SHARP NORM INEQUALITY FOR BOUNDED SUBMARTINGALES

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ABSTRACT. Let $\alpha \in [0,1]$ be a fixed number and $f = (f_n)$ be a nonnegative submartingale bounded from above by 1. Assume $g = (g_n)$ is a process satisfying, with probability 1,

$$|dg_n| \leq |df_n|, \quad |\mathbb{E}(dg_{n+1}|\mathcal{F}_n)| \leq \alpha \mathbb{E}(df_{n+1}|\mathcal{F}_n), \qquad n = 0, 1, 2, \ldots.$$

We provide a sharp bound for the first moment of the process g. A related estimate for stochastic integrals is also established.

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1. INTRODUCTION

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration, a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$. Throughout the paper, α is a fixed number belonging to the interval $[0,1]$. Let $f = (f_n)_{n\geq 0}$, $g = (g_n)_{n\geq 0}$ denote adapted real-valued integrable processes, such that f is a submartingale and g is α-subordinate to f: for any $n = 0, 1, 2, \ldots$ we have, almost surely,

$$|dg_n| \leq |df_n| \tag{1.1}$$

and

$$|\mathbb{E}(dg_{n+1}|\mathcal{F}_n)| \leq \alpha \mathbb{E}(df_{n+1}|\mathcal{F}_n). \tag{1.2}$$

Here $df = (df_n)_{n\geq 0}$ and $dg = (dg_n)$ stand for the difference sequences of f and g, given by

$$df_0 = f_0,\ df_n = f_n - f_{n-1},\ \ dg_0 = g_0,\ dg_n = g_n - g_{n-1}, \qquad n = 1, 2, \ldots.$$

The main objective of this paper is to provide some bounds on the size of the process g under some additional assumptions on boundedness of f. Let us provide some information about related estimates which have appeared in the literature. Let Φ be an increasing convex function on $[0,\infty)$ such that $\Phi(0) = 0$, the integral $\int_0^\infty \Phi(t)e^{-t}dt$ is finite and Φ is twice differentiable on $(0,\infty)$ with a strictly convex first derivative satisfying $\Phi'(0+) = 0$. For example, one can take $\Phi(t) = t^p$, $p > 2$, or $\Phi(t) = e^{at} - 1 - at$ for $a \in (0,1)$.

2000 *Mathematics Subject Classification.* Primary: 60642. Secondary: 60H05.

Key words and phrases. Martingale, submartingale, stochastic integral, norm inequality, differential subordination.

In [2] Burkholder proved a sharp Φ-inequality

$$\sup_n \mathbb{E}\Phi(|g_n|) < \frac{1}{2}\int_0^\infty \Phi(t)e^{-t}dt$$

under the assumption that f is a martingale (and so is g, by (1.2)), which is bounded in absolute value by 1. This inequality was later extended in [5] to the submartingale case: if f is a nonnegative submartingale bounded from above by 1 and g is 1-subordinate to f, then we have a sharp estimate

$$\sup_n \mathbb{E}\Phi\Big(\frac{|g_n|}{2}\Big) < \frac{2}{3}\int_0^\infty \Phi(t)e^{-t}dt.$$

Finally, Kim and Kim proved in [8], that if the 1-subordination is replaced by α-subordination, then we have

$$\mathbb{E}\Phi\Big(\frac{|g_n|}{1+\alpha}\Big) < \frac{1+\alpha}{2+\alpha}\int_0^\infty \Phi(t)e^{-t}dt, \tag{1.3}$$

if f is a nonnegative submartingale bounded by 1.

There are other related results, concerning tail estimates of g. Let us state here Hammack's inequality, an estimate we will need later on. In [7] it is proved that if f is a submartingale bounded in absolute value by 1 and g is 1-subordinate to f, then, for $\lambda \geq 4$,

$$\mathbb{P}(\sup_n |g_n| \geq \lambda) \leq \frac{(8+\sqrt{2})e}{12}\exp(-\lambda/4). \tag{1.4}$$

For other similar results, see papers by Burkholder [3], Hammack [7] and the author [9].

A natural question arises: what can be said about the Φ-inequalities for other functions Φ? The purpose of this paper is to give the answer for $\Phi(t) = t$. The main result can be stated as follows.

Theorem 1.1. *Suppose f is a nonnegative submartingale such that $\sup_n f_n \leq 1$ almost surely and let g be α-subordinate to f. Then*

$$||g||_1 \leq \frac{(\alpha+1)(2\alpha^2+3\alpha+2)}{(2\alpha+1)(\alpha+2)}. \tag{1.5}$$

The constant on the right is the best possible.

In a special case $\alpha = 1$, this leads to an interesting inequality for stochastic integrals. Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, filtered by a nondecreasing family $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-algebras of $\mathcal{F}$ and assume that $\mathcal{F}_0$ contains all the events A with $\mathbb{P}(A) = 0$. Let $X = (X_t)_{t\geq 0}$ be an adapted nonnegative right-continuous submartingale with left limits, satisfying $\mathbb{P}(X_t \leq 1) = 1$ for all t and let $H = (H_t)$ be a predictable process with values in $[-1, 1]$. Let $Y = (Y_t)$ be an Itô stochastic integral of H with respect to X, that is,

$$Y_t = H_0X_0 + \int_{(0,t]} H_s dX_s.$$

Let $||Y||_1 = \sup_t ||Y_t||_1$.

Theorem 1.2. *For X, Y as above, we have*

$$||Y||_1 \leq \frac{14}{9} \tag{1.6}$$

and the constant is the best possible. It is already the best possible if H is assumed to take values in the set $\{-1,1\}$.

The proofs are based on Burkholder's techniques, developed in [2] and [3]. These enable to reduce the proof of the submartingale inequality (1.5) to finding a special function, satisfying some convexity-type properties or, equivalently, to solving a certain boundary value problem.

The paper is organized as follows. In the next section we introduce the special function corresponding to the moment inequality and study its properties. Section 3 contains the proofs of inequalities (1.5) and (1.6). The sharpness of these estimates is postponed to the last section, Section 4.

2. The special function

Let S denote the strip $[0,1]\times\mathbb{R}$. Consider the following subsets of S.

$$\begin{aligned}
D_1&=\{(x,y)\in S: x\leq \frac{\alpha}{2\alpha+1},\ x+|y|>\frac{\alpha}{2\alpha+1}\},\\
D_2&=\{(x,y)\in S: x\geq \frac{\alpha}{2\alpha+1},\ -x+|y|>-\frac{\alpha}{2\alpha+1}\},\\
D_3&=\{(x,y)\in S: x\geq \frac{\alpha}{2\alpha+1},\ -x+|y|\leq-\frac{\alpha}{2\alpha+1}\},\\
D_4&=\{(x,y)\in S: x\leq \frac{\alpha}{2\alpha+1},\ x+|y|\leq\frac{\alpha}{2\alpha+1}\}.
\end{aligned}$$

Consider a function $H:\mathbb{R}^2\to\mathbb{R}$ defined by

$$H(x,y)=(|x|+|y|)^{1/(\alpha+1)}((\alpha+1)|x|-|y|).$$

Let $u:S\to\mathbb{R}$ be given by

$$u(x,y)=-\alpha x+|y|+\alpha+\exp\Big[-\frac{2\alpha+1}{\alpha+1}\Big(x+|y|-\frac{\alpha}{2\alpha+1}\Big)\Big]\Big(x+\frac{1}{2\alpha+1}\Big)$$

if $(x,y)\in D_1$,

$$u(x,y)=-\alpha x+|y|+\alpha+\exp\Big[-\frac{2\alpha+1}{\alpha+1}\Big(-x+|y|+\frac{\alpha}{2\alpha+1}\Big)\Big](1-x)$$

if $(x,y)\in D_2$,

$$u(x,y)=-(1-x)\log\Big[\frac{2\alpha+1}{\alpha+1}(1-x+|y|)\Big]+(\alpha+1)(1-x)+|y|$$

if $(x,y)\in D_3$ and

$$u(x,y)=-\frac{\alpha^2}{(2\alpha+1)(\alpha+2)}\Big[1+\Big(\frac{2\alpha+1}{\alpha}\Big)^{(\alpha+2)/(\alpha+1)}H(x,y)\Big]+\frac{2\alpha^2}{2\alpha+1}+1$$

if $(x,y)\in D_4$.

The key properties of the function u are described in the two lemmas below.

Lemma 2.1. *The following statements hold true.*

(i) The function u has continuous partial derivatives in the interior of S.

(ii) We have

$$u_x\leq-\alpha|u_y|. \tag{2.1}$$

(iii) For any real numbers x, h, y, k such that x, $x+h \in [0,1]$ and $|h| \geq |k|$ we have

$$u(x+h, y+k) \leq u(x,y) + u_x(x,y)h + u_y(x,y)k. \tag{2.2}$$

Proof. Let us first compute the partial derivatives in the interiors D_i^o of the sets D_i, $i \in \{1, 2, 3, 4\}$. We have that $u_x(x,y)$ equals

$$\begin{cases} -\alpha + \exp\Big[-\frac{2\alpha+1}{\alpha+1}\big(x+|y|-\frac{\alpha}{2\alpha+1}\big)\Big]\Big(-\frac{2\alpha+1}{\alpha+1}x+\frac{\alpha}{\alpha+1}\Big), & (x,y)\in D_1^o,\\ -\alpha + \exp\Big[-\frac{2\alpha+1}{\alpha+1}\big(-x+|y|+\frac{\alpha}{2\alpha+1}\big)\Big]\Big(-\frac{2\alpha+1}{\alpha+1}x+\frac{\alpha}{\alpha+1}\Big), & (x,y)\in D_2^o,\\ \log\big[\frac{2\alpha+1}{\alpha+1}(1-x+|y|)\big]+\frac{1-x}{1-x+|y|}-(\alpha+1), & (x,y)\in D_3^o,\\ -\alpha\Big(\frac{2\alpha+1}{\alpha}\Big)^{1/(\alpha+1)}(x+|y|)^{-\alpha/(\alpha+1)}\big(x+\frac{\alpha}{\alpha+1}|y|\big), & (x,y)\in D_4^o, \end{cases}$$

while $u_y(x,y)$ is given by

$$\begin{cases} y'-\frac{2\alpha+1}{\alpha+1}\exp\Big[-\frac{2\alpha+1}{\alpha+1}\Big(x+|y|-\frac{\alpha}{2\alpha+1}\Big)\Big]\Big(x+\frac{1}{2\alpha+1}\Big)y', & (x,y)\in D_1^o,\\ y'-\frac{2\alpha+1}{\alpha+1}\exp\Big[-\frac{2\alpha+1}{\alpha+1}\Big(-x+|y|+\frac{\alpha}{2\alpha+1}\Big)\Big](1-x)y', & (x,y)\in D_2^o,\\ \frac{y}{1-x+|y|}, & (x,y)\in D_3^o,\\ \Big(\frac{2\alpha+1}{\alpha}\Big)^{1/(\alpha+1)}(x+|y|)^{-\alpha/(\alpha+1)}\frac{\alpha}{\alpha+1}y, & (x,y)\in D_4^o. \end{cases}$$

Here $y' = y/|y|$ is the sign of y. Now we turn to the properties (i) - (iii).

(i) This follows immediately by the formulas for u_x, u_y above.

(ii) We have that $u_x(x,y) + \alpha|u_y(x,y)|$ equals

$$\begin{cases} -\exp\Big[-\frac{2\alpha+1}{\alpha+1}\Big(x+|y|-\frac{\alpha}{2\alpha+1}\Big)\Big](2\alpha+1)x, & (x,y)\in D_1,\\ -\exp\Big[-\frac{2\alpha+1}{\alpha+1}\Big(-x+|y|+\frac{\alpha}{2\alpha+1}\Big)\Big]\big(\frac{2\alpha+1}{\alpha+1}x(1-\alpha)+\frac{2\alpha^2}{\alpha+1}\big), & (x,y)\in D_2,\\ -\alpha+\log\Big[\frac{2\alpha+1}{\alpha+1}(1-x+|y|)\Big]-\frac{|y|(1-\alpha)}{1-x+|y|}, & (x,y)\in D_3,\\ -\alpha\Big(\frac{2\alpha+1}{\alpha}\Big)^{1/(\alpha+1)}(x+|y|)^{-\alpha/(\alpha+1)}x, & (x,y)\in D_4 \end{cases}$$

and all the expressions are clearly nonpositive.

(iii) There is a well-known procedure to establish (2.2). Fix x, y, h and k satisfying the conditions of (iii) and consider a function $G = G_{x,y,h,k} : t \mapsto u(x+th, y+tk)$, defined on $\{t : 0 \leq x+th \leq 1\}$. The inequality (2.2) reads $G(1) \leq G(0) + G'(0)$, so in order to prove it, it suffices to show that G is concave. Since u is of class C^1, it is enough to check $G''(t) \leq 0$ for those t, for which $(x+th, y+tk)$ belongs to the interior of D_1, D_2, D_3 or D_4. Furthermore, by translation argument (we have $G''_{x,y,h,k}(t) = G''_{x+th,y+tk,h,k}(0)$), we may assume $t = 0$.

If $(x,y) \in D_1^o$, we have

$$G''(0) = \frac{2\alpha+1}{\alpha+1}\exp\Big[-\frac{2\alpha+1}{\alpha+1}\Big(x+|y|-\frac{\alpha}{2\alpha+1}\Big)\Big]\times$$
$$\times(h+k)\Big\{\Big[\frac{2\alpha+1}{\alpha+1}\Big(x+\frac{1}{2\alpha+1}\Big)-2\Big]h+\frac{2\alpha+1}{\alpha+1}\big(x+\frac{1}{2\alpha+1}\big)k\Big\},$$

which is nonpositive; this is due to

$$|h| \geq |k|, \quad \frac{2\alpha+1}{\alpha+1}\Big(x+\frac{1}{2\alpha+1}\Big)-2 \leq -1 \quad \text{and} \quad \frac{2\alpha+1}{\alpha+1}\big(x+\frac{1}{2\alpha+1}\big) \leq 1.$$

If $(x,y) \in D_2^o$, then

$$G''(0) = \frac{2\alpha+1}{\alpha+1}\exp\Big[-\frac{2\alpha+1}{\alpha+1}\Big(-x+|y|+\frac{\alpha}{2\alpha+1}\Big)\Big]\times$$
$$\times(h-k)\Big\{\Big[\frac{2\alpha+1}{\alpha+1}(1-x)-2\Big]h-\frac{2\alpha+1}{\alpha+1}(1-x)k\Big\} \leq 0,$$

since

$$|h| \geq |k|, \quad \frac{2\alpha+1}{\alpha+1}(1-x)-2 \leq -1 \quad \text{and} \quad \frac{2\alpha+1}{\alpha+1}(1-x) \leq 1.$$

For $(x,y) \in D_3^o$ we have

$$G''(0) = \frac{-h+k}{1-x+|y|}\Big[\Big(2-\frac{1-x}{1-x+|y|}\Big)h+\frac{1-x}{1-x+|y|}k\Big] \leq 0,$$

because

$$|h| \geq |k|, \quad 2-\frac{1-x}{1-x+|y|} \geq 1 \quad \text{and} \quad \frac{1-x}{1-x+|y|} \leq 1.$$

Finally, for $(x,y) \in D_4^o$, this follows by the result of Burkholder: the function $t \mapsto -H(x+th, y+tk)$ is concave, see page 17 of [3]. □

Lemma 2.2. *Let* $(x,y) \in S$.

(i) We have

$$u(x,y) \geq |y|. \tag{2.3}$$

(ii) If $|y| \leq x$, *then*

$$u(x,y) \leq u(0,0) = \frac{(\alpha+1)(2\alpha^2+3\alpha+2)}{(2\alpha+1)(\alpha+2)}. \tag{2.4}$$

Proof. (i) Since for any $(x,y) \in S$ the function $G(t) = u(x+t, y+t)$ defined on $\{t : x+t \in [0,1]\}$ is concave, it suffices to prove (2.3) on the boundary of the strip S. Furthermore, by symmetry, we may restrict ourselves to $(x,y) \in \partial S$ satisfying $y \geq 0$. We have, for $y \in [0, \alpha/(2\alpha+1)]$,

$$u(0,y) \geq -\frac{\alpha^2}{(2\alpha+1)(\alpha+2)} + \frac{2\alpha^2}{2\alpha+1} + 1 \geq 1 \geq y,$$

while for $y > \alpha/(2\alpha+1)$, the inequality $u(0,y) \geq y$ is trivial. Finally, note that we have $u(1,y) = y$ for $y \geq 0$. Thus (2.3) follows.

(ii) As one easily checks, we have $u_y(x,y) \geq 0$ for $y \geq 0$ and hence, by symmetry, it suffices to prove (2.4) for $x = y$. The function $G(t) = u(t,t)$, $t \in [0,1]$, is concave and satisfies $G'(0+) = 0$. Thus $G \leq G(0)$ and we are done. □

3. Proofs of the inequalities (1.5) and (1.6)

Proof of inequality (1.5): Let f, g be as in the statement and fix a nonnegative integer n. Furthermore, fix $\beta \in (0,1)$ and set $f' = \beta f$, $g' = \beta g$. Clearly, g' is α-subordinate to f', so the inequality (2.2) implies that, with probability 1,

$$u(f'_{n+1}, g'_{n+1}) \leq u(f'_n, g'_n) + u_x(f'_n, g'_n)df'_{n+1} + u_y(f'_n, g'_n)dg'_{n+1}. \tag{3.1}$$

Both sides are integrable: indeed, since f is bounded by 1, so is f'; furthermore, we have $\mathbb{P}(|df_k| \leq 1) = 1$ and hence $\mathbb{P}(|dg_k| \leq 1) = 1$ by (1.1). This gives $|g'_n| = \beta|g_n| \leq \beta n$ almost surely and now it suffices to note that u is locally bounded on $[0,\beta] \times \mathbb{R}$ and the partial derivatives u_x, u_y are bounded on this set.

Therefore, taking the conditional expectation of (3.1) with respect to $\mathcal{F}_n$ yields

$$\begin{aligned}\mathbb{E}(u(f'_{n+1},g'_{n+1})|\mathcal{F}_n) &\leq u(f'_n,g'_n)+u_x(f'_n,g'_n)\mathbb{E}(df'_{n+1}|\mathcal{F}_n)+u_y(f'_n,g'_n)\mathbb{E}(dg'_{n+1}|\mathcal{F}_n)\\ &\leq u(f'_n,g'_n)+u_x(f'_n,g'_n)\mathbb{E}(df'_{n+1}|\mathcal{F}_n)+|u_y(f'_n,g'_n)|\cdot|\mathbb{E}(dg'_{n+1}|\mathcal{F}_n)|.\end{aligned}$$

By α-subordination, this can be further bounded from above by

$$u(f'_n,g'_n)+(u_x(f'_n,g'_n)+\alpha|u_y(f'_n,g'_n)|)\mathbb{E}(df'_{n+1}|\mathcal{F}_n)\leq u(f'_n,g'_n),$$

the latter inequality being a consequence of (2.1). Thus, taking expectation, we obtain

$$\mathbb{E}u(f'_{n+1},g'_{n+1})\leq \mathbb{E}u(f'_n,g'_n). \tag{3.2}$$

Combining this with (2.3), we get

$$\mathbb{E}|g'_n|\leq \mathbb{E}u(f'_n,g'_n)\leq \mathbb{E}u(f'_0,g'_0).$$

But $|g'_0|\leq f'_0$ by (1.1); hence (2.4) implies

$$\beta\mathbb{E}|g_n|=\mathbb{E}|g'_n|\leq \frac{(\alpha+1)(2\alpha^2+3\alpha+2)}{(2\alpha+1)(\alpha+2)}.$$

Since n and $\beta\in(0,1)$ were arbitrary, the proof is complete. □

Proof of the inequality (1.6): This follows by approximation argument. See Section 16 of [2], where it is shown how similar inequalities for stochastic integrals are implied by their discrete-time analogues combined with the result of Bichteler [1]. □

4. Sharpness

We start with the inequality (1.5). For $\alpha=0$ simply take constant processes $f=g=(1,1,1,\dots)$ and note that both sides are equal in (1.5). Suppose then, that α is a positive number. We will construct an appropriate example; this will be done in a few steps. Denote $\gamma=\alpha/(2\alpha+1)$ and fix $\varepsilon>0$.

Step 1. Using the ideas of Choi [6] (which go back to Burkholder's examples from [4]), one can show that there exists a pair (F,G) of processes starting from $(0,0)$ such that F is a nonnegative submartingale, G is α-subordinate to F and, for some N, (F_{3N},G_{3N}), takes values in the set $\{(\gamma,0),(0,\pm\gamma)\}$ with

$$\left|\mathbb{P}((F_{3N},G_{3N})=(\gamma,0))-\frac{1}{\alpha+2}\right|\leq\varepsilon,\quad \left|\mathbb{P}((F_{3N},G_{3N})=(0,\gamma))-\frac{\alpha+1}{2(\alpha+2)}\right|\leq\varepsilon$$

and $\mathbb{P}((F_{3N},G_{3N})=(0,\gamma))=\mathbb{P}((F_{3N},G_{3N})=(0,-\gamma))$. Furthermore, if $\alpha=1$, then G can be taken to be a ±1 transform of F, that is, $dF_n=\pm dG_n$ for any nonnegative integer n.

Step 2. Consider the following two-dimensional Markov process (f,g), with a certain initial distribution concentrated on the set $\{(\gamma,0),\ (0,\gamma),\ (0,-\gamma)\}$. To describe the transity function, let M be a (large) nonnegative integer and $\delta\in(0,\gamma/3)$; both numbers will be specified later. Assume for $k=0,\,1,\,2,\,\dots,\,M-1$ and any $\hat\varepsilon\in\{-1,1\}$, the conditions below are satisfied.

- The state $(0,\hat\varepsilon(\gamma+k(\alpha+1)\delta))$ leads to $(\delta,\hat\varepsilon(\gamma+k(\alpha+1)\delta+\alpha\delta))$ with probability 1.
- The state $(\delta,\hat\varepsilon(\gamma+k(\alpha+1)\delta+\alpha\delta))$ leads to $(0,\hat\varepsilon(\gamma+(k+1)(\alpha+1)\delta))$ with probability $1-\delta/\gamma$ and to $(\gamma,\hat\varepsilon(k+1)(\alpha+1)\delta)$ with probability δ/γ.

- The state $(\gamma, \hat{\varepsilon}(k+1)(\alpha+1)\delta)$ leads to $(1, \hat{\varepsilon}((k+1)(\alpha+1)\delta+1-\gamma))$ with probability
$$\frac{(\alpha+1)\delta}{2-2\gamma+(\alpha+1)\delta}$$
and to $(\gamma-(\alpha+1)\delta/2, \hat{\varepsilon}(k+1/2)(\alpha+1)\delta)$ with probability
$$1-\frac{(\alpha+1)\delta}{2-2\gamma+(\alpha+1)\delta}.$$
- The state $(\gamma-(\alpha+1)\delta/2, \hat{\varepsilon}(k+1/2)(\alpha+1)\delta)$ leads to $(0, \hat{\varepsilon}(\gamma+k(\alpha+1)\delta))$ with probability $(\alpha+1)\delta/(2\gamma)$ and to $(\gamma, \hat{\varepsilon}k(\alpha+1)\delta)$ with probability $1-(\alpha+1)\delta/(2\gamma)$.
- The state $(\gamma, 0)$ leads to $(1, 1-\gamma)$ with probability γ and to $(0, -\gamma)$ with probability $1-\gamma$.
- The state $(0, \hat{\varepsilon}(\gamma+M(\alpha+1)\delta))$ is absorbing.
- The states lying on the line $x=1$ are absorbing.

It is easy to check that f is a nonnegative submartingale bounded by 1 and g satisfies
$$|dg_n| \leq |df_n| \quad \text{and} \quad |\mathbb{E}(dg_n|\mathcal{F}_{n-1})| \leq \alpha\mathbb{E}(df_n|\mathcal{F}_{n-1}), \quad n=1,\ 2,\ \ldots$$
almost surely. Furthermore, if $\alpha=1$, then g is a ±1 transform of f: $df_n = \pm dg_n$ for $n \geq 1$ (note that this fails for $n=0$).

Step 3. Let $(\mathcal{G}_n)$ be the natural filtration generated by the process (f, g) and set $K = \gamma + M(1+\alpha)\delta$. Introduce the stopping time
$$\tau = \inf\{k : f_k = 1 \quad \text{or} \quad g_k = \pm K\}.$$
The purpose of this step is to establish a bound for the first moment of τ.

Let n be a nonnegative integer and set $\kappa = 4^{-3\delta M/(2\gamma)}$. We will prove that
$$\mathbb{P}(\tau \leq n+2M+1|\mathcal{G}_n) \geq \kappa\gamma. \tag{4.1}$$
We will need the following estimate
$$\left(1-\frac{3\delta}{2\gamma}\right)^M \geq \kappa, \tag{4.2}$$
which immediately follows from the facts that the function $h:(0,1/2] \to \mathbb{R}_+$ given by $h(x)=(1-x)^{1/x}$ is decreasing and $\delta < \gamma/3$.

Let $A \neq \emptyset$ be an atom of $\mathcal{G}_n$. We will consider three cases.

1°. If we have $f_n = 0$ or $f_n = \delta$ on A, consider the event
$$A' = A \cap \{|g_{n+k+1}| \geq |g_{n+k}|,\ k=0,\ 1,\ \ldots, 2M-1\}.$$
Clearly, in view of the transity function described above, we have $A' \subseteq \{|g_{n+2M}| = K\} \subseteq \{\tau \leq n+2M\}$ and
$$\mathbb{P}(\tau \leq n+2M+1|\mathcal{G}_n) \geq \mathbb{P}(\tau \leq n+2M|\mathcal{G}_n) \geq \frac{\mathbb{P}(A')}{\mathbb{P}(A)} \geq (1-\delta/\gamma)^M > \kappa > \kappa\gamma \quad \text{on } A,$$
in view of (4.2).

2°. If we have $f_n = \gamma$ or $f_n = \gamma-(\alpha+1)\delta/2$ on A, consider the event
$$A' = A \cap \{|g_{n+k+1}| < |g_{n+k}| \quad \text{or} \quad (f_{n+k+1}, g_{n+k+1}) = (1, 1-\gamma), \quad k=0,\ 1,\ \ldots\}.$$
In other words, A' contains those paths of $(f_{n+k}, g_{n+k})_{k\geq0}$, for which $|g|$ decreases to 0 and then, in the next step, (f,g) moves to $(1, 1-\gamma)$. It follows from the definition

of the transity function, that, on A, it is impossible for $|g|$ to be decreasing $2M+1$ times in a row; that is to say, we have $f_{n+2M+1}=1$ on A' and hence

$$\mathbb{P}(\tau\leq n+2M+1|\mathcal{G}_n)\geq\frac{\mathbb{P}(A')}{\mathbb{P}(A)}\geq\left[\left(1-\frac{(\alpha+1)\delta}{2\gamma}\right)\left(1-\frac{(\alpha+1)\delta}{2-2\gamma+(\alpha+1)\delta}\right)\right]^M\gamma$$
$$=\left(1-\frac{(2\alpha+1)\delta}{(2+(2\alpha+1)\delta)\gamma}\right)^M\gamma\geq\left(1-\frac{3\delta}{2\gamma}\right)^M\gamma\geq\kappa\gamma,$$

by (4.2).

3°. Finally, if $f_n=1$ on A, we have

$$\mathbb{P}(\tau\leq n+2M+1|\mathcal{G}_n)=1\geq\kappa\gamma.$$

Therefore the inequality (4.1) is established. It implies that

$$\mathbb{P}(\tau>n+2M+1)\leq(1-\kappa\gamma)\mathbb{P}(\tau>n),$$

which leads to

$$\mathbb{E}\tau\leq\frac{2M+1}{\kappa\gamma}<\frac{2K}{\kappa\gamma\delta}=\frac{2K}{\gamma\delta}\cdot 4^{3(K-\gamma)/2\gamma(1+\alpha)}. \tag{4.3}$$

This implies that $\tau<\infty$ with probability 1 and the pointwise limits f_∞, g_∞ exist almost surely.

Step 4. Let us establish an exponential bound for $\mathbb{P}(f_\infty=0)$. We have $\{f_\infty=0\}\subseteq\{g_\infty\geq K\}$ and g is clearly 1-subordinate to f (as it is α-subordinate to f). Therefore, we may use Hammack's result (1.4): we have

$$\mathbb{P}(f_\infty=0)\leq\frac{(8+\sqrt{2})e}{12}\exp(-K/4) \tag{4.4}$$

provided $K\geq 4$.

Step 5. Consider a process $(u(f_n,g_n))_n$ and observe the following.

- For $y\geq\gamma$, the function $G(t)=u(t,y-t)$, $t\in[0,1]$, is continuously differentiable and linear on $[0,\gamma]$.
- For $y\geq-\gamma$, the function $G(t)=u(t,y+t)$, $t\in[0,1]$, is continuously differentiable and linear on $[\gamma,1]$.
- For $y\geq\gamma$, the function $G(t)=u(t,y+\alpha t)$, $t\in[0,1]$, satisfies $G'(0+)=0$.
- The function u is locally bounded on $\overline{D_1\cup D_2}$ and its partial derivatives are bounded on this set.

These four facts, together with symmetry of u, imply that there exists a constant $\eta(\delta,K)$ such that $\eta(\delta,K)/\delta\to 0$ as $\delta\to 0$ and, for any n,

$$u(f_{n+1},g_{n+1})\geq u(f_n,g_n)+u_x(f_n,g_n)df_{n+1}+u_y(f_n,g_n)dg_{n+1}-\eta(\delta,K)\chi_{\{\tau>n\}}.$$

Both sides of this inequality are integrable: indeed, it suffices to use the fourth property above and the fact that (f_n,g_n) is bounded and belongs to $\overline{D_1\cup D_2}$. Therefore, we may take expectation to obtain

$$\mathbb{E}u(f_{n+1},g_{n+1})\geq\mathbb{E}u(f_n,g_n)-\eta(\delta,K)\mathbb{P}(\tau>n).$$

This implies

$$\mathbb{E}u(f_\infty,g_\infty)\geq\mathbb{E}u(f_0,g_0)-\eta(\delta,K)\mathbb{E}\tau,$$

or

$$\mathbb{E}|g_\infty|+\left\{\alpha+\exp\big[-\frac{2\alpha+1}{\alpha+1}\big(K-\frac{\alpha}{2\alpha+1}\big)\big]\cdot\frac{1}{2\alpha+1}\right\}\mathbb{P}(f_\infty=0)$$
$$\geq\mathbb{E}u(f_0,g_0)-\eta(\delta,K)\mathbb{E}\tau.$$

By (4.4), we may fix $K \geq 4$ such that

$$\Big\{\alpha + \exp\big[-\frac{2\alpha+1}{\alpha+1}\big(K + \frac{\alpha}{2\alpha+1}\big)\big] \cdot \frac{1}{2\alpha+1}\Big\}\mathbb{P}(f_\infty = 0) \leq \varepsilon.$$

Now we specify the numbers δ and M, as promised at the beginning of Step 2. By (4.3), we may choose $\delta > 0$ such that $\eta(\delta, K)\mathbb{E}\tau \leq \varepsilon$ and, clearly, we may also ensure that $M = (K-\gamma)/(1+\alpha)\delta$ is an integer. Thus we obtain

$$\mathbb{E}|g_\infty| \geq \mathbb{E}u(f_0, g_0) - 2\varepsilon. \tag{4.5}$$

Step 6. Now we put all the things together. Let $(f,g) = ((f_n, g_n))_{n\geq 0}$ be a process which coincides with (F, G) from Step 1 for $n \leq 3N$ and which, for $n > 3N$, conditionally on $\mathcal{F}_{3N}$, moves according to the transities described in Step 2. We have, by (4.5),

$$\mathbb{E}|g_\infty| \geq \mathbb{E}u(F_{3N}, G_{3N}) - 2\varepsilon.$$

But, since u is nonnegative (due to (2.3)),

$$\begin{aligned}\mathbb{E}u(F_{3N}, G_{3N}) &\geq u(\gamma, 0)\big(\frac{1}{\alpha+2} - \varepsilon\big) + u(0,\gamma)\big(\frac{\alpha+1}{\alpha+2} - \varepsilon\big)\\ &= \frac{(\alpha+1)(2\alpha^2+3\alpha+2)}{(2\alpha+1)(\alpha+2)} - (u(\gamma,0) + u(0,\gamma))\varepsilon.\end{aligned}$$

Since ε was arbitrary, this implies that the constant in (1.5) is the best possible. This also establishes the sharpness of the estimate (1.6), even in the special case $H \in \{-1, 1\}$: if $\alpha = 1$, then the processes f, g constructed above satisfy $|df_k| = |dg_k|$ for all k. The proofs of Theorems 1.1 and 1.2 are complete.

Acknowledgement:

This result was obtained while the author was visiting Université de Franche-Comté in Besançon, France.

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