldots$, where $v = (v_n)$ is a real-valued predictable sequence bounded in absolute value by 1. In the special case when each term

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v_n is deterministic and takes values in $\{-1,1\}$, we will say that g is a ± 1-transform of f. Obviously, for such f and g, we have that g is differentially subordinate to f.

The notion of differential subordination was introduced by Burkholder [Bu2] (though this concept appears in his earlier papers, see e.g. [Bu1]). He also studied various estimates of g in terms of f. These include the following weak type $(1,1)$ and moment estimates. Denote $f^* = \sup_n |f_n|$ and $||f||_p = \sup_n ||f_n||_p$ for $p \in [1,\infty]$.

Theorem 1.1 (Burkholder)**.** *Assume g is differentially subordinate to f.*

(i) For any $\lambda > 0$,

$$\lambda \mathbb{P}(g^* \geq \lambda) \leq 2||f||_1 \tag{1.1}$$

and the constant 2 is the best possible.

(ii) For any $1 < p < \infty$,

$$||g||_p \leq (p^* - 1)||f||_p, \tag{1.2}$$

where $p^* = \max\{p, p/(p-1)\}$. The constant $p^* - 1$ is the best possible.

The result above has been extended in many directions. For more information on the subject and connections with the harmonic analysis, the reader is referred to the survey [Bu4]. For more recent results, see [BW], [BW2] and [Bu3] – [W].

The moment inequality (1.2) fails to hold for $p = 1$. However, the following logarithmic estimate is valid. For $K > 0$, let $L(K) \in (0,\infty]$ denote the smallest constant L such that the inequality

$$||g||_1 \leq K \sup_n \mathbb{E}|f_n| \log |f_n| + L, \tag{1.3}$$

holds for all f, g with g being differentially subordinate to f.

Here is the main result of [O2].

Theorem 1.2. *We have*

$$L(K) = \begin{cases} \infty & \text{if } 0 < K \leq 1, \\ \frac{K^2}{2(K-1)} \exp(-K^{-1}) & \text{if } 1 < K < 2, \\ K \exp(K^{-1} - 1) & \text{if } K \geq 2. \end{cases}$$

In the paper we provide another extension of Theorem 1.1. Namely, for any p, q lying in $[1,\infty)$ and satisfying $p < q$, we will determine the best constants $C_{p,q}$ such that for any martingales f, g, with g being differentially subordinate to f,

$$||g||_p \leq C_{p,q}||f||_q. \tag{1.4}$$

In fact, this estimate will follow from a more general one. We will derive the best constant $L_{p,q}$ such that for any f, g as above, we have

$$||g||_p^p \leq ||f||_q^q + L_{p,q}, \tag{1.5}$$

where $1 \leq p < q < 2$ or $2 < p < q < \infty$.

The proof of (1.5) depends heavily on the techniques which were invented by Burkholder. As exhibited in [Bu4], the problem of proving an inequality for differentially subordinated martingales is equivalent to the problem of finding certain special function, which has some convex-type properties. To be more specific, let us outline the proof of (1.5). This inequality is equivalent to

$$\mathbb{E}V_{p,q}(f_n, g_n) \leq L_{p,q}, \qquad n = 0,\, 1,\, 2,\, \ldots,$$

where $V_{p,q} : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ is given by $V_{p,q}(x, y) = |y|^p - |x|^q$. We will construct a function $U_{p,q} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, which satisfies

(i) $U_{p,q} \geq V_{p,q}$ on $\mathcal{H} \times \mathcal{H}$,

(ii) $\mathbb{E}U_{p,q}(f_n, g_n) \leq \mathbb{E}U_{p,q}(f_{n-1}, g_{n-1})$ for $n \geq 1$,

(iii) $U_{p,q}(x, y) \leq L_{p,q}$ if $|y| \leq |x|$.

This will yield the claim: indeed, if $n = 0, 1, 2, \ldots$, then

$$\mathbb{E}V_{p,q}(f_n, g_n) \leq \mathbb{E}U_{p,q}(f_n, g_n) \leq \mathbb{E}U_{p,q}(f_0, g_0) \leq L_{p,q}.$$

A few words about the organization of the paper. In the next section we study special solutions to a certain family of differential equations. Then, in Section 3, we introduce two basic functions, which, integrated against special kernels, lead to special functions $U_{p,q}$ corresponding to the moment inequalites (1.5). Section 4 is devoted to the study of the properties of these functions. In the next two sections we establish the moment inequalities (1.4) and (1.5) as well as certain extensions of the estimate (1.3) above. The final section addresses the problem of the strictness and sharpness of these inequalities.

2. A differential equation

As explained above, the inequality (1.5) holds if one manages to find an appropriate majorant $U_{p,q}$ of $V_{p,q}$. How can we construct such a special function? We will present the idea of the search in the case $1 < p < 2$.

It is natural to try to look at the limit case $p = q$ in the real valued setting: $\mathcal{H} = \mathbb{R}$. Unfortunately, then (1.5) is not valid with any finite $L_{p,p}$ unless $p = 2$ (this is clear from Theorem 1.1). However, (1.4) does hold and can be used to gain some insight into the problem. For $x, y \in \mathbb{R}$, let

$V_{p,p}(x,y) = |y|^p - (p-1)^{-p}|x|^p$ and

$$U_{p,p}(x,y) = p^{2-p}(|y| - (p-1)^{-1}|x|)(|x|+|y|)^{p-1}.$$

Then, as shown by Burkholder (see [Bu4]), the functions $U_{p,p}$ and $V_{p,p}$ satisfy the conditions (i), (ii) and (iii) from the end of the previous section and thus (1.4) holds. Let us take a closer look at the properties of these functions. First, the equality $V_{p,p}(x,y) = U_{p,p}(x,y)$ holds if and only if $(|x|,|y|)$ lies on the half line $\{(\alpha,\beta) \in [0,\infty)\times[0,\infty) : \beta = (p-1)^{-1}\alpha\}$. Second, the condition (ii) is guaranteed by the following geometrical property of $U_{p,p}$: this function is concave along any line of slope $a \in [-1,1]$. Moreover, if we restrict ourselves to the first quadrant, we see that extremal lines (along which $U_{p,p}$ is linear) are those of slope -1. Finally, $U_{p,p}$ are symmetric: $U_{p,p}(x,y) = U_{p,p}(|x|,|y|)$ and $(U_{p,p})_y(x,0) = 0$ for all x.

It is natural to expect that $U_{p,q}$ should have similar properties. Let us assume that the equality $V_{p,q}(x,y) = U_{p,q}(x,y)$ takes place if and only if $(|x|,|y|)$ lies on a certain curve $\{(\alpha,\beta) \in [0,\infty)\times[0,\infty) : \beta = h(\alpha)\}$ and suppose that in the first quadrant $U_{p,q}$ is linear along the lines of slope -1. These two conditions imply that for $x > 0$ and $t \in [-x, h(x)]$,

$$\begin{aligned} U_{p,q}(x+t, h(x)-t) &= V_{p,q}(x,h(x)) + \big[(V_{p,q})_x(x,h(x)) - (V_{p,q})_y(x,h(x))\big]t \\ &= (h(x))^q - x^p + (q(h(x))^{q-1} - px^{p-1})t. \end{aligned}$$

Finally, the condition $(U_{p,q})_y(x,0) = 0$ leads to the following differential equation

$$p(2-p)h'(x) + p = q(q-1)x^{q-2}h(x)^{2-p}. \tag{2.1}$$

If we manage to find an appropriate solution, we will obtain the explicit formula for the function $U_{p,q}$, at least on the set $\{(x,y) : |x|+|y| \geq h(0)\}$.

Now we turn to rigorous reasoning. The purpose of this section is to establish, for certain values of p and q, the existence of an increasing solution of (2.1) on $[0,\infty)$, satisfying $h(0) > 0$ and $h'(x) \to 0$, $h(x) \to \infty$, as $x \to \infty$. As we will see below, if 2 lies between p and q, then the constant $C_{p,q}$ is one. Therefore, while studying (1.4), two possibilities are of interest: $1 \leq p < q < 2$ or $2 < p < q < \infty$. Till the end of Section 4, we assume that one of them takes place.

We consider the cases $1 = p < q < 2$, $1 < p < q < 2$ and $2 < p < q < \infty$ separately.

2.1. **The case $1 = p < q < 2$.** This is straightforward. There is an explicit formula for the solutions of (2.1), defined on the whole half line $[0,\infty)$. It reads

$$h(t) = \exp(qt^{q-1}) \left[\int_t^\infty \exp(-qs^{q-1}) ds + \alpha \right], \quad \alpha \in \mathbb{R},$$

and it is easy to check that the solution h we are interested in corresponds to the case $\alpha = 0$.

2.2. **Duality.** Before we turn to the remaining two cases, let us make here a crucial observation. Suppose $p > 1$. A deeper insight into the equation (2.1) reveals the following dual structure. Let us write $(2.1)_{p,q}$ to indicate the dependence on p and q. If the function h is a solution to $(2.1)_{p,q}$, then the function G, given by

$$G(x) = \beta h((\alpha x)^{1/(q-1)})^{p-1}, \tag{2.2}$$

where

$$\begin{aligned} \alpha &= \left(\frac{q}{p}\right)^{p(q-1)/(p-q)} \left(\frac{p-1}{q-1}\right)^{(p-1)(q-1)/(p-q)}, \\ \beta &= \left(\frac{p}{q}\right)^{q(p-1)/(p-q)} \left(\frac{q-1}{p-1}\right)^{(p-1)(q-1)/(p-q)}, \end{aligned} \tag{2.3}$$

is a solution to $(2.1)_{p',q'}$. Here $p' = p/(p-1)$, $q' = q/(q-1)$ denote the harmonic conjugates of p and q. To see this, note that (2.1) implies

$$p(2-p)h'((\alpha x)^{1/(q-1)}) + p = q(q-1)(\alpha x)^{(q-2)/(q-1)} h((\alpha x)^{1/(q-1)})^{2-p}.$$

Dividing throughout by the right-hand side, we obtain

$$\frac{(2-p)p}{(p-1)q}\frac{1}{\alpha\beta}G'(x) + \frac{p}{q(q-1)}\beta^{(2-p)/(p-1)}G(x)^{(p-2)/(p-1)} \cdot (\alpha x)^{(2-q)/(q-1)} = 1$$

and the particular choice (2.3) of the parameters α, β gives

$$(2 - \frac{p}{p-1})G'(x) + 1 = \frac{\frac{q}{q-1} \cdot \frac{1}{q-1} x^{(2-q)/(q-1)} G(x)^{(p-2)/(p-1)}}{p/(p-1)},$$

i.e. $(2.1)_{p',q'}$ for $h = G$.

2.3. **The case $1 < p < q < 2$.** We have the following.

Theorem 2.1. *There exists an increasing solution $\hat{h} : [0, \infty) \to [0, \infty)$ of $(2.1)_{p,q}$ satisfying $\hat{h}(0) > 0$ and $\hat{h}'(t) \to 0$, $\hat{h}(t) \to \infty$, as $t \to \infty$.*

Proof. In fact, we do not know if the equation $(2.1)_{p,q}$ has any solutions given on the whole half line $[0, \infty)$. Let h be a function, defined on a certain subinterval of $[0, \infty)$, on which it satisfies the differential equation. This function can be extended to its maximal domain, which is a certain interval I. Note that since $(2-p)h'(x) + 1 > 0$, we have that I contains its left end; therefore, $I = [a, b]$, $[a, b)$ or $[a, \infty)$. Observe that $a = 0$. Indeed, if it was not the case, then we would have $h(a) = 0$ (by maximality of the domain) and $h'(a+) = -\frac{1}{2-p} < 0$, a contradiction.

It is convenient to divide the remaining part of the proof into a few steps.

Step 1. For a fixed $z \in (0, \infty)$, by Picard-Lindelöf's theorem, there is a unique solution $h^{(z)}$ to $(2.1)_{p,q}$, satisfying the condition

$$q(q-1)z^{q-2}\left[h^{(z)}(z)\right]^{2-p} = p, \tag{2.4}$$

or, equivalently, $(h^{(z)})'(z) = 0$ (again, it is extended to its maximal domain). We will show that

$$h^{(z)} \text{ is strictly concave,} \tag{2.5}$$

$$h^{(z)}(0) > 0. \tag{2.6}$$

Let us omit the upper index and write h instead of $h^{(z)}$. We have the identity

$$h''(x) = \frac{q(q-1)}{(2-p)p} h(x)^{1-p} x^{q-3}[(2-p)h'(x)x - (2-q)h(x)] \tag{2.7}$$

Denote the expression in the square brackets by $F(x)$ and derive

$$F'(x) = (2-p)h''(x)x + (q-p)h'(x). \tag{2.8}$$

Suppose that there is $S \in (0, z)$ for which $h''(S) = 0$. Then by (2.7) we have $h'(S) > 0$ and hence $F'(S) > 0$. It is easy to see that this implies $F'(x) > 0$ and $h''(x) > 0$ for $x > S$ and hence h' has no zeros larger than S, a contradiction. Therefore $h''(x) \neq 0$ for $x < z$. However, by (2.7), we have $h''(z) < 0$ and thus $h''(x) < 0$ for $x \leq z$. On the other hand, (2.8) gives $F'(z) < 0$ and, in consequence, $F'(x) < 0$ and $h''(x) < 0$ for $x > z$. This gives (2.5).

Now note that, by (2.1),

$$\begin{aligned}\left[\frac{p(2-p)}{p-1} h(x)^{p-1} - qx^{q-1}\right]' &= p(2-p)h(x)^{p-2}h'(x) - q(q-1)x^{q-2} \\ &= -ph(x)^{p-2} < 0\end{aligned} \tag{2.9}$$

and hence

$$\frac{p(2-p)}{p-1} h(0)^{p-1} > \frac{p(2-p)}{p-1} h(z)^{p-1} - qz^{q-1}.$$

However, by (2.4), the right-hand side equals

$$\frac{p(2-p)}{p-1}\left(\frac{p}{q(q-1)}\right)^{(p-1)/(2-p)} z^{(p-1)(2-q)/(2-p)} - qz^{q-1},$$

which is nonnegative for small z, as

$$\frac{(p-1)(2-q)}{2-p} < q - 1.$$

Thus we have proved $h^{(z)}(0) > 0$ for small z. However, this is sufficient: by (2.4), the bigger z, the bigger $h^{(z)}(z)$ and, in consequence, the bigger $h^{(z)}(0)$.

Step 2. Now we will provide a uniform bound for $h^{(z)}$, $z \in (0, \infty)$. Let

$$w = \left(\frac{p(3-q)}{q(q-1)}\right)^{1/(q-p)}$$

and denote by h_w the unique solution of (2.1) satisfying $h_w(w) = w$ (extended to its maximal domain). It is easy to check that

$$(2-p)h_w'(w)w - (2-q)h_w(w) = 0$$

and hence $h_w''(w) = 0$, by (2.7). Thus $h_w \neq h^{(z)}$ for any z. Furthermore, by (2.9) applied to $h = h_w$, we have, for any $x > 0$,

$$\frac{p(2-p)}{p-1}h_w(0)^{p-1} > \frac{p(2-p)}{p-1}h_w(x)^{p-1} - qx^{q-1}$$

and hence the maximal domain of h_w equals $[0, \infty)$. Now if $h > h_w$ is any solution of (2.1), then $h' > h_w' > 0$ and therefore, h does not coincide with $h^{(z)}$ for any z. In other words, we have $\sup h^{(z)} \leq h_w$.

Step 3. The function we search for in the theorem is defined by

$$\hat{h}(x) = \sup\{h^{(z)}(x) : z > x\} = \lim_{z\to\infty} h^{(z)}(x).$$

Note that $\hat{h}$ is finite due to Step 3. Clearly, $\hat{h}$ is a solution to (2.1) and satisfies $\hat{h}(0) > 0$. Furthermore, it is nondecreasing, since for any $x < z$ we have $(h^{(z)})'(x) > 0$. In fact, it is increasing, since otherwise we would have $h'(z) = 0$ for some z and thus the function would coincide with $h^{(z)}$, which is impossible. To complete the proof, note that $\hat{h}'(x) \leq \hat{h}'(1)$ for $x > 1$ and, by (2.1),

$$\hat{h}(x) \leq \left(\frac{p}{q(q-1)}((2-p)\hat{h}'(1)+1)x^{2-q}\right)^{1/(2-p)},$$

which implies $\hat{h}' \to 0$ as $t \to \infty$. Combined with $(2.1)_{p,q}$, this immediately yields $\hat{h}(t) \to \infty$ as $t \to \infty$. □

2.4. **The case $2 < p < q < \infty$.** Then $q \in (p, \infty)$. We have the following fact.

Theorem 2.2. *There exists an increasing solution $\hat{h} : [0, \infty) \to [0, \infty)$ of $(2.1)_{q,p}$ satisfying $\hat{h}'(t) \downarrow 0$ as $t \to \infty$.*

Proof. We use duality. Let h be the solution of $(2.1)_{q',p'}$ such as in Theorem 2.1 and let $\hat{h}$ be its dual defined by the transformation (2.2). Then $\hat{h}$ is given on the whole half line $[0, \infty)$, $\hat{h}(0) > 0$, $\hat{h}(t) \to \infty$ as $t \to \infty$ and

$$\hat{h}'(x) = \beta\alpha(q'-1)^{-1}(x')^{2-q'}h(x')^{p'-2}h'(x'),$$

where $x' = (\alpha x)^{1/(q-1)}$. It suffices to note that, by $(2.1)_{q',p'}$,

$$(x')^{2-q'}h(x')^{p'-2} \leq \frac{q'(q'-1)}{p'},$$

so $h'(x') \to 0$ as $x \to \infty$. The proof is complete. □

In all the considerations below, we will denote the solution $\hat{h}$ of (2.1) by h. We conclude this section by introducing another function to be used later: let $H : [h(0), \infty) \to [0, \infty)$ be the inverse to $t \mapsto t + h(t)$. We have the following trivial equalities:

$$h(H(t)) + H(t) = t, \tag{2.10}$$

$$h'(H(t)) + 1 = \frac{1}{H'(t)}. \tag{2.11}$$

Now, as explained at the beginning of this section, we are able to provide the formula for the function $U_{p,q}$. The next step would be to check the conditions (i), (ii) and (iii) listed at the end of Section 1. However, the verification of the second condition is quite elaborate; to overcome this difficulty, we will take a different approach.

3. Simple special functions and the integration argument

In order to prove inequality (1.1), Burkholder invented a special function $W_1 : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ given by

$$W_1(x, y) = \begin{cases} |y|^2 - |x|^2 & \text{if } |x| + |y| \leq 1, \\ 1 - 2|x| & \text{if } |x| + |y| > 1 \end{cases}$$

and showed (cf. [Bu4]) that W_1 has the following property: if x, y, k_x, $k_y \in \mathcal{H}$ satisfy $|k_y| \leq |k_x|$, then the function $G : \mathbb{R} \to \mathbb{R}$ given by

$$G(t) = G_{x,y,k_x,k_y}(t) = W_1(x + tk_x, y + tk_y) \tag{3.1}$$

is concave. As explained in [Bu4] (see also the proof of Lemma 3.1 below), this property implies that W_1 satisfies the condition (ii) mentioned at the end of Section 1. More specifically, if f, g are martingales such that for some $n \geq 1$ we have $|dg_n| \leq |df_n|$, then

$$\mathbb{E}W_1(f_n, g_n) \leq \mathbb{E}W_1(f_{n-1}, g_{n-1}). \tag{3.2}$$

As a consequence, if g is differentially subordinate to f, then the sequence $(\mathbb{E}W_1(f_n, g_n))$ is nonincreasing. This property is preserved by integration against positive kernels. To be more precise, let $w : \mathbb{R}_+ \to \mathbb{R}_+$ be such that for all x, $y \in \mathcal{H}$,

$$\int_0^\infty w(t)|W_1(x/t, y/t)|\mathrm{d}t < \infty.$$

Now if we take $\beta \in \mathbb{R}$ and set

$$U(x,y) = \int_0^\infty w(t)W_1(x/t, y/t)dt + \beta, \tag{3.3}$$

then the sequence $(\mathbb{E}U(f_n, g_n))$ is nonicreasing, provided all the expectations exist. Furthermore, we have that $W_1(x,y) \leq 0$ for all x, y satisfying $|y| \leq |x|$, which implies $U(x,y) \leq \beta$ for such x, y, and, in consequence, for f and g as above, we have

$$\mathbb{E}U(f_n, g_n) \leq \mathbb{E}U(f_0, g_0) \leq \beta, \quad n = 0,\, 1,\, 2,\, \ldots. \tag{3.4}$$

Using function W_1 and the integration, we will obtain the special functions corresponding to the moment inequalities in case $1 \leq p < 2$. To cover other possible choices of the parameter p, we need a dual function to W_1. Let us introduce $W_\infty : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ by

$$W_\infty(x,y) = \begin{cases} 0 & \text{if } |x| + |y| \leq 1, \\ (|y|-1)^2 - |x|^2 & \text{if } |x| + |y| > 1. \end{cases}$$

The function W_∞ has the concavity property described above. Precisely, we have the following.

Lemma 3.1. *Fix x, y, k_x, $k_y \in \mathcal{H}$ such that $|k_y| \leq |k_x|$. The function $G : \mathbb{R} \to \mathbb{R}$ given by*

$$G(t) = G_{x,y,k_x,k_y}(t) = W_\infty(x + tk_x, y + tk_y) \tag{3.5}$$

is concave.

Proof. For t such that $|y + tk_y| \neq 0$ and $|x + tk_x| + |y + tk_y| > 1$, we have

$$G''(t) = 2(|k_y|^2 - |k_x|^2) - 2 \cdot \frac{|k_y|^2 - ((y+tk_y)', k_y)^2}{|y + tk|} \leq 0,$$

where we have used the notation $z' = z/|z|$ for $z \in \mathcal{H} \setminus \{0\}$. If $|x + tk_x| + |y + tk_y| < 1$, then $G''(t) = 0$. For the remaining t we easily check that $G'(t+) \leq G'(t-)$. □

As previously, this implies the following property: if g is differentially subordinate to f and f belongs to L^2, then the sequence $(\mathbb{E}W_\infty(f_n, g_n))$ is nonincreasing. This property carries over to function U defined by the formula (3.3), with W_1 replaced by W_∞. As $W_\infty(x,y) \leq 0$ for x, y satisfying $|y| \leq |x|$, we have, by the arguments used above,

$$\mathbb{E}U(f_n, g_n) \leq \mathbb{E}U(f_0, g_0) \leq \beta, \quad n = 0,\, 1,\, 2,\, \ldots.$$

We conclude this section with a result which will be used in the proof of strictness of our main estimates.

Lemma 3.2. *Let $n \geq 1$ be fixed and suppose that f, g are martingales such that g is differentially subordinate to f. Assume further that $|f_{n-1}|+|g_{n-1}| < 1$ almost surely and $\mathbb{P}(|f_n|+|g_n| > 1) > 0$. Then we have the strict inequality*

$$\mathbb{E}W_1(f_n, g_n) < \mathbb{E}W_1(f_{n-1}, g_{n-1}) \tag{3.6}$$

and, if f is square integrable,

$$\mathbb{E}W_\infty(f_n, g_n) < \mathbb{E}W_\infty(f_{n-1}, g_{n-1}). \tag{3.7}$$

Proof. Let x, y, k_x, $k_y \in \mathcal{H}$ satisfy $|x| + |y| < 1$ and $|k_y| \leq |k_x|$. The concavity of the function G given by (3.1) gives $G(1) \leq G(0) + G'(0)$, or

$$\begin{aligned} W_1(x + k_x, y + k_y) &\leq W_1(x, y) + (W_{1x}(x, y), h) + (W_{1y}(x, y), k) \\ &= |y|^2 - |x|^2 - 2(x, k_x) + 2(y, k_y). \end{aligned}$$

Now, to establish (3.6), it suffices to show that if $|x + k_x| + |y + k_y| > 1$, then the above estimate is strict. Indeed, if we prove this, we plug $x = f_{n-1}$, $y = g_{n-1}$, $k_x = df_n$ and $k_y = dg_n$, thus obtaining an inequality which is strict on a set of positive measure. Taking expectation of both sides yields (3.6). So, assume that $|x + k_x| + |y + k_y| > 1$ and observe that the desired estimate can be rewritten in the form

$$1 - 2|x + k_x| < |y + k_y|^2 - |x + k_x|^2 + |k_x|^2 - |k_y|^2,$$

or

$$(|x + k_x| + |y + k_y| - 1)(|x + k_x| - |y + k_y| - 1) < |k_x|^2 - |k_y|^2.$$

If $|x + k_x| < |y + k_y| + 1$, then this inequality holds: the left-hand side is negative and the right hand side is nonnegative. If, conversely, $|x + k_x| \geq |y+k_y|+1$, then, using trivial bounds $|k_x| \geq ||x+k_x|-|x||$, $|k_y| \leq |y+k_y|+|y|$, we get

$$\begin{aligned} |k_x|^2 - |k_y|^2 &\geq (|x + k_x| + |y + k_y| - |x| + |y|)(|x + k_x| - |y + k_y| - |x| - |y|) \\ &> (|x + k_x| + |y - k_y| - 1)(|x + k_x| - |y - k_y| - 1). \end{aligned}$$

The reasoning for W_∞ is similar: we must show that if x, y, k_x, $k_y \in \mathcal{H}$ satisfy $|x| + |y| < 1$, $|x + k_x| + |y + k_y| > 1$ and $|k_y| \leq |k_x|$, then

$$W_\infty(x + k_x, y + k_y) < W_\infty(x, y) + (W_{\infty x}(x, y), k_x) + (W_{\infty y}(x, y), k_y),$$

or, equivalently,

$$(|x + k_x| + |y + k_y| - 1)(-|x + k_x| + |y + k_y| - 1) < 0.$$

This follows immediately from

$$-|x + k_x| + |y + k_y| - 1 \leq |x| + |y| - 1 + |k_y| - |k_x| < 0. \quad \square$$

4. Special functions

Here we determine the kernels w and numbers β, which give us the special functions $U_{p,q}:\mathcal{H}\times\mathcal{H}\to\mathbb{R}$ corresponding to the moment inequality (1.5). For $1\leq p<q<2$, let h be the solution to $(2.1)_{p,q}$,

$$w_{p,q}(t)=\frac{p(2-p)}{2}h(H(t))^{p-3}h'(H(t))H'(t)t^2 \tag{4.1}$$

and

$$U_{p,q}(x,y)=\int_{h(0)}^\infty w_{p,q}(t)W_1(x/t,y/t)dt+\frac{(2-p)h(0)^p}{2}. \tag{4.2}$$

For $2<p<q<\infty$, let h be the solution to $(2.1)_{q,p}$,

$$w_{p,q}(t)=\frac{q(q-2)}{2}h(H(t))^{q-3}h'(H(t))H'(t)t^2$$

and

$$U_{p,q}(x,y)=\int_{h(0)}^\infty w_{p,q}(t)W_\infty(x/t,y/t)dt+\frac{q}{2}h(0)^{q-2}(|y|^2-|x|^2)+\frac{q-2}{2}h(0)^q. \tag{4.3}$$

Note that the formulas for $w_{p,q}$ in the two cases are essentially the same: one only has to switch the parameters p and q and change the sign.

Let us turn to the explicit formulas for $U_{p,q}$. It can be verified that the formulas presented in the two lemmas below agree with those introduced at the beginning of Section 2. First we deal with the case $1\leq p<q<2$.

Lemma 4.1. *Let $1\leq p<q<2$. We have*

$$U_{p,q}(x,y)=p\frac{|y|^2-|x|^2}{2h(0)^{2-p}}+\frac{(2-p)h(0)^p}{2} \tag{4.4}$$

if $|x|+|y|\leq h(0)$, and

$$\begin{aligned}U_{p,q}(x,y)=&p|y|h(H(|x|+|y|))^{p-1}-(p-1)h(H(|x|+|y|))^p\\&-H(|x|+|y|)^q-qH(|x|+|y|)^{q-1}\big(|x|-H(|x|+|y|)\big),\end{aligned} \tag{4.5}$$

if $|x|+|y|>h(0)$.

Proof. Using the formula for W_1 we see that if $|x|+|y|<h(0)$, then

$$U_{p,q}(x,y)=(|y|^2-|x|^2)\int_{h(0)}^\infty\frac{w_{p,q}(t)}{t^2}dt+\frac{(2-p)h(0)^p}{2}.$$

Now we have, for any $s>0$,

$$\int_s^\infty\frac{w_{p,q}(t)}{t^2}dt=\frac{p}{2}\int_s^\infty\left[-h(H(t))^{p-2}\right]'dt=\frac{p}{2}h(H(s))^{p-2} \tag{4.6}$$

and hence (4.4) is valid. Suppose then, that $|x|+|y|\geq h(0)$. We have

$$\begin{aligned} U_{p,q}(x,y) &= \int_{h(0)}^{|x|+|y|} w_{p,q}(t)dt - 2|x|\int_{h(0)}^{|x|+|y|}\frac{w_{p,q}(t)}{t}dt \\ &\quad + (|y|^2-|x|^2)\int_{|x|+|y|}^{\infty}\frac{w_{p,q}(t)}{t^2}dt + \frac{(2-p)h(0)^p}{2} \end{aligned} \tag{4.7}$$

and we need to calculate the first and the second integral. First note that as H is inverse to $t\mapsto t+h(t)$, we have

$$\begin{aligned} \int_{h(0)}^{s} h(H(t))^{p-2}dt &= \int_{h(0)}^{s} h(H(t))^{p-2}H'(t)(1+h'(H(t)))dt \\ &= \int_{h(0)}^{s} h(H(t))^{p-2}H'(t)dt \\ &\quad + \int_{h(0)}^{s} h(H(t))^{p-2}h'(H(t))H'(t)dt. \end{aligned}$$

The equation (2.1), applied to $x=H(t)$, is equivalent to

$$h(H(t))^{p-2}H'(t) = \frac{q(q-1)H(t)^{q-2}H'(t)}{p} + (p-2)h(H(t))^{p-2}h'(H(t))H'(t),$$

which combined with the previous equality gives

$$\begin{aligned} &\int_{h(0)}^{s} h(H(t))^{p-2}dt \\ &= \int_{h(0)}^{s}\frac{q(q-1)H(t)^{q-2}H'(t)}{p}dt + (p-1)\int_{h(0)}^{s} h(H(t))^{p-2}h'(H(t))H'(t)dt \\ &= \frac{qH(s)^{q-1}}{p} + h(H(s))^{p-1} - h(0)^{p-1}. \end{aligned} \tag{4.8}$$

Therefore, integrating by parts,

$$\begin{aligned} \int_{h(0)}^{s}\frac{w_{p,q}(t)}{t}dt &= \frac{p}{2}\int_{h(0)}^{s} t[-h(H(t))^{p-2}]'dt \\ &= -\frac{p}{2}sh(H(s))^{p-2} + \frac{p}{2}h(0)^{p-1} + \frac{p}{2}\int_{h(0)}^{s} h(H(t))^{p-2}dt \\ &= -\frac{p}{2}sh(H(s))^{p-2} + \frac{qH(s)^{q-1}}{2} + \frac{p}{2}h(H(s))^{p-1}, \end{aligned} \tag{4.9}$$

where in the last passage we have used (4.8). Finally, integration by parts gives

$$\int_{h(0)}^{s} w_{p,q}(t)dt = \frac{p}{2}\left[-s^2h(H(s))^{p-2} + h(0)^p\right] + p\int_{h(0)}^{s} th(H(t))^{p-2}dt$$

and

$$\int_{h(0)}^{s} th(H(t))^{p-2}dt = \int_{h(0)}^{s} \big[h(H(t)) + H(t)\big]h(H(t))^{p-2}dt$$
$$= \int_{h(0)}^{s} h(H(t))^{p-1}dt + \int_{h(0)}^{s} H(t)h(H(t))^{p-2}dt = I_1 + I_2.$$

Again integrating by parts, we obtain

$$I_1 = sh(H(s))^{p-1} - h(0)^p - (p-1)\int_{h(0)}^{s} th(H(t))^{p-2}h'(H(t))H'(t)dt$$

and, using the equality $t = h(H(t)) + H(t)$,

$$-(p-1)\int_{h(0)}^{s} th(H(t))^{p-2}h'(H(t))H'(t)dt = J_1 + J_2 + J_3,$$

where

$$J_1 = -(p-1)\int_{h(0)}^{s} h(H(t))^{p-1}h'(H(t))H'(t)dt = \frac{p-1}{p}\left[-h(H(s))^p + h(0)^p\right],$$

$$J_2 = -\int_{h(0)}^{s} h(H(t))^{p-2}h'(H(t))H(t)H'(t)dt$$

and

$$J_3 = (2-p)\int_{h(0)}^{s} h(H(t))^{p-2}h'(H(t))H(t)H'(t)dt.$$

We have that $I_2 + J_2$ equals

$$\int_{h(0)}^{s} h(H(t))^{p-2}H(t)\big[1 - h'(H(t))H'(t)\big]dt = \int_{h(0)}^{s} h(H(t))^{p-2}H(t)H'(t)dt,$$

so, by (2.1), $I_2 + J_2 + J_3$ is given by

$$\frac{1}{p}\int_{h(0)}^{s} q(q-1)H(t)^{q-2}H(t)H'(t)dt = \frac{1}{p}\int_{h(0)}^{s} \left[(q-1)H(t)^q\right]' dt$$
$$= \frac{(q-1)H(s)^q}{p}.$$

Summarizing, we have shown that

(4.10)
$$\int_{h(0)}^{s} w_{p,q}(t)dt = \frac{p}{2}\left[-s^2h(H(s))^{p-2} + h(0)^p\right] + p(sh(H(s))^{p-1} - h(0)^p)$$
$$+ (p-1)\left(-h(H(s))^p + h(0)^p\right) + (q-1)H(s)^q.$$

Now insert the expressions (4.6), (4.9) and (4.10) in (4.7) to obtain (4.5). The proof is complete. □

Analogous arguments lead to the following formula for $U_{p,q}$ in the case $2 < p < q < \infty$.

Lemma 4.2. *Let $2 < p < q < \infty$. We have*

$$U_{p,q}(x,y) = q\frac{|y|^2-|x|^2}{2h(0)^{2-q}} + \frac{q-2}{2}h(0)^q, \tag{4.11}$$

if $|x|+|y| \leq h(0)$, *and*

$$\begin{aligned} U_{p,q}(x,y) =& H(|x|+|y|)^p + pH(|x|+|y|)^{p-1}(|y| - H(|x|+|y|)) \\ & - h(H(|x|+|y|))^q + qh(H(|x|+|y|))^{q-1}(|y| - H(|x|+|y|)), \end{aligned} \tag{4.12}$$

if $|x|+|y| > h(0)$.

Proof. If $|x|+|y| \leq h(0)$, then the integral part in the definition of $U_{p,q}$ vanishes and (4.11) is valid. If $|x|+|y| > h(0)$, then

$$\begin{aligned} U_{p,q}(x,y) =& (|y|^2-|x|^2)\int_{h(0)}^{|x|+|y|} \frac{w_{p,q}(t)}{t^2}dt - 2|y|\int_{h(0)}^{|x|+|y|}\frac{w_{p,q}(t)}{t}dt \\ & + \int_{h(0)}^{|x|+|y|} w_{p,q}(t)dt + q\frac{|y|^2-|x|^2}{2h(0)^{2-q}} + \frac{q-2}{2}h(0)^q \end{aligned} \tag{4.13}$$

and the first integral is given by

$$\int_{h(0)}^{|x|+|y|}\left(\frac{q}{2}h(H(t))^{q-2}\right)' dt = \frac{q}{2}\left(h(H(|x|+|y|))^{q-2} - h(0)^{q-2}\right).$$

To compute the remaining two integrals, we recall the similarity of the formulas for $w_{p,q}$ in the cases $1 \leq p < q < 2$, $2 < p < q < \infty$ and see that the arguments leading to (4.9) and (4.10) are still valid (one only needs to replace p by q, q by p and change the sign). Now insert the equalities to (4.13) to obtain (4.12). □

Now we turn to the majorization property. Recall that $V_{p,q} : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ is given by $V_{p,q}(x,y) = |y|^p - |x|^q$.

Lemma 4.3. *For any $(x,y) \in \mathcal{H}\times\mathcal{H}$ we have $U_{p,q}(x,y) \geq V_{p,q}(x,y)$.*

Proof. Clearly, it suffices to prove the lemma for $\mathcal{H} = \mathbb{R}$. We will present all the details in the case $1 \leq p < q < 2$ only; the remaining case can be established in the same manner. For the convenience of the reader, the proof is split into a few parts.

Step 1: the reduction to the set $|x|+|y| \geq h(0)$. If $|x|+|y| < h(0)$, then, for fixed x, the expression $U_{p,q}(x,y) - V_{p,q}(x,y)$, considered as a function of $|y|$, is nonincreasing (simply calculate the derivative and use the fact that $|y| \leq h(0)$). Hence $U_{p,q}(x,y) - V_{p,q}(x,y) \geq U_{p,q}(x,z) - V_{p,q}(x,z)$, where $z \in \mathcal{H}$ satisfies $|x|+|z| = h(0)$. Therefore, it suffices to establish the majorization on the set $|x|+|y| \geq h(0)$.

Step 2: the case $p = 1$. The majorization follows from convexity of the function $t \mapsto t^q$: indeed, the inequality $U_{p,q}(x,y) \geq V_{p,q}(x,y)$ is equivalent to

$$|x|^q - H(|x|+|y|)^q - qH(|x|+|y|)^{q-1} \cdot (|x| - H(|x|+|y|)) \geq 0.$$

Therefore, till the end of the proof, we assume that $p \neq 1$.

Step 3: the reduction to the case $y = 0$. Fix $r \geq h(0)$ and suppose $|x| + |y| = r$. Denoting $s = |y|$ we see that the inequality $U_{p,q}(x,y) \geq V_{p,q}(x,y)$ is equivalent to

$$\begin{aligned} F(s) = & psh(H(r))^{p-1} - (p-1)h(H(r))^p \\ & - H(r)^q - qH(r)^{q-1}h(H(r)) - s^p + (r-s)^q \geq 0. \end{aligned} \tag{4.14}$$

We have $F(h(H(r))) = F'(h(H(r))) = 0$. Furthermore, the second derivative of F, equal to $F''(s) = -p(p-1)s^{p-2} + q(q-1)(r-s)^{q-2}$, is negative on $(0, s_0)$ and positive on (s_0, r) for some $s_0 \in (0, r)$. Therefore, to show (4.14), it suffices to prove that $F(0) \geq 0$, or $U_{p,q}(x,0) \geq V_{p,q}(x,0)$.

Step 4: the proof of $U_{p,q}(x,0) > V_{p,q}(x,0)$ for large $|x|$. As $q < 2$, we have, for large s,

$$\begin{aligned} \frac{s^q - (H(s))^q - q(H(s))^{q-1}h(H(s))}{h(H(s))^p} &> \frac{q(q-1)}{2} s^{q-2} h(H(s))^{2-p} \\ &= \frac{q(q-1)}{2} H(s)^{q-2} h(H(s))^{2-p} \cdot \left(\frac{s}{H(s)} \right)^{q-2}. \end{aligned}$$

But, by (2.1),

$$\frac{q(q-1)}{2} H(s)^{q-2} h(H(s))^{2-p} \geq \frac{p}{2}$$

and, by (2.10) and Theorem 2.1,

$$\frac{s}{H(s)} = 1 + \frac{h(H(s))}{H(s)} \to 1, \quad \text{as } s \to \infty.$$

Since $p/2 > p - 1$, we see that

$$\frac{s^q - (H(s))^q - q(H(s))^{q-1}h(H(s))}{h(H(s))^p} > p - 1$$

for large s. This is equivalent to $U_{p,q}(x,0) > V_{p,q}(x,0)$ with $|x| = s$.

Step 5: $U_{p,q}(x,0) \geq V_{p,q}(x,0)$: the general case. Suppose the inequality does not hold for all x. Let T denote the largest t satisfying $U_{p,q}(t,0) = V_{p,q}(t,0)$ (its existence is guaranteed by the continuity of $U_{p,q}$ and $V_{p,q}$ and the previous step). By the considerations above, we have $U_{p,q} \geq V_{p,q}$ on the set $|x| + |y| \geq T$. Consider the processes $f = (f_0, f_1, f_2)$, $g = (g_0, g_1, g_2)$ on a probability space $([0,1], \mathcal{B}([0,1]), |\cdot|)$ such that $(f_0, g_0) \equiv (T, 0)$ and

$$df_1 = dg_1 = \delta\chi_{[0,t_0]} - h(H(T))\chi_{(t_0,1]},$$

$$df_2 = -dg_2 = \delta\chi_{[0,t_1]} - (h(H(T)) - \delta)\chi_{(t_1,t_0]},$$

where

$$t_0 = \frac{h(H(T))}{h(H(T)) + \delta}, \quad t_1 = t_0 \cdot \frac{h(H(T)) - \delta}{h(H(T))} = \frac{h(H(T)) - \delta}{h(H(T)) + \delta}.$$

It is straightforward to check that f and g are martingales and g is differentially subordinate to f. Note that we have $U_{p,q}(f_2, g_2) \geq V_{p,q}(f_2, g_2)$ almost surely, as $|f_2| + |g_2| \geq T$ with probability 1. We have, by (3.2) with W replaced by $U_{p,q}$,

$$\begin{aligned} -T^q =& U_{p,q}(T, 0) = \mathbb{E}U_{p,q}(f_0, g_0) \geq \mathbb{E}U_{p,q}(f_2, g_2) \geq \mathbb{E}V_{p,q}(f_2, g_2) \\ =& \frac{2\delta}{h(H(T)) + \delta}(h(H(T))^p - H(T)^q) - \frac{h(H(T)) - \delta}{h(H(T)) + \delta}(T + 2\delta)^q, \end{aligned}$$

which is equivalent to

$$\frac{(T + 2\delta)^q - T^q}{2\delta} - \frac{(T + 2\delta)^q}{h(H(T)) + \delta} + \frac{H(T)^q}{h(H(T)) + \delta} \geq \frac{h(H(T))^p}{h(H(T)) + \delta}.$$

Letting $\delta \to 0$ and muiltiplying throughout by $h(H(T))$ yields

$$h(H(T))^p \leq H(T)^q - T^q - qT^{q-1}h(H(T)). \tag{4.15}$$

But we have

$$H(T)^q - T^q - qT^{q-1}h(H(T)) \leq T^q - H(T)^q - qH(T)^{q-1}h(H(T)). \tag{4.16}$$

To see this, substitute $H(T) = a > 0$, $T - H(T) = b > 0$ and observe that (4.16) becomes

$$2[(a + b)^q - a^q] - [qa^{q-1} + q(a + b)^{q-1}]b \geq 0.$$

Now calculate the derivative of the left hand side with respect to b: it is equal to

$$q(a + b)^{q-1} - qa^{q-1} - q(q - 1)(a + b)^{q-2}b \geq 0,$$

due to mean value theorem. Thus (4.16) follows.

To conclude the proof, observe that the inequalities (4.15) and (4.16) give

$$h(H(T))^p \leq T^q - H(T)^q - qH(T)^{q-1}h(H(T))$$

and since $p < 2$, this implies

$$(p - 1)h(H(T))^p < T^q - H(T)^q - qH(T)^{q-1}h(H(T)),$$

a contradiction: the inequality above is equivalent to $U_{p,q}(T, 0) < V_{p,q}(T, 0)$.

□

Remark 4.4. A careful study of the proof shows that $U_{p,q}(x, y) = V_{p,q}(x, y)$ if and only if $|y| = h(|x|)$.

5. Proofs of moment inequalities

Now we are ready to establish the inequalities announced in the introduction. Suppose $1 \le p < q < \infty$ and let

$$L_{p,q} = \begin{cases} \frac{1}{2}(2-p)h(0)^p & \text{if } 1 \le p < q < 2, \\ \frac{1}{2}(q-2)h(0)^q & \text{if } 2 < p < q < \infty \end{cases} \tag{5.1}$$

and

$$C_{p,q} = \begin{cases} L_{p,q}^{(q-p)/pq} \left(\frac{q-p}{p}\right)^{1/q} \left(\frac{q}{q-p}\right)^{1/p} & \text{if } 1 \le p < q < 2 \\ & \text{ or } 2 < p < q < \infty, \\ 1 & \text{otherwise.} \end{cases} \tag{5.2}$$

Theorem 5.1. *Assume g is differentially subordinate to f.*

(i) If $1 \le p < q < 2$ or $2 < p < q < \infty$, then (1.5) *holds.*

(ii) If $1 \le p < q < \infty$, then the inequality (1.4) *holds.*

Remark 5.2. (i) If $p = 1$, then there are explicit formulas for the constants $C_{p,q}$, given by

$$C_{1,q} = \begin{cases} \left[\frac{q}{2}\Gamma\left(\frac{q}{q-1}\right)\right]^{(q-1)/q} & \text{if } q \in (1,2), \\ 1 & \text{if } q \ge 2. \end{cases}$$

(ii) It is easy to check that $C_{p,q} = C_{q',p'}$ for $1 < p < q < \infty$.

Proof of Theorem 5.1. With no loss of generality we may assume $0 < ||f||_q < \infty$, which, by Burkholder's inequality (1.2), implies $||g||_p \le ||g||_q < \infty$. Since $H(s) \le s$ and $h(H(s)) \le s$, we have that the variables $U_{p,q}(f_n, g_n)$, $n = 0, 1, 2, \ldots$ are integrable and, by Lemma 4.3 and (3.4), we have, for any $n = 0, 1, 2, \ldots$,

$$\mathbb{E}V_{p,q}(f_n, g_n) \le \mathbb{E}U_{p,q}(f_n, g_n) \le \mathbb{E}U_{p,q}(f_0, g_0) \le L_{p,q}.$$

It suffices to let $n \to \infty$ to obtain (1.5). Now we turn to (1.4). If $p \le 2 \le q$, the inequality is clear: we have $||g||_p \le ||g||_2 \le ||f||_2 \le ||f||_q$. Suppose that $1 \le p < q < 2$. As already proved, we have

$$||g||_p^p \le ||f||_q^q + L_{p,q}.$$

Since for $\lambda > 0$, $g \cdot \lambda^{1/(q-p)}$ is differentially subordinate to $f \cdot \lambda^{1/(q-p)}$, we obtain

$$||g||_p^p \le \lambda ||f||_q^q + \frac{L_{p,q}}{\lambda^{p/(q-p)}}. \tag{5.3}$$

It can be easily checked that the right-hand side, as a function of λ, attains its minimum for

$$\lambda = \left(\frac{p}{q-p} \cdot \frac{L_{p,q}}{||f||_q^q}\right)^{(q-p)/q}$$

and the minimum is equal to

$$\left[L_{p,q}^{(q-p)/pq}\cdot\left(\frac{q-p}{p}\right)^{1/q}\left(\frac{q}{q-p}\right)^{1/p}||f||_q\right]^p.$$

The case $2 < p < q < \infty$ is dealt with exactly in the same manner. □

Theorem 5.3. *Let* $1 < p < \infty$*. Then*

$$\lim_{p',q'\to p} C_{p',q'} = p^* - 1.$$

Proof. The inequalities (1.2) and (1.4) are sharp even if $\mathcal{H} = \mathbb{R}$, f is simple and g is a ± 1 transform of f (see [Bu2] and Theorem 7.2 below). Recall that f is simple if for any n the variable f_n takes only a finite number of values and there is a deterministic N such that $f_N = f_{N+1} = f_{N+2} = \ldots$ with probability 1. Note that if f is simple, then so is its ± 1 transform g.

Fix $1 < p < \infty$ and a pair (f, g) as above. Let $1 \leq p_n < q_n < \infty$, $n = 0, 1, 2, \ldots$, satisfy $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = p$. Observe that we have $||g||_{p_n} \leq C_{p_n,q_n}||f||_{q_n}$, $\lim_{n\to\infty}||g||_{p_n} = ||g||_p$ and $\lim_{n\to\infty}||f||_{q_n} = ||f||_p$ (there are no problems with integrability, as the martingales are simple). Consequently, $||g||_p \leq \liminf_{n\to\infty} C_{p_n,q_n}||f||_p$ and hence

$$p^* - 1 \leq \liminf_{n\to\infty} C_{p_n,q_n}.$$

On the other hand, for any n we have $||g||_{p_n} \leq ||g||_{q_n} \leq (q_n^* - 1)||f||_{q_n}$, which implies $C_{p_n,q_n} \leq q_n^* - 1$ and

$$\limsup_{n\to\infty} C_{p_n,q_n} \leq p^* - 1.$$

This completes the proof. □

6. Related estimates for the first moment

The method above can be extended to establish some related inequalities. Suppose $\Phi : [0,\infty) \to [0,\infty)$ is a function satisfying the following conditions: Φ is an increasing convex function belonging to C^2 such that $\Phi(0) = \Phi'(0+) = 0$,

$$\Phi''(t) \downarrow 0 \text{ as } t \to \infty \tag{6.1}$$

and

$$\int_0^\infty \exp(-\Phi'(s))ds < \infty. \tag{6.2}$$

For example, one can take $\Phi(t) = t^q$, $q \in (1,2)$, or $\Phi(t) = t\log(1+t)$. Note that the above conditions imply that Φ is strictly convex.

The purpose of this section is to establish the inequality

$$||g||_1 \leq \sup_n \mathbb{E}\Phi(|f_n|) + L(\Phi) \tag{6.3}$$

for any pair (f, g) of $\mathcal{H}$-valued martingales such that g is differentially subordinate to f. Here $L(\Phi)$ is a constant which depends only on Φ. Let

$$h(t) = \exp(\Phi'(t)) \int_t^\infty \exp(-\Phi'(s))ds \tag{6.4}$$

and let H be an inverse function to $t \mapsto t + h(t)$. The special function $U_{1,\Phi}$: $\mathcal{H} \times \mathcal{H} \to \mathbb{R}$ corresponding to (6.3) is defined as follows: if $|x| + |y| \leq h(0)$, then

$$U_{1,\Phi}(x, y) = \frac{|y|^2 - |x|^2}{2h(0)} + \frac{h(0)}{2}, \tag{6.5}$$

while for $|x| + |y| > h(0)$ and $|y| \leq h(|x|)$, set

(6.6)

$$U_{1,\Phi}(x, y) = |y| - \Phi(H(|x| + |y|)) - \Phi'(H(|x| + |y|))\big(|x| - H(|x| + |y|)\big).$$

Finally, if $|y| > h(|x|)$ (then $|x| + |y| > h(0)$), define

$$U_{1,\Phi}(x, y) = |y| - \Phi(|x|). \tag{6.7}$$

If one takes $\Phi(t) = t^q$, $1 < q < 2$, one obtains an alternative special function for the moment inequalities. One can check (arguing as in the proof of Lemma 4.1) that we have the following integral identity: for $|y| \leq h(|x|)$,

$$U_{1,\Phi}(x, y) = \int_{h(0)}^\infty w_{1,\Phi}(t) W_1(x/t, y/t)dt, \tag{6.8}$$

where

$$w_{1,\Phi}(t) = \frac{1}{2} h(H(t))^{-2} h'(H(t)) H'(t) t^2.$$

The reason for which we modify the function on the set $|y| > h(|x|)$ (and use the formula (6.7) instead of (6.6)) is to avoid the problem of integrability of the variables $U_{1,\Phi}(f_n, g_n)$, $n = 0, 1, 2, \ldots$.

The key properties of the function $U_{1,\Phi}$ are gathered in the lemma below. The function $V_{1,\Phi} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is given by $V_{1,\Phi}(x, y) = |y| - \Phi(|x|)$.

Lemma 6.1. *(i) For any $(x, y) \in \mathcal{H} \times \mathcal{H}$ we have $U_{1,\Phi}(x, y) \geq V_{1,\Phi}(x, y)$. Both sides are equal if and only if $|y| \geq h(|x|)$.*

(ii) Assume that x, y, k_x, $k_y \in \mathcal{H}$ satisfy $|k_y| \leq |k_x|$. Then the function $G : \mathbb{R} \to \mathbb{R}$, given by $G(t) = U_{1,\Phi}(x + tk_x, y + tk_y)$, is concave.

Proof. (i) Arguing as in the proof of Lemma 4.3, we see that it suffices to establish the majorization for x, y satisfying $|x| + |y| > h(0)$. If $|y| \geq$

$h(|x|)$, then we have equality: $U_{1,\Phi}(x,y) = V_{1,\Phi}(x,y)$. For remaining x, y, the estimate is equivalent to

$$\Phi(|x|) - \Phi(H(|x|+|y|)) - \Phi'(H(|x|+|y|))\big(|x| - H(|x|+|y|)\big) \geq 0,$$

which follows from convexity of Φ. In fact, the latter inequality is strict, since Φ is strictly convex.

(ii) By (6.8), the function G has this property on the set of those t, for which $|y+tk_y| < h(|x+tk_x|)$. Let us deal with the inequality $G''(t) < 0$ for t such that $|y+tk_y| > h(|x+tk_x|)$, $|x+tk_x| > 0$. By translation argument, we may assume $t = 0$. We have

$$G''(0) = \frac{|k_y|^2 - (y', k_y)^2}{|y|} - \Phi''(|x|)(x', k_x)^2 - \Phi'(|x|)\frac{|k_x|^2 - (x', k_x)^2}{|x|},$$

which, as $\Phi'(|x|) \geq \Phi''(|x|)|x|$, can be bounded from above by

$$\begin{aligned} \frac{|k_y|^2}{|y|} - \Phi''(|x|)|k_x|^2 &\leq \frac{|k_y|^2}{h(|x|)} - \Phi''(|x|)|k_x|^2 \\ &= \frac{|k_y|^2 - |k_x|^2}{h(|x|)} + \frac{(1 - \Phi''(|x|)h(|x|))|k_x|^2}{h(|x|)}. \end{aligned}$$

It suffices to show that both summands appearing in the last expression are nonpositive. This is clear for the first one, as $|k_y| \leq |k_x|$. To deal with the second one, note that by concavity of Φ', for any $s > t$ we have

$$\Phi'(s) \leq \Phi'(t) + \Phi''(t)(s-t) \tag{6.9}$$

and hence, for any t,

$$h(t) = \int_t^\infty \exp(\Phi'(t) - \Phi'(s))ds \geq \int_t^\infty \exp(-\Phi''(t)(s-t))ds = \frac{1}{\Phi''(t)}.$$

The final observation is that we have $G'(t+) \leq G'(t-)$ if $|y+tk_y| = h(|x+tk_x|)$ or $|x+tk_x| = 0$; hence G is concave. □

Now we can prove the following result.

Theorem 6.2. *Assume g is differentially subordinate to f. Then the inequality* (6.3) *holds with*

$$L(\Phi) = \frac{h(0)}{2} = \frac{1}{2}\int_0^\infty \exp(-\Phi'(s))ds. \tag{6.10}$$

Proof. We proceed as previously. With no loss of generality we may assume that $\sup_n \mathbb{E}\Phi(|f_n|) < \infty$, which guarantees integrability of $U_{1,\Phi}(f_n, g_n)$, $n = 0, 1, 2, \ldots$, and we have

$$\mathbb{E}V_{1,\Phi}(f_n, g_n) \leq \mathbb{E}U_{1,\Phi}(f_n, g_n) \leq \mathbb{E}U_{1,\Phi}(f_0, g_0) \leq L(\Phi)$$

for all n. Let $n \to \infty$ to complete the proof. □

As an example, we will establish a family of log log-estimates for the first moment of the martingale g. As a motivation, note that the inequality (1.3) holds with some finite L if and only if $K > 1$. Therefore, there is a natural question what happens in the limit case $K = 1$. The (partial) answer is contained in the following theorem.

Theorem 6.3. *If g is differentially subordinate to f, then, for any $K > 1$,*

$$\begin{aligned} ||g||_1 \leq& \sup_n \left[\mathbb{E}|f_n|\log(|f_n|+1) + K\mathbb{E}\log\log(|f_n|+e)\right] \\ &+ \frac{1}{2}\int_0^\infty \frac{1}{(s+1)(\log(s+e))^K}\exp\left(-\frac{s}{s+1} - \frac{Ks}{(s+e)\log(s+e)}\right) ds. \end{aligned} \tag{6.11}$$

To prove this result, let Φ be given by $\Phi(t) = t\log(t+1) + Kt\log\log(t+e)$. Then the assertion follows from Theorem 6.2, once we have checked that this function enjoys all the needed properties, described at the beginning of this section. Clearly, Φ is convex, of class C^2 and we have $\Phi(0) = \Phi'(0+) = 0$. Furthermore,

Lemma 6.4. *For any $K > 0$ the condition* (6.1) *is valid.*

Proof. It is easy to see that $\Phi''(t) \to 0$ as $t \to \infty$. Hence it suffices to show that $\Phi''' < 0$. Let $\psi_1 = t\log(t+1)$ and $\psi_2(t) = t\log\log(t+e)$. We have

$$\psi_1'''(t) = -\frac{2}{(t+1)^3} - \frac{1}{(t+1)^2} < 0$$

and

$$\psi_2'''(t) = \frac{-3e\log(t+e) - 3e\log^2(t+e) - t\log^2(t+e) + 2t}{((t+e)\log(t+e))^3}.$$

If $t \in (0, 3e]$, then $2t \leq 6e \leq 3e\log(t+e) + 3e\log^2(t+e)$, which implies that the derivative is negative. This is also true for $t > 3e$, which follows from estimate $2t < t\log^2(4e) < t\log^2(t+e)$. □

The last property we need is the following.

Lemma 6.5. *If $K > 1$, then the condition* (6.2) *holds.*

Proof. We have

$$\begin{aligned} \Phi'(t) =& \log(t+1) + K\log\log(t+e) + \frac{t}{t+1} + K\frac{t}{(t+e)\log(t+e)} \\ &\geq \log(t+1) + K\log\log(t+e). \end{aligned} \tag{6.12}$$

Therefore

$$\int_0^\infty \exp(-\Phi'(s))ds < \int_0^\infty \frac{ds}{(s+1)(\log(s+e))^K} < \infty. \quad \square$$

Therefore we may use Theorem 6.2 to the function Φ. One easily checks that the constant $L(\Phi)$ is given by the expression appearing on the right hand side of (6.11).

7. Strictness and sharpness

This is the final section of the paper. We will address here the strictness of the estimates studied above as well as the optimality of the constants $L_{p,q}$, $C_{p,q}$ and $L(\Phi)$. First we will establish the following result.

Theorem 7.1. *The inequality* (1.5) *is strict for all* $1 \leq p < q < 2$ *and* $2 < p < q < \infty$*, unless both sides are infinite. The inequality* (1.4) *is strict for* $1 \leq p < q < 2$ *and* $2 < p < q < \infty$*, unless both sides are infinite or equal to* 0*. If* $2 \in [p, q]$*, then both sides of* (1.4) *may be equal for some nontrivial* f *and* g*. The inequality* (6.3) *is strict unless both sides are infinite.*

Proof. We start from showing that equality in (1.5) cannot be attained. First we consider the case $1 \leq p < q < 2$. Let h be the corresponding solution to $(2.1)_{p,q}$ and fix martingales f, g such that $||f||_q < \infty$ and g is differentially subordinate to f. Our goal is to prove that $||g||_p < ||f||_q + L_{p,q}$. We may assume that $(f_0, g_0) \equiv (0, 0)$, which can be shown by the following standard argument. Take a Rademacher variable ε, independent of $\sigma(\bigcup_{n=1}^{\infty} \mathcal{F}_n)$ (enlarging the probability space, if necessary) and replace (f, g) by the pair $(\hat{f}, \hat{g})$ given by $(\hat{f}_0, \hat{g}_0) = (0, 0)$ and $\hat{f}_n = \varepsilon f_{n-1}$, $\hat{g}_n = \varepsilon g_{n-1}$, relative to the filtration $\{\emptyset, \Omega\} \subset \sigma(\mathcal{F}_0, \varepsilon) \subset \sigma(\mathcal{F}_1, \varepsilon) \subset \ldots$. Then $\hat{f}$, $\hat{g}$ have the required starting property and $||\hat{f}||_q = ||f||_q$, $||\hat{g}||_p = ||g||_p$.

We will consider two cases: first, assume that that there is an integer N such that $\mathbb{P}(|f_N| + |g_N| > h(0)) > 0$ and pick the smallest N with such a property. Note that $N \geq 1$, since f and g start from 0. Then, for t belonging to some interval of the form $(h(0), T)$, we have $|f_{N-1}/t| + |g_{N-1}/t| < 1$ almost surely and $\mathbb{P}(|f_N/t| + |g_N/t| > 1) > 0$. By Lemma 3.1 we have, for these t,

$$\mathbb{E}W_1(f_N/t, g_N/t) < \mathbb{E}W_1(f_{N-1}/t, g_{N-1}/t).$$

Since the kernel $w_{p,q}$ defined in (4.1) is positive, we obtain, by (4.2),

$$\mathbb{E}U_{p,q}(f_N, g_N) < \mathbb{E}U_{p,q}(f_{N-1}, g_{N-1}).$$

This implies the strictness:

$$||g||_p^p - ||f||_q^q = \lim_{n \to \infty} \mathbb{E}V_{p,q}(f_n, g_n) \leq \mathbb{E}U_{p,q}(f_N, g_N) < \mathbb{E}U_{p,q}(f_0, g_0) = L_{p,q}.$$

We turn to the second case when for all n we have $|f_n| + |g_n| \leq h(0)$ almost surely. Then the martingales f, g are bounded, so in particular they converge in L^2 to, say, f_∞ and g_∞. We have $|f_\infty| + |g_\infty| \leq h(0)$ almost surely and, by (1.2), $||g_\infty||_2^2 \leq ||f_\infty||_2^2$, so

$$\mathbb{P}(|f_\infty| > 0) > 0 \quad \text{or} \quad |f_\infty| = |g_\infty| = 0. \tag{7.1}$$

This implies

$$||g||_p^p - ||f||_q^q = ||g_\infty||_p^p - ||f_\infty||_q^q = \mathbb{E}V_{p,q}(f_\infty, g_\infty) < \mathbb{E}U_{p,q}(f_\infty, g_\infty) \leq L_{p,q},$$

where the strict inequality comes from Remark 4.4 and (7.1): the only x, y satisfying $|x|+|y| \leq h(0)$ and $|y| = h(|x|)$, are given by $x = 0$ and $|y| = h(0)$. This shows that there are no f, g with $||f||_q < \infty$, for which both sides of (1.5) are equal. The case $2 < p < q < \infty$ can be studied in the same manner, using the second inequality from Lemma 3.1. Clearly, this implies that (1.4) is strict for $1 \leq p < q < 2$ and $2 < p < q < \infty$, unless $||f||_q = ||g||_p = 0$ or $||f||_q = ||g||_p = \infty$. On the other hand, if $2 \in [p, q]$, then equality in (1.4) can be attained: take, for example, $f = g \equiv 1$.

We conclude by observing that (6.3) is also strict unless both sides are infinite. Although $U_{1,\Phi}$ is not defined using integral representation, it majorizes the integral on the right of (6.8), with equality for $|y| \leq h(|x|)$ and hence also for $|x| + |y| \leq h(0)$. This is all we need to apply the above argument with the function W_1. The strictness follows. □

Now we turn to the optimality of the constants.

Theorem 7.2. *(i) The constants $L_{p,q}$, $C_{p,q}$ given by* (5.1) *and* (5.2) *are the best possible even in the case $\mathcal{H} = \mathbb{R}$ and when g is ± 1 transform of f.*

(ii) If Φ satisfies the condition

$$\lim_{t\to\infty} \frac{\Phi(t + h(t)) - \Phi(t) - \Phi'(t)h(t)}{\exp(\Phi'(t))} = 0, \tag{7.2}$$

then the constant $L(\Phi)$ defined by (6.10) *is the best possible even in the case $\mathcal{H} = \mathbb{R}$ and when g is ± 1 transform of f.*

The proof will be based on a construction of an appropriate example, which will work for (i) in the case $1 \leq p < q < 2$, and for (ii) simultaneously. Suppose h is the solution to $(2.1)_{p,q}$ for some $1 \leq p < q < 2$ or is given by (6.4). Let U, V denote $U_{p,q}$, $V_{p,q}$ or $U_{1,\Phi}$, $V_{1,\Phi}$, respectively. Furthermore, in the first case, assume that $\Phi(t) = t^q$, $t \in [0, \infty)$, and observe that (7.2) is valid, which is a consequence of $h(t)/t \to 0$, as $t \to \infty$ (the latter convergence is due to $h'(t) \to 0$ as $t \to \infty$).

We start with the following auxiliary result.

Lemma 7.3. *Assume $\mathcal{H} = \mathbb{R}$ and let $x > h(0)$, $\delta \in (0, h(0))$. Then*

$$\begin{aligned} U(x+\delta,\delta) =& \frac{\delta}{x+2\delta-H(x+2\delta)} U(H(x+2\delta), x+2\delta-H(x+2\delta)) \\ &+ \frac{x+\delta-H(x+2\delta)}{x+2\delta-H(x+2\delta)} U(x+2\delta,0) \end{aligned} \tag{7.3}$$

and

$$\begin{aligned} U(x,0) =& \frac{\delta}{x-H(x)+\delta} U(H(x), x-H(x)) \\ &+ \frac{x-H(x)}{x-H(x)+\delta} U(x+\delta,\delta) + R(x,\delta), \end{aligned} \tag{7.4}$$

with $R(x,\delta)$ satisfying

$$\lim_{\delta\to 0} \sup_{h(0)\leq x\leq T} \frac{R(x,\delta)}{\delta} = 0 \tag{7.5}$$

for any $T > h(0)$.

Proof. The first equality follows immediately from the fact that for any $x \geq h(0)$, the function $t \mapsto U(x-t,t)$, $t \in [0, x-H(x)]$, is linear. This property is also needed in the second part; we may write

$$\begin{aligned} \frac{R(x,\delta)}{\delta} =& \frac{-U(H(x), x-H(x)) + U(x,0)}{x-H(x)+\delta} \\ &+ \frac{x-H(x)}{x-H(x)+\delta} \cdot \frac{U(x,0)-U(x+\delta,\delta)}{\delta} \end{aligned}$$

and since $U_y(x,0) = 0$, we have

$$U(H(x), x-H(x)) - U(x,0) = (H(x)-x)U_x(x,0).$$

Therefore,

$$\frac{R(x,\delta)}{\delta} = \frac{x-H(x)}{x-H(x)+\delta}\left(U_x(x,0) - \frac{U(x+\delta,\delta)-U(x,0)}{\delta}\right)$$

and, by the mean value property,

$$\begin{aligned} \left|\frac{R(x,\delta)}{\delta}\right| &\leq \left|U_x(x,0) - \frac{U(x+\delta,0)-U(x,0)}{\delta} - \frac{U(x+\delta,\delta)-U(x+\delta,0)}{\delta}\right| \\ &\leq |U_x(x,0) - U_x(\xi,0)| + |U_y(x+\delta,\eta)| \end{aligned}$$

for some $\xi \in (x, x+\delta)$, $\eta \in (0,\delta)$. Now it is easy to see that (7.5) is valid for any T: this follows from uniform continuity of the function $U_x(\cdot,0)$ on $[h(0), T+1]$ and the uniform continuity of U_y on $[h(0), T+1] \times [0, h(0)]$. □

Now let us study the following example. Let $T > h(0)$ be a fixed (large) positive number. Let N be an integer and δ be a positive number such that $T = h(0) + 2N\delta$. Eventually we will let δ go to 0 (and hence N to ∞).

Consider a Markov martingale (f,g), determined uniquely by the following conditions.

(i) It starts from $(h(0)/2, h(0)/2)$ almost surely.

(ii) From $(h(0)/2, h(0)/2)$ it moves either to $(h(0), 0)$, or to $(0, h(0))$.

(iii) For $h(0) \leq x < T$, the state $(x, 0)$ leads to $(H(x), h(H(x)))$ or to $(x+\delta, -\delta)$.

(iv) For $h(0) \leq x < T$, the state $(x+\delta, -\delta)$ leads to $(H(x+2\delta), -h(H(x+2\delta)))$ or to $(x+2\delta, 0)$.

(v) For $x \geq T$, the states $(x, 0)$ are absorbing.

(vi) All the states lying on the curves $y = \pm h(x)$ are absorbing.

Note that in (ii)–(iv) we do not need to specify the transity probabilities, as they are determined by the martingale property. Note that g is a ± 1 transform of f.

From the proof of (1.5) we know that the sequence $(\mathbb{E}U(f_n, g_n))$ is non-increasing. The lemma below states that it is almost constant.

Lemma 7.4. *We have*

$$\mathbb{E}U(f_{2N}, g_{2N}) \geq U(h(0)/2, h(0)/2) - N \sup_{h(0)\leq x\leq T} |R(x,\delta)|.$$

Proof. We start from the observation that, by Lemma 7.3, we have

$$\mathbb{E}U(f_{2n}, g_{2n}) = \mathbb{E}U(f_{2n+1}, g_{2n+1})$$

for $n = 0, 1, \ldots, N$, and

$$\mathbb{E}U(f_{2n+1}, g_{2n+1}) = \mathbb{E}U(f_{2n+2}, g_{2n+2}) + R(x_{2n+1}, \delta)$$

for $n = 0, 1, 2, \ldots, N-1$, where we have used the notation $x_n = h(0) + (n-1)\delta$. This gives

$$U(h(0)/2, h(0)/2) = \mathbb{E}U(f_0, g_0) = \mathbb{E}U(f_{2N}, g_{2N}) + \sum_{n=0}^{N-1} R(x_{2n+1}, \delta)$$

and the estimate follows. □

It is easy to see from the conditions (iii) and (iv), that the process (f_n, g_n) moves from $(x, 0)$ to $(x+2\delta, 0)$ in two steps with probability

$$\left(1 - \frac{\delta}{x - H(x) + \delta}\right)\left(1 - \frac{\delta}{x + 2\delta - H(x+2\delta)}\right).$$

Hence the probability p_δ that (f_n, g_n) ever reaches $(T, 0)$ equals

$$\frac{1}{2}\prod_{n=0}^{N-1}\left(1 - \frac{\delta}{x_{2n+1} - H(x_{2n+1}) + \delta}\right)\left(1 - \frac{\delta}{x_{2n+1} + 2\delta - H(x_{2n+1}+2\delta)}\right).$$

In the next lemma we study the limit behavior of p_δ, as $\delta \to 0$.

Lemma 7.5. *We have*

$$\lim_{\delta\to 0} p_\delta = \frac{1}{2}\exp(-\Phi'(H(T))).$$

Proof. We have that

$$\frac{1}{2}\exp\left[\sum_{k=0}^{N-1}\left(\frac{\delta}{x_{2n+1}-H(x_{2n+1})+\delta}+\frac{\delta}{x_{2n+1}+2\delta-H(x_{2n+1}+2\delta)}\right)\right]\cdot p_\delta^{-1}$$

converges to 1 as $\delta \to 0$, and the Riemann sum

$$\sum_{k=0}^{N-1}\left(\frac{\delta}{x_{2n+1}-H(x_{2n+1})+\delta}+\frac{\delta}{x_{2n+1}+2\delta-H(x_{2n+1}+2\delta)}\right)$$

converges, as $\delta \to 0$, to

$$-\int_{h(0)}^T \frac{1}{x-H(x)}dx = -\int_{h(0)}^T \frac{1}{h(H(x))}dx.$$

Substitution $y = H(x)$ (so $x = y + h(y)$) transforms the integral to

$$-\int_0^{H(T)} \frac{1+h'(y)}{h(y)}dy = -\int_0^{H(T)} \Phi''(y)dy = -\Phi'(H(T)).$$

This yields the claim. □

Proof of Theorem 7.2. For convenience, let us divide the proof into a few parts.

Step 1. Sharpness of (1.5) for $1 \leq p < q < 2$ and sharpness of (6.3).

Note that on the set $\{(x, y) : y = \pm h(x)\}$ the functions U and V coincide. The variable (f_{2N}, g_{2N}) belongs to this set unless we have $(f_{2N}, g_{2N}) = (T, 0)$, which occurs with probability p_δ. Hence we may write

$$\mathbb{E}V(f_{2N}, g_{2N}) = \mathbb{E}U(f_{2N}, g_{2N}) + (V(T,0) - U(T,0))p_\delta$$

$$\geq U(h(0)/2, h(0)/2) - N \sup_{h(0)\leq x\leq T} |R(x,\delta)| + (V(T,0) - U(T,0))p_\delta,$$

where we have used Lemma 7.4. Now let us tend with δ to 0. By (7.5) and the equation $T = h(0) + 2N\delta$, we have

$$N \sup_{h(0)\leq x\leq T} |R(x,\delta)| = \frac{T-h(0)}{2\delta} \sup_{h(0)\leq x\leq T} |R(x,\delta)| \to 0. \tag{7.6}$$

Furthermore, by Lemma 4.3 and Lemma 7.5, we have

$$0 \leq (V(T,0) - U(T,0))p_\delta \to -\frac{\Phi(T) - \Phi(H(T)) - \Phi'(H(T))(T - H(T))}{2\exp(\Phi'(H(T))}. \tag{7.7}$$

Now we proceed as follows. By (7.2), for a fixed $\varepsilon > 0$ we may choose such a T, that the expression appearing on the right is bigger than $-\varepsilon$. Keeping this T fixed, we use (7.6) and (7.7) to choose $\delta > 0$, for which

$$\mathbb{E}|g_{2N}| - \mathbb{E}\Phi(|f_{2N}|) = \mathbb{E}V(f_{2N}, g_{2N}) \geq U(h(0)/2, h(0)/2) - 3\varepsilon.$$

However, $U(h(0)/2, h(0)/2)$ equals $L_{p,q}$ or $L(\Phi)$, depending on the choice of h in the example; hence, as ε is arbitrary, these constants are best possible.

Step 2. The optimality of $C_{p,q}$, $1 \leq p < q < \infty$. If $1 \leq p \leq 2 \leq q < \infty$, then, obviously, $C_{p,q} = 1$ is the best possible (take $f = g \equiv 1$). Now, by duality and Remark 5.2 (ii), it suffices to deal with the case $1 \leq p < q < 2$. For fixed $\varepsilon > 0$, let f^ε, g^ε be real martingales such that g^ε is ± 1 transform of f^ε and

$$||g^\varepsilon||_p^p > ||f^\varepsilon||_q^q + L_{p,q} - \varepsilon. \tag{7.8}$$

Let

$$\lambda = \left(\frac{p}{q-p} \cdot \frac{L_{p,q} - \varepsilon}{||f||_q^q}\right)^{(q-p)/q}.$$

The martingale $\tilde{g} = g^\varepsilon \cdot \lambda^{1/(q-p)}$ is a ± 1 transform of $\tilde{f} = f^\varepsilon \cdot \lambda^{1/(q-p)}$. Multiply both sides of (7.8) by $\lambda^{p/(q-p)}$ to obtain

$$||\tilde{g}||_p^p > \left[(L_{p,q} - \varepsilon)^{(q-p)/pq} \cdot \left(\frac{q-p}{p}\right)^{1/q} \left(\frac{q}{q-p}\right)^{1/p}\right]^p ||\tilde{f}||_q^p.$$

It is clear that the expression in the square brackets can be made arbitrarily close to $C_{p,q}$. This implies $C_{p,q}$ can not be replaced by a smaller constant in (1.4).

Step 3. Sharpness of (1.5) for $2 < p < q < \infty$. This is an immediate consequence of Step 2. If $L_{p,q}$ were not optimal for some p, q, then $C_{p,q}$ would not be the best either; this can be easily seen by examining the proof of Theorem 5.1. □

We conclude the paper by proving that the loglog inequality established in the previous section is sharp. Let $\Phi(t) = t\log(t+1) + Kt\log\log(t+e)$ and let h be given by (6.4).

Lemma 7.6. *If $K > 1$, then the condition* (7.2) *is valid.*

Proof. By de l'Hospital rule, it suffices to show that

$$\lim_{t\to\infty} \frac{\Phi'(t+h(t))(1+h'(t)) - \Phi'(t)(1+h'(t)) - \Phi''(t)h(t)}{\exp(\Phi'(t))\Phi''(t)} = 0.$$

Since $1 + h'(t) = h(t)\Phi''(t)$, this can be rewritten in the form

$$\lim_{t\to\infty} \frac{[\Phi'(t+h(t)) - \Phi'(t) - 1]h(t)}{\exp(\Phi'(t))} = 0.$$

As $h(t)/\exp(\Phi'(t)) = \int_t^\infty \exp(-\Phi'(s))ds$ tends to 0, we must show that

$$\lim_{t\to\infty}(\Phi'(t+h(t)) - \Phi'(t)) \int_t^\infty \exp(-\Phi'(s))ds = 0. \tag{7.9}$$

We have, by (6.12),

$$\begin{aligned}\int_t^\infty \exp(-\Phi'(s))ds &\leq \int_t^\infty \frac{ds}{(s+1)(\log(s+e))^K}\\ &\leq \int_t^\infty \frac{e\,ds}{(s+e)(\log(s+e))^K} = \frac{e(\log(t+e))^{1-K}}{K-1}.\end{aligned} \tag{7.10}$$

Furthermore, as

$$\Phi'(t) \leq \log(t+1) + K\log\log(t+e) + K + 1,$$

we have

$$\begin{aligned}h(t) &\leq \exp(\Phi'(t)) \cdot \frac{e(\log(t+e))^{1-K}}{K-1}\\ &\leq e^{K+1}(t+1)(\log(t+e))^K \cdot \frac{e(\log(t+e))^{1-K}}{K-1}\\ &= \frac{e^{K+2}}{K-1}(t+1)\log(t+e).\end{aligned} \tag{7.11}$$

Therefore, by (6.12),

$$\begin{aligned}0 \leq \Phi'(t+h(t)) - \Phi'(t) \leq& K\log\log(t+h(t)+e) - K\log\log(t+e)\\ &+ \log(t+h(t)+1) - \log(t+1) + (K+1).\end{aligned}$$

Now, by (7.10) and (7.11),

$$\begin{aligned}&(\log(t+h(t)+1) - \log(t+1)) \int_t^\infty \exp(-\Phi'(s))ds\\ &\leq \log\left(\frac{e^{K+2}}{K-1}\log(t+e) + 1\right) \frac{e(\log(t+e))^{1-K}}{K-1} \to 0\end{aligned}$$

as $t \to \infty$. Furthermore, again by (7.11), we may write

$$\begin{aligned}&\log\log(t+h(t)+e) - \log\log(t+e)\\ &\qquad = \log\left(1 + \frac{\log(t+h(t)+e) - \log(t+e)}{\log(t+e)}\right)\\ &\qquad \leq \log\left(1 + \frac{\log\left(\frac{e^{K+2}}{K-1}\log(t+e)\right)}{\log(t+e)}\right) \to 0,\end{aligned}$$

as $t \to 0$. This yields (7.9) and, in consequence, (7.2). □

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Department of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland

E-mail address: ados@mimuw.edu.pl