Sharp maximal bound for continuous martingales

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Abstract

Let X, Y be continuous-path martingales satisfying the condition $[X,X]_t \geq [Y,Y]_t$ for all $t \geq 0$. We prove that

$$||\sup_{t\geq 0} Y_t||_1 \leq \frac{3}{2}||\sup_{t\geq 0}|X_t|||_1$$

and the constant $3/2$ is the best possible.

Key words: Martingale, stochastic integral, maximal inequality, differential subordination

1. Introduction

The purpose of this note is to prove a sharp inequality for martingales satisfying certain domination relation. However, to present the motivation and related results from the literature, we start with the setting of stochastic integrals.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, filtered by a non-decreasing right-continuous family $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-fields of $\mathcal{F}$. Let $X = (X_t)_{t\geq 0}$ be an adapted right-continuous martingale with limits from the left, taking values in $\mathbb{R}$, and let $Y = H \cdot X$ be the Itô integral of H with respect to X. That is, for any $t \geq 0$,

$$Y_t = H_0X_0 + \int_{0+}^t H_s dX_s.$$

Here $H = (H_s)$ is a predictable process, taking values in $[-1, 1]$. The maximal function and the one-sided maximal function of X are defined by $|X|^* = \sup_{t\geq 0}|X_t|$ and $X^* = \sup_{t\geq 0} X_t$, respectively. We will also use the notation $|X|^*_s = \sup_{0\leq t\leq s}|X_t|$ and $X^*_s = \sup_{0\leq t\leq s} X_t$ for $s \geq 0$. Furthermore, we shall write $||X||_p = \sup_{t\geq 0}||X_t||_p$ for $1 \leq p \leq \infty$.

We will be interested in comparing the sizes of the maximal functions of X and Y. Burkholder (1984) introduced a method of handling such inequalities and exploited it to prove the following.

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Theorem 1.1. *Let X and Y be as above. Then*

$$||Y||_1 \leq \gamma|| \, |X|^* \, ||_1, \tag{1}$$

where $\gamma = 2.536\ldots$ is the solution to the equation

$$3 - \gamma = \exp\left(\frac{1-\gamma}{2}\right).$$

The inequality is sharp.

The author showed that if X is assumed to be nonnegative, the optimal constant in (1) equals $2 + (3e)^{-1} = 2.1226\ldots$ (see Osękowski (2008)). The following related inequality

$$|| \, Y^* \, ||_1 \leq \beta|| \, |X|^* \, ||_1$$

was studied in Osękowski (2009a). It was proved that the optimal constant $\beta = 2.0856\ldots$ is the unique number from $(0, 8/3)$ satisfying

$$2\log\left(\frac{8}{3} - \beta\right) = 1 - \beta.$$

If, in addition, $X \geq 0$, then the best constant decreases to $14/9 = 1.555\ldots$. We will study this inequality under different assumptions. Let X, Y be two $(\mathcal{F}_t)$-martingales on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $[X, X]$ and $[Y, Y]$ denote their quadratic variation processes, respectively (see e.g. Chapter VII in Dellacherie and Meyer (1982)). Here we allow the martingales to be vector-valued: if X takes values in $\mathbb{R}^\nu$ for some $\nu > 1$, then we set $[X, X]_t = \sum_{j=1}^{\nu}[X^j, X^j]_t$. We impose the following condition on X and Y:

$$[Y, Y]_t \leq [X, X]_t \qquad \text{for every } t \geq 0. \tag{2}$$

Note that this holds in the setting of stochastic integrals described above: indeed, for any $t \geq 0$, $[X, X]_t - [Y, Y]_t = \int_0^t (1 - |H_s|^2) d[X, X]_s \geq 0$. The condition (2) is closely related to *differential subordination*: we say that Y is differentially subordinate to X if the process $([X, X] - [Y, Y])_{t\geq 0}$ is nonnegative and nondecreasing (see Bañuelos and Wang (1995) and Wang (1995)).

We are ready to state the main result of the paper.

Theorem 1.2. *Suppose that X and Y are continuous-path martingales such that X takes values in $\mathbb{R}^\nu$, $\nu \geq 1$, Y takes values in $\mathbb{R}$ and that (2) holds. Then*

$$|| \, Y^* \, ||_1 \leq \frac{3}{2}|| \, |X|^* \, ||_1 \tag{3}$$

and the constant is the best possible. It is optimal even if $\nu = 1$, X is a stopped Brownian motion and $Y = H \cdot X$ for some predictable H taking values in $\{-1, 1\}$.

Some remarks relating the above result to those presented earlier are in order. First, let us stress that we have the additional continuity of the paths of X and Y. On the other hand, we see that a sharp inequality for stochastic integrals is extended, with unchanged constant, to a much more general setting described by (2). It is even less restrictive than the differential subordination, the condition which is usually imposed while studying estimates of this type (see e.g. Wang (1995), Suh (2005) and Osękowski (2009b)).

A few words about the organization of the paper. In the proof we exploit Burkholder's method: the annouced inequality follows from the existence of a certain special function. Such an object is introduced in the next section and we establish (3) there. In the final part of the paper we prove that the constant 3/2 cannot be replaced by a smaller one.

2. Proof of (3)

Let $U : \mathbb{R}^\nu \times \mathbb{R} \times (0,\infty) \times \mathbb{R} \to \mathbb{R}$ be given by the formula

$$U(x,y,z,w) = \frac{(y-w+z)^2 - |x|^2}{2z} + w - z.$$

The key properties of this function are listed in the lemma below. The easy proof is omitted.

Lemma 2.1. *(i) We have the majorization*

$$U(x,y,z,w) \geq w - \frac{3}{2}z, \tag{4}$$

provided $|x| \leq z$.

(ii) For any $x \in \mathbb{R}^\nu$, $y \in \mathbb{R}$, $z > 0$ *and* $w > 0$,

$$U_z(x,y,|x|,w) \leq 0 \qquad \text{if } |x| \neq 0, \tag{5}$$

and

$$U_w(x,y,z,y) = 0. \tag{6}$$

Proof of (3). We start with a reduction step. Take a positive integer N and consider the stopping time $\tau_N = \inf\{t : \sqrt{N^{-2} + |X_t|^2} + |Y_t| \geq N\}$. Let X^N, Y^N be martingales given by

$$X_t^N = (X_{\tau\wedge N}, N^{-1})1_{\{\tau_N>0\}} \in \mathbb{R}^{\nu+1}, \quad Y_t^N = Y_{\tau\wedge N}1_{\{\tau_N>0\}}, \quad t \geq 0.$$

Clearly, (2) is valid for these processes. Moreover, it suffices to establish (3) for X^N and Y^N. Indeed, having this done, we obtain that $||(Y^N)^*||_1 \leq \frac{3}{2}(||\,|X|^*\,||_1 + N^{-1})$, and the inequality for initial X and Y follows from Lebesgue's monotone convergence theorem (when $N \to \infty$). Thus, we may assume that X and Y are bounded and $|X_0|$ is bounded away from 0. This will guarantee the integrability of all the variables appearing below.

Denote $Z_t = (X_t, Y_t, |X|_t^*, Y_t^*)$ for $t \geq 0$. The function U is of class C^2, so we may apply Itô's formula to obtain

$$U(Z_t) = U(Z_0) + I_1 + I_2 + \frac{1}{2}I_3, \tag{7}$$

where

$$I_1 = \int_{0+}^t U_x(Z_s)dX_s + \int_{0+}^t U_y(Z_s)dY_s,$$

$$I_2 = \int_{0+}^t U_z(Z_s)d|X|_s^* + \int_{0+}^t U_w(Z_s)dY_s^*,$$

$$I_3 = \int_{0+}^t U_{xx}(Z_s)d[X,X]_s + 2\int_{0+}^t U_{xy}(Z_s)d[X,Y]_s + \int_{0+}^t U_{yy}(Z_s)d[Y,Y]_s.$$

Here $U_x(Z_s) = (U_{x_1}(Z_s), U_{x_2}(Z_s), \ldots, U_{x_\nu}(Z_s))$, $U_{xx}(Z_s) = (U_{x_ix_j}(Z_s))_{i,j=1}^\nu$, $U_{xy}(Z_s) = (U_{x_1y}(Z_s), U_{x_2y}(Z_s), \ldots, U_{x_\nu y}(Z_s))$, and we have used the notation

$$\int_{0+}^t U_{xx}(Z_s)d[X,X]_s = \sum_{i,j=1}^\nu \int_{0+}^t U_{x_ix_j}(Z_s)d[X^i,X^j]_s$$

and similar notation for the second integral in I_3. We will show that the expectation of the right-hand side of (7) is nonpositive. We have that $U(Z_0) = Y_0 - |X_0|$. By the properties of stochastic integrals, I_1 has mean 0. Furthermore, by the continuity of paths, the measure dX_s^* is concentrated on the set $\{s : X_s = |X|_s^*\}$, and $U_z(X_s, Y_s, |X|_s^*, Y_s^*)$ is nonpositive there, by (5). Similarly, (6) implies that U_w vanishes on the support of the measure dY_s^*; thus $I_2 \leq 0$. To deal with I_3, note that integration by parts, together with (2) and the fact that the process $(1/|X|_t^*)$ is nonincreasing, yield

$$\begin{aligned} I_3 &= -\int_{0+}^t \frac{1}{|X|_s^*}d([X,X]_s - [Y,Y]_s) \\ &= \int_{0+}^t ([X,X]_s - [Y,Y]_s)d\frac{1}{|X|_s^*} - \frac{[X,X]_t - [Y,Y]_t}{|X|_t^*} + \frac{[X,X]_0 - [Y,Y]_0}{|X|_0^*} \\ &\leq \frac{[X,X]_0 - [Y,Y]_0}{|X|_0^*} = \frac{|X_0|^2 - |Y_0|^2}{|X_0|}. \end{aligned}$$

Plugging the above estimates to (7) we get

$$\mathbb{E}U(Z_t) \leq \mathbb{E}\left[Y_0 - |X_0| + \frac{|X_0|^2 - |Y_0|^2}{2|X_0|}\right] = -\mathbb{E}\frac{(|X_0| - Y_0)^2}{2|X_0|} \leq 0.$$

It suffices to use (4) and let $t \to \infty$ to complete the proof.

3. Sharpness

We will construct an example showing that the constant 3/2 in (3) is optimal. Fix a positive parameter δ. Suppose that $B = (B_t)_{t\geq 0}$ is a one-dimensional Brownian motion starting from 1 and let $(\mathcal{F}_t)_{t\geq 0}$ stand for the completion of its natural filtration. Consider a sequence $(\tau_n)_{n\geq 0}$ of stopping times of B, given by $\tau_0 = \inf\{t : B_t = B_t^*/2\}$ and, by induction, for $k = 0, 1, 2, \ldots$,

$$\tau_{2k+1} = \begin{cases} \inf\{t > \tau_{2k} : B_t/B_{\tau_0}^* \in \{-\delta, 1\}\} & \text{if } B_{\tau_{2k}} \neq -B_{\tau_0}^*, \\ \tau_{2k} & \text{if } B_{\tau_{2k}} = -B_{\tau_0}^*, \end{cases}$$

$$\tau_{2k+2} = \begin{cases} \inf\{t > \tau_{2k+1} : B_t/B_{\tau_0}^* \in \{-1, 0\}\} & \text{if } B_{\tau_{2k+1}} \neq B_{\tau_0}^*, \\ \tau_{2k+1} & \text{if } B_{\tau_{2k+1}} = B_{\tau_0}^*. \end{cases}$$

Now take $\tau = \lim_{n\to\infty} \tau_n$ and observe that $\tau = \inf\{t > \tau_0 : |B_t| = B^*_{\tau_0}\}$, which guarantees the integrability of $|B|^*_\tau$: indeed, we have $|B|^*_\tau = B^*_{\tau_0}$, and the latter random variable belongs to L^p for any $p < 2$, see e.g. Wang (1991).

Now let $X_t = B_{\tau\wedge t}$ for $t \geq 0$ and let $H = (H_t)_{t\geq 0}$ be a process defined by

$$H_t = 1_{[0,\tau_0)}(t) + \sum_{n=1}^{\infty}(-1)^n 1_{[\tau_{n-1},\tau_n)}(t).$$

Note that H is predictable and takes values in $\{-1, 1\}$. Finally, let Y be the Itô integral of H with respect to X.

Let us write some elementary facts which follow immediately from the properties of the Brownian motion. First, we have that $\mathbb{P}(\tau > \tau_0) = 1$ and

$$\mathbb{P}(\tau = \tau_1|\mathcal{F}_{\tau_0}) = \mathbb{P}(B_{\tau_1} = B^*_{\tau_0}|\mathcal{F}_{\tau_0}) = \frac{1+2\delta}{2(1+\delta)}. \tag{8}$$

Moreover, for $n \geq 1$, $\{\tau > \tau_n\} = \{|B_{\tau_k}/B^*_{\tau_0}| \neq 1 \text{ for all } k = 1, 2, \ldots, n\}$, which implies that for $k \geq 1$,

$$\mathbb{P}(\tau > \tau_{2k-1}|\mathcal{F}_{\tau_0}) = \frac{1}{2}(1+\delta)^{-k}(1-\delta)^{k-1}, \quad \mathbb{P}(\tau > \tau_{2k}|\mathcal{F}_{\tau_0}) = \frac{1}{2}(1+\delta)^{-k}(1-\delta)^{k}.$$

Hence, for $n \geq 1$,

$$\mathbb{P}(\tau = \tau_{n+1} > \tau_n|\mathcal{F}_{\tau_0}) = \frac{1}{2}(1+\delta)^{-k-1}(1-\delta)^k\delta, \tag{9}$$

where $k = \lfloor n/2 \rfloor$.

The second observation is that $Y^*_{\tau_0} = B^*_{\tau_0}$, since $Y_t = B_t$ for $t \leq \tau_0$. Furthermore, for $n \geq 1$, on the set $\{\tau > \tau_n\}$ we have

$$\begin{aligned} Y^*_{\tau_n} = Y_{\tau_n} &= B_{\tau_0} - (B_{\tau_1} - B_{\tau_0}) + (B_{\tau_2} - B_{\tau_1}) + \ldots + (-1)^n(B_{\tau_n} - B_{\tau_{n-1}}) \\ &= B^*_{\tau_0}(1+n\delta). \end{aligned}$$

Indeed, the first two summands are $B^*_{\tau_0}/2$ and $B^*_{\tau_0}(1/2+\delta)$, respectively, and the remaining $n-1$ ones are equal to $\delta B^*_{\tau_0}$. Thus, we may write that for $n \geq 0$,

$$Y^*_\tau \geq Y^*_{\tau_n} = B^*_{\tau_0}(1+n\delta) \quad \text{on the set } \{\tau = \tau_{n+1} > \tau_n\}. \tag{10}$$

Now we are ready to estimate the first moment of Y^*. By (8), (9) and (10),

$$\begin{aligned} \mathbb{E}(Y^*|\mathcal{F}_{\tau_0}) = \mathbb{E}(Y^*_\tau|\mathcal{F}_{\tau_0}) &\geq B^*_{\tau_0}\sum_{n=0}^{\infty}(1+n\delta)\mathbb{P}(\tau = \tau_{n+1} > \tau_n|\mathcal{F}_0) \\ &= B^*_{\tau_0} \cdot \frac{6+3\delta-3\delta^2}{4+4\delta} = |X|^* \cdot \frac{6+3\delta-3\delta^2}{4+4\delta}. \end{aligned}$$

Since $\delta > 0$ was arbitrary, we see that no constant smaller than $3/2$ suffices in (3). The proof is complete.

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