LOGARITHMIC ESTIMATES FOR SUBMARTINGALES AND THEIR DIFFERENTIAL SUBORDINATES

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Abstract. In the paper we determine, for any $K > 0$ and $\alpha \in [0, 1]$, the optimal constant $L(K, \alpha) \in (0, \infty]$ for which the following holds. If X is a nonnegative submartingale and Y is α-strongly differentially subordinate to X, then

$$\sup_t \mathbb{E}|Y_t| \leq K \sup_t \mathbb{E} X_t \log^+ X_t + L(K, \alpha).$$

Related sharp inequalities for martingales are also established. As an application, we obtain logarithmic estimates for smooth functions on Euclidean domains.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, filtered by a nondecreasing family $(\mathcal{F}_n)_{n\geq 0}$ of sub-σ-fields of $\mathcal{F}$. Let $f = (f_n)$, $g = (g_n)$ be adapted sequences of integrable real-valued random variables. The difference sequences $df = (df_n)$, $dg = (dg_n)$ are given by the equalities

$$f_n = \sum_{k=0}^{n} df_k, \qquad g_n = \sum_{k=0}^{n} dg_k, \qquad n = 0, 1, 2, \ldots.$$

The following notion of *differential subordination* is due to Burkholder [2]: we say that g is differentially subordinate to f, if for any n we have $|dg_n| \leq |df_n|$. If f, g are martingales, then this condition implies many interesting estimates: see [5] and references therein, and [13], [14], [15] for some recent progress. For example, we have the following moment inequality, proved by Burkholder in [2]. We use the notation $||f||_p = \sup_n ||f_n||_p$, $p \in [1, \infty]$.

Theorem 1.1. *Let f, g be two martingales such that g is differentially subordinate to f. Then for $1 < p < \infty$,*

$$||g||_p \leq \max\{p-1, (p-1)^{-1}\}||f||_p \tag{1.1}$$

and the constant is the best possible.

For $p = 1$ the inequality does not hold with any finite constant. However, the author established in [13] the following substitute.

Theorem 1.2. *Let f, g be two martingales such that g is differentially subordinate to f. Then for $K > 1$,*

$$||g||_1 \leq K \sup_n \mathbb{E}|f_n| \log |f_n| + L_0(K), \tag{1.2}$$

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where

$$L_0(K) = \begin{cases} \frac{K^2}{2(K-1)} \exp(-K^{-1}) & \text{if } K < 2, \\ K \exp(K^{-1} - 1) & \text{if } K \geq 2. \end{cases}$$

The constant is the best possible. Furthermore, for $K \leq 1$ the inequality does not hold in general with any universal $L_0(K) < \infty$.

We will work with a wider class of processes to study which we need a domination stronger than the differential subordination. Let α be a fixed nonnegative number. Following Choi [7] (see also [6]), we say that the sequence g is *α-strongly differentially subordinate to f* (α-subordinate in short), if g is differentially subordinate to f and, in addition, for any $n \geq 1$ we have $|\mathbb{E}(dg_n|\mathcal{F}_{n-1})| \leq \alpha|\mathbb{E}(df_n|\mathcal{F}_{n-1})|$ almost surely.

We have the following extension of Theorem 1.1, established by Choi in [7] and in the earlier paper [6] by Burkholder in the particular case $\alpha = 1$.

Theorem 1.3. *Assume that $\alpha \in [0, 1]$, f is a nonnegative submartingale and g is α-subordinate to f. Then for $1 < p < \infty$,*

$$||g||_p \leq \max\{(\alpha + 1)p - 1, (p - 1)^{-1}\}||f||_p \tag{1.3}$$

and the constant is the best possible.

There is a natural question about the submartingale version of Theorem 1.2. For a fixed $\alpha \in [0, 1]$, let $K_0 = K_0(\alpha)$ be given by (2.1) below. For $K > 1$, let $c = c(K, \alpha)$ be given by (2.2) or (2.4), depending on whether $K < K_0$ or $K \geq K_0$. Here is one of the main results of the present paper.

Theorem 1.4. *Assume that f is a nonnegative submartingale and g is α-subordinate to f. Then for $K > 1$,*

$$||g||_1 \leq K \sup_n \mathbb{E} f_n \log^+ f_n + L(K, \alpha), \tag{1.4}$$

where

$$L(K, \alpha) = \begin{cases} \dfrac{c}{\alpha + 2} + \dfrac{(\alpha + 1)^2}{c(2\alpha + 1)^2(\alpha + 2)} + \dfrac{\alpha(2\alpha^2 + 5\alpha + 3)}{(\alpha + 2)(2\alpha + 1)} & \text{if } K < K_0, \\ \dfrac{(\alpha + 1)(2\alpha^2 + 3\alpha + 2)}{(2\alpha + 1)(\alpha + 2)} & \text{if } K \geq K_0. \end{cases} \tag{1.5}$$

The constant $L(K, \alpha)$ is the best possible. Furthermore, for $K \leq 1$ the inequality does not hold in general with any universal $L(K, \alpha) < \infty$.

Therefore, as in Theorem 1.2, there are two different expressions for the constants $L(K, \alpha)$ depending on whether K is small or large. However, comparing (1.2) and (1.4), we see a slight difference: in the second estimate we have the positive part of the logarithm. We have not been able to find optimal constant without this small change. On the other hand, we can show the following "Log$^+$" version of Theorem 1.2. Here the process f may take negative values.

Theorem 1.5. *Let f, g be two martingales such that g is differentially subordinate to f. Then for $K > 1$,*

$$||g||_1 \leq K \sup_n \mathbb{E}|f_n| \log^+ |f_n| + L(K, 0). \tag{1.6}$$

The constant $L(K, 0)$ is the best possible. Furthermore, for $K \leq 1$ the inequality does not hold in general with any universal $L(K) < \infty$.

The next result we obtain is the following sharp logarithmic inequality for one-sided maximal function of g. Let $g_n^* = \sup_{0\leq k\leq n} g_k$ and $g^* = \sup_n g_n^*$. Let K_1 be the solution to the equation $K_1 - 3 + \log(K_1 - 1) = 0$.

Theorem 1.6. *Let f, g be two martingales such that g is differentially subordinate to f. Then for $K > 1$,*

$$||g^*||_1 \leq K \sup_n \mathbb{E}|f_n| \log^+ |f_n| + L^*(K), \tag{1.7}$$

where

$$L^*(K) = \begin{cases} c(K,0) + (2c(K,0))^{-1} & \text{if } 1 < K \leq K_0(0), \\ 1 + \dfrac{c(K,0)}{2} + \dfrac{c(K,0)(\log c(K,0))^2}{2} & \text{if } K_0(0) \leq K \leq K_1, \\ 1 + e^{-1} & \text{if } K > K_1. \end{cases} \tag{1.8}$$

The constant $L^*(K)$ is the best possible. Furthermore, for $K \leq 1$ the inequality does not hold in general with any universal $L^*(K) < \infty$.

In fact, we will prove the theorems above in a more general continuous-time setting. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$, and let X, Y be two real-valued right-continuous semimartingales with limits from the left. Set $X_{0-} = Y_{0-} = 0$. Let $[X, X]$ and $[X, Y]$ stand for the quadratic variance of X and quadratic covariance process of X, Y, respectively; see e.g. Dellacherie and Meyer [9]. Following [1] and [16], we say that Y is differentially subordinate to X, if the process $[X, X] - [Y, Y]$ is nonnegative and nondecreasing. This is consistent with the previous definition: if f, g are discrete-time sequences and we treat them as continuous-time semimartingales (via $X_t = f_{\lfloor t \rfloor}$, $Y_t = g_{\lfloor t \rfloor}$, $t \geq 0$), then

$$[X, X]_t - [Y, Y]_t = \sum_{k=0}^{\lfloor t \rfloor} (|df_k|^2 - |dg_k|^2)$$

is nonnegative and nondecreasing if and only if $|dg_n| \leq |df_n|$ for all n.

We will now explain how to extend the notion of α-subordination to the continuous-time setting. Write the Doob-Meyer decomposition for X and Y:

$$X_t = X_0 + M_t + A_t, \qquad Y_t = Y_0 + N_t + B_t, \qquad t \geq 0, \tag{1.9}$$

where M, N are local martingales and A, B are the finite variation processes. If X is a submartingale, we assume that A is predictable (determined uniquely). In general, the decompositions may not be unique. We say that Y is α-strongly differentially subordinate to X, if Y is differentially subordinate to X and there is such a decomposition (1.9), for which the process $(\alpha|A|_t - |B|_t)$ is nonnegative and nondecreasing. Here $|A|_t$ denotes the total variation of A on the interval $[0, t]$. Again, one easily checks that in the discrete-time case, this reduces to the previous definition of α-subordination.

Let us reformulate the results of the present paper in our new, more general setting. We use the notation $||X||_1 = \sup_t ||X_t||_1$ and $X_t^* = \sup_{0\leq s\leq t} X_s$, $X^* = \sup_t X_t^*$.

Theorem 1.7. *(i) If X is a nonnegative submartingale and Y is α-subordinate to X, then for $K > 1$,*

$$||Y||_1 \leq K \sup_t \mathbb{E} X_t \log^+ X_t + L(K, \alpha), \tag{1.10}$$

and the constant $L(K,\alpha)$ is the best possible. For $K \le 1$ there is no $L(K,\alpha)<\infty$ for which (1.10) *is valid in general.*

(ii) If X is a martingale and Y is differentially subordinate to X, then for $K>1$,

$$||Y||_1 \le K \sup_t \mathbb{E}|X_t| \log^+ |X_t| + L(K,0), \tag{1.11}$$

and the constant $L(K,0)$ is the best possible. For $K \le 1$ there is no $L(K)<\infty$ for which (1.11) *is valid in general.*

(iii) If X is a martingale and Y is differentially subordinate to X, then for $K>1$,

$$||Y^*||_1 \le K \sup_t \mathbb{E}|X_t| \log^+ |X_t| + L^*(K), \tag{1.12}$$

and the constant $L^*(K)$ is the best possible. For $K \le 1$ there is no $L^*(K)<\infty$ for which (1.12) *is valid in general.*

The paper depends heavily on the techniques invented by Burkholder (see e.g. [5] and [6]): the announced inequalities will be established exploiting some special functions, which have certain convexity-type properties. These functions are introduced and studied in the next section. Then, in Section 3, we provide the proofs of the estimates (1.10), (1.11), (1.12), and show the optimality of the constants in Section 4. The final part of the paper is devoted to some applications concerning smooth functions on Euclidean domains.

2. Special functions

Let us start with some technical lemma to be needed later.

Lemma 2.1. *Let α be a fixed number belonging to $[0,1]$.*

(i) There is a unique $K_0 = K_0(\alpha) \in (1,\infty)$ such that

$$K_0 + \log(K_0 - 1) = \alpha + 1 + \log\left(\frac{2\alpha+1}{\alpha+1}\right). \tag{2.1}$$

(ii) If $K \in (1, K_0)$, then there is a unique $c = c(K,\alpha) \in (\frac{\alpha+1}{2\alpha+1}, (K-1)^{-1})$ satisfying

$$\alpha + \frac{\alpha+1}{2\alpha+1}\cdot\frac{1}{c} - \log(c(K-1)) - K = 0. \tag{2.2}$$

Proof. (i) As a function of $K_0 \in (1,\infty)$, the left-hand side of (2.1) is increasing and tends to $-\infty$ as $K_0 \to 1-$ and to ∞ as $K_0 \to \infty$.

(ii) Denoting the left-hand side of (2.2) by $F(c)$, we see that the function F is continuous, decreasing on $(0,\infty)$, $\lim_{c\to\infty} F(c) = -\infty$ and, by (2.1),

$$\begin{aligned} F((\alpha+1)/(2\alpha+1)) &= \alpha + 1 - \log\frac{\alpha+1}{2\alpha+1} - \log(K-1) - K \\ &= \log(K_0 - 1) + K_0 - \log(K-1) - K > 0. \end{aligned}$$

Hence there is unique c such that $F(c) = 0$. The bound $c < (K-1)^{-1}$ follows directly from the estimate

$$F((K-1)^{-1}) = (2\alpha^2 - 1 - \alpha K)/(2\alpha+1) < 0. \quad \square$$

Let $V_K : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be given by

$$V_K(x, y) = |y| - Kx \log^+ x. \tag{2.3}$$

Now we will introduce the special functions corresponding to the inequality (1.4). Suppose $\alpha \in [0, 1]$ is fixed and assume first that $1 < K < K_0(\alpha)$. Consider the following subsets of $[0,\infty) \times \mathbb{R}$:

$$\begin{aligned}
D_1 &= \{(x, y) : x + |y| \le c - (2\alpha + 1)^{-1}\},\\
D_2 &= \{(x, y) : x + |y| > c - (2\alpha + 1)^{-1},\ x \le \alpha/(2\alpha + 1)\},\\
D_3 &= \{(x, y) : -x + |y| \ge c - 1,\ \alpha/(2\alpha + 1) < x \le 1\},\\
D_4 &= \{(x, y) : -x + |y| < c - 1,\ c - (2\alpha + 1)^{-1} \le x + |y| \le c + 1\},\\
D_5 &= \{(x, y) : c + 1 < x + |y| \le K/(K - 1)\},\\
D_6 &= \{(x, y) : K/(K - 1) - x < |y| \le x/(K - 1)\},\\
D_0 &= [0,\infty) \times \mathbb{R} \setminus (D_1 \cup D_2 \cup \ldots \cup D_6)
\end{aligned}$$

(see Figure 1). Note that D_5 is nonempty, due to Lemma 2.1 (ii). Let

$$p = p_{K,\alpha} = \frac{\alpha + 1}{c(\alpha + 2)}\left(c - \frac{1}{2\alpha + 1}\right)^{\alpha/(\alpha+1)},$$

$$\lambda = \frac{\alpha + 1}{c(2\alpha + 1)} \exp\left(-1 + \frac{2\alpha + 1}{\alpha + 1}\right),$$

and introduce $U = U_{K,\alpha} : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ by

$$U(x,y)=\begin{cases}
p_{K,\alpha}(x + |y|)^{1/(\alpha+1)}(-(\alpha + 1)x + |y|) + L(K, \alpha) & \text{on } D_1,\\
-\alpha x + |y| + \alpha + \lambda \exp\left[-\frac{2\alpha+1}{\alpha+1}\left(x + |y| - \frac{\alpha}{2\alpha+1}\right)\right]\left(x + \frac{1}{2\alpha+1}\right) & \text{on } D_2,\\
-\alpha x + |y| + \alpha + \lambda \exp\left[-\frac{2\alpha+1}{\alpha+1}\left(-x + |y| + \frac{\alpha}{2\alpha+1}\right)\right](1 - x) & \text{on } D_3,\\
\frac{|y|^2-x^2}{2c} + \left(\frac{1}{c} - \log(c(K - 1)) - K\right)(x - 1) + \frac{c}{2} + \frac{1}{2c} & \text{on } D_4,\\
|y| - (x - 1)[\log(x + |y| - 1) + K + \log(K - 1)] & \text{on } D_5,\\
K|y| - x - Kx \log[(K - 1)(x + |y|)/K] & \text{on } D_6,\\
|y| - Kx \log x & \text{on } D_0.
\end{cases}$$

Now assume that $K \ge K_0(\alpha)$ and let

$$c = c(K, \alpha) = \left[\exp(\alpha + 1 - K)\frac{\alpha + 1}{(2\alpha + 1)(K - 1)}\right]^{1/2}. \tag{2.4}$$

Consider the following subsets of $[0,\infty) \times \mathbb{R}$:

$$\begin{aligned}
D_1 &= \{(x, y) : x + |y| \le \alpha/(2\alpha + 1)\},\\
D_2 &= \{(x, y) : x + |y| > \alpha/(2\alpha + 1),\ x \le \alpha/(2\alpha + 1)\},\\
D_3 &= \{(x, y) : -x + |y| \ge \alpha/(2\alpha + 1),\ \alpha/(2\alpha + 1) < x \le 1\},\\
D_4 &= \{(x, y) : c - 1 \le -x + |y| < \alpha/(2\alpha + 1),\ x \le 1\},\\
D_5 &= \{(x, y) : |x - 1| + |y| < c\},\\
D_6 &= \{(x, y) : c + 1 \le x + |y| \le K/(K - 1)\},\\
D_7 &= \{(x, y) : K/(K - 1) - x < |y| \le x/(K - 1)\},\\
D_0 &= [0,\infty) \times \mathbb{R} \setminus (D_1 \cup D_2 \cup \ldots \cup D_7)
\end{aligned}$$

(see Figure 1). Set

$$p = p_{K,\alpha} = \alpha^{\alpha/(\alpha+1)}(2\alpha + 1)^{1/(\alpha+1)}(\alpha + 2)^{-1}$$

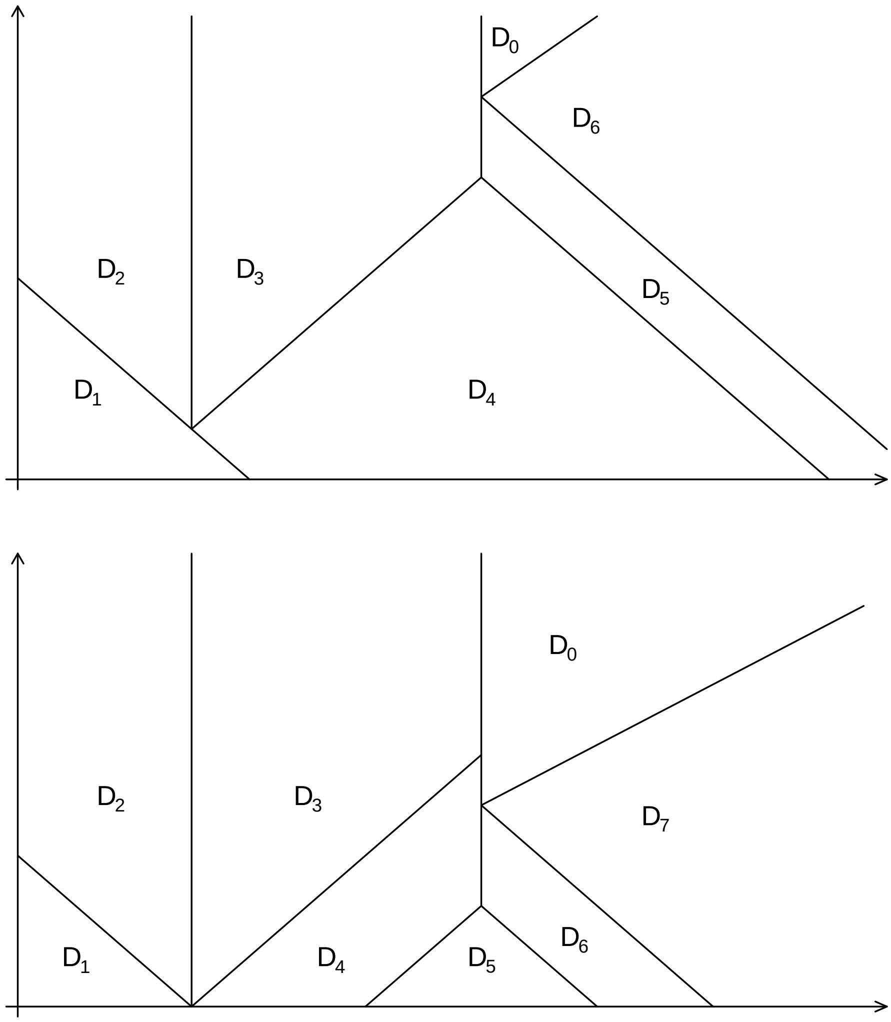

Figure 1. The sets D_i, intersected with $\mathbb{R}^2_+$, in case $1 < K < K_0$ (upper picture) and $K \geq K_0$ (lower picture).

(with the convention $0^0 = 1$) and let $U = U_{K,\alpha} : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be given by

$$U(x,y) = \begin{cases} p_{K,\alpha}(x+|y|)^{1/(\alpha+1)}(-(\alpha+1)x+|y|) + L(K,\alpha) & \text{on } D_1, \\ -\alpha x + |y| + \alpha + \exp\left[-\frac{2\alpha+1}{\alpha+1}\left(x+|y|-\frac{\alpha}{2\alpha+1}\right)\right]\left(x+\frac{1}{2\alpha+1}\right) & \text{on } D_2, \\ -\alpha x + |y| + \alpha + \exp\left[-\frac{2\alpha+1}{\alpha+1}\left(-x+|y|+\frac{\alpha}{2\alpha+1}\right)\right](1-x) & \text{on } D_3, \\ -(1-x)\log\left[\frac{2\alpha+1}{\alpha+1}(1-x+|y|)\right] + (\alpha+1)(1-x)+|y| & \text{on } D_4, \\ \frac{|y|^2-x^2}{2c} + (\frac{1}{c} - \log(c(K-1)) - K)(x-1) + \frac{c}{2} + \frac{1}{2c} & \text{on } D_5, \\ |y| - (x-1)[\log(x+|y|-1) + K + \log(K-1)] & \text{on } D_6, \\ K|y| - x - Kx\log[(K-1)(x+|y|)/K] & \text{on } D_7, \\ |y| - Kx\log x & \text{on } D_0. \end{cases}$$

Wherever possible, we will skip the lower indices and write U, V instead of $U_{K,\alpha}$, V_K; usually it will be clear from the context which K and α we are working with. In the sequence of lemmas below, we present the key properties of the function U.

Lemma 2.2. *Let $\alpha \in [0,1]$, $K > 1$. The function U is continuous on $[0,\infty)\times\mathbb{R}$. Furthermore, the partial derivative U_y is continuous on $(0,\infty)\times\mathbb{R}$ and the partial derivative U_x is continuous on $(0,\infty)\times\mathbb{R}\setminus\{(x,y): x=1,\ |y|\geq c\}$.*

Proof. We omit the tedious calculations. One easily verifies the continuity in the interiors of D_i and hence one needs to check that the functions agree at the common boundaries. □

Lemma 2.3. *Let $\alpha\in[0,1]$, $K>1$. For any $y\in\mathbb{R}$ and $|\gamma|\leq 1$, the function $t\mapsto U(t,y+\gamma t)$ is concave. Furthermore, if $|\gamma|\leq\alpha$, then the function is nonincreasing.*

Proof. First we slightly reformulate the statement. Fix x, y, $h>0$ and $k\in\mathbb{R}$ such that $|k|\leq h$. Introduce a function $G=G_{x,y,h,k}$ defined on the set $\{t: x+th\geq 0\}$ by the formula $G(t)=U(x+th,y+tk)$. We will show the following three conditions:

1° if (x,y) belongs to the interior D_i^o of some D_i, then $G''(0)\leq 0$.

2° $G'_{1,y,h,k}(0-)\geq G'_{1,y,h,k}(0+)$ for any $y>c$.

3° $G'_{0,y,h,k}(0+)\leq 0$ if $|k|\leq\alpha h$.

This clearly will yield the claim: 1°, 2°, together with Lemma 2.2 imply the concavity, and then 3° gives the monotonicity property. We will only present the detailed proof in the case $K\geq K_0$; for $K<K_0$ the conditions 1°, 2° and 3° can be established in a similar manner.

We start with the property 1°. If $(x,y)\in D_1^o$, this follows by the result of Burkholder: the function $t\mapsto(|x+th|+|y+tk|)^{1/(\alpha+1)}(-(\alpha+1)|x+th|+|y+tk|)$ is concave, see page 17 of [5]. If (x,y) lies in the interior of D_2, then

$$G''(0)=\frac{2\alpha+1}{\alpha+1}\exp\Big[-\frac{2\alpha+1}{\alpha+1}\Big(x+|y|-\frac{\alpha}{2\alpha+1}\Big)\Big]\times$$
$$\times(h+k)\Big\{\Big[\frac{2\alpha+1}{\alpha+1}\Big(x+\frac{1}{2\alpha+1}\Big)-2\Big]h+\frac{2\alpha+1}{\alpha+1}(x+\frac{1}{2\alpha+1})k\Big\}\leq 0.$$

The latter inequality is valid, because

$$|k|\leq h,\quad \frac{2\alpha+1}{\alpha+1}\Big(x+\frac{1}{2\alpha+1}\Big)-2\leq -1\quad\text{and}\quad\frac{2\alpha+1}{\alpha+1}\Big(x+\frac{1}{2\alpha+1}\Big)\leq 1.$$

If $(x,y)\in D_3^o$, then we derive that

$$G''(0)=\frac{2\alpha+1}{\alpha+1}\exp\Big[-\frac{2\alpha+1}{\alpha+1}\Big(-x+|y|+\frac{\alpha}{2\alpha+1}\Big)\Big]\times$$
$$\times(h-k)\Big\{\Big[\frac{2\alpha+1}{\alpha+1}(1-x)-2\Big]h-\frac{2\alpha+1}{\alpha+1}(1-x)k\Big\}\leq 0,$$

which follows from

$$|k|\leq h,\quad \frac{2\alpha+1}{\alpha+1}(1-x)-2\leq -1\quad\text{and}\quad \frac{2\alpha+1}{\alpha+1}(1-x)\leq 1.$$

For $(x,y)\in D_4^o$ we have

$$G''(0)=\frac{-h+k}{1-x+|y|}\Big[\Big(2-\frac{1-x}{1-x+|y|}\Big)h+\frac{1-x}{1-x+|y|}k\Big]\leq 0,$$

a consequence of

$$|k| \leq h, \quad 2 - \frac{1-x}{1-x+|y|} \geq 1 \quad \text{and} \quad \frac{1-x}{1-x+|y|} \leq 1.$$

If $(x, y) \in D_5^o$, then $G''(0) = c^{-1}(k^2 - h^2) \leq 0$. If (x, y) lies in the interior of D_6, one easily checks that

$$G''(0) = (x+y-1)^{-2}(h+k)[(-x-2y+1)h + (x-1)k]$$

is nonpositive, as $|k| \leq h$ and $-x-2y+1 \leq 1-x < 0$. If $(x, y) \in D_7^o$, then

$$G''(0) = K(x+y)^{-2}(h+k)[h(-x-2y) + xk] \leq 0,$$

since $|k| \leq h$ and $-x-2y \leq -x < 0$. Finally, on D_0^o, $G''(0) = -Kx^{-1}h^2 \leq 0$.

2° Since U_y is continuous, it suffices to show that for $y > c$ we have

$$U_x(1-, y) \geq U_x(1+, y). \tag{2.5}$$

However,

$$U_x(1-, y) = \begin{cases} \log[\frac{2\alpha+1}{\alpha+1}y] - \alpha - 1 & \text{if } y \in (c, \alpha/(2\alpha+1)), \\ -\alpha - \exp[-\frac{2\alpha+1}{\alpha+1}y + 1] & \text{if } y \geq \alpha/(2\alpha+1) \end{cases}$$

is nondecreasing as a function of y, while

$$U_x(1+, y) = \begin{cases} -\log|y| - K - \log(K-1) & \text{if } y \in (c, (K-1)^{-1}), \\ -K & \text{if } y \geq (K-1)^{-1} \end{cases}$$

is nonincreasing. In consequence, it suffices to check (2.5) for $y = c$, and an easy computation shows that for this choice of y both sides of the estimate are equal.

3° The inequality can be rewritten as $U_x(0+, y) + \alpha|U_y(0+, y)| \leq 0$, and we easily check that in fact we have equality here. □

Lemma 2.4. *Let $\alpha \in [0, 1]$, $K > 1$.*

(i) We have the majorization

$$U(x, y) \geq V(x, y) \tag{2.6}$$

for $x \geq 0$, $y \in \mathbb{R}$.

(ii) If $|y| \leq x$, then

$$U(x, y) \leq L(K, \alpha). \tag{2.7}$$

Proof. (i) Since $U(x, y) = U(x, -y)$ on $[0, \infty) \times \mathbb{R}$, we may assume that $y \geq 0$. The inequality (2.6) is an immediate consequence of the following three facts: for fixed $x \geq 0$,

1° $\lim_{y\to\infty}(U(x, y) - V(x, y)) \geq 0$,

2° $\lim_{y\to\infty}(U_y(x, y) - V_y(x, y)) = 0$,

3° the function $y \mapsto U(x, y)$ is convex and $y \mapsto V(x, y)$ is linear on $[0, \infty)$.

The details are left to the reader.

(ii) We use the previous lemma: the function $t \mapsto U(xt, yt)$ is concave and since $U_x(0+, 0) = U_y(0+, 0) = 0$, it is nonincreasing. Hence $U(x, y) \leq U(0, 0) = L(K, \alpha)$. □

The final property of the functions U is the following.

Lemma 2.5. *Let $K > 1$. Then*

$$\sup_{x \geq 0} [U_{K,0}(x,0) + x] = L^*(K), \tag{2.8}$$

where $L^*(K)$ is given by (1.8).

Proof. Straightforward analysis of the derivative. □

3. Proofs of (1.10), (1.11) and (1.12)

For any semimartingale X there exists a unique continuous local martingale part X^c of X satisfying

$$[X,X]_t = [X^c, X^c]_t + \sum_{0 \leq s \leq t} |\triangle X_s|^2$$

for all $t \geq 0$ (here $\triangle X_s = X_s - X_{s-}$, $s \geq 0$). Furthermore, $[X^c, X^c] = [X,X]^c$, the pathwise continuous part of $[X,X]$. We will need Lemma 1 from [16], which can be stated as follows. Recall that we have set $X_{0-} = Y_{0-} = 0$.

Lemma 3.1. *If X and Y are semimartingales, then Y is differentially subordinate to X if and only if Y^c is differentially subordinate to X^c and for any $t \geq 0$ we have $|\triangle Y_t| \leq |\triangle X_t|$.*

Proof of the inequality (1.10). Obviously, we may restrict ourselves to those submartingales X, for which

$$\sup_t \mathbb{E} X_t \log^+ X_t < \infty. \tag{3.1}$$

It suffices to prove that for any $t > 0$, $\mathbb{E}|Y_t| \leq K \mathbb{E} X_t \log^+ X_t + L(K,\alpha)$. The main tool used in the proof is the Itô's formula. Since U is not of class C^2, we need to approximate it by sufficiently smooth function: fix $0 < \varepsilon < 1$ and let $g^\varepsilon : \mathbb{R}^2 \to [0,\infty)$ be a nonnegative C^∞ function, supported on the ball of center $(0,0)$ and radius ε, satisfying $\int_{\mathbb{R}^2} g^\varepsilon = 1$. Let $U^\varepsilon, V^\varepsilon : [\varepsilon, \infty) \times \mathbb{R} \to \mathbb{R}$ be given by

$$U^\varepsilon(x,y) = \int_{[-\varepsilon,\varepsilon]\times\mathbb{R}} U(x-u, y-v) g^\varepsilon(u,v) \mathrm{d}u\mathrm{d}v$$

and

$$V^\varepsilon(x,y) = \int_{[-\varepsilon,\varepsilon]\times\mathbb{R}} V(x-u, y-v) g^\varepsilon(u,v) \mathrm{d}u\mathrm{d}v.$$

It is clear that the properties described in Lemma 2.3 remain valid for the function U^ε. In consequence, since this function is of class C^∞, for any h, k such that $|k| \leq |h|$, we have

$$U^\varepsilon_{xx} h^2 + 2U^\varepsilon_{xy} hk + U^\varepsilon_{yy} k^2 \leq 0 \qquad \text{on } (\varepsilon,\infty) \times \mathbb{R} \tag{3.2}$$

and

$$U^\varepsilon_x + \alpha |U^\varepsilon_y| \leq 0 \qquad \text{on } (\varepsilon,\infty) \times \mathbb{R}. \tag{3.3}$$

Now fix $\delta \in (\varepsilon\sqrt{2}, \sqrt{2})$, $t \geq 0$ and apply Itô's formula to obtain

$$U^\varepsilon(\delta + X_t, Y_t) = I_0 + I_1 + I_2/2 + I_3, \tag{3.4}$$

where

$$
\begin{aligned}
I_0 &= U^\varepsilon(\delta + X_0, Y_0),\\
I_1 &= \int_{0+}^t U_x^\varepsilon(\delta + X_{s-}, Y_{s-})\mathrm{d}X_s + \int_{0+}^t U_y^\varepsilon(\delta + X_{s-}, Y_{s-})\mathrm{d}Y_s,\\
I_2 &= \int_{0+}^t U_{xx}^\varepsilon(\delta + X_{s-}, Y_{s-})\mathrm{d}[X_s^c, X_s^c] + 2\int_{0+}^t U_{xy}^\varepsilon(\delta + X_{s-}, Y_{s-})\mathrm{d}[X_s^c, Y_s^c]\\
&\quad + \int_{0+}^t U_{yy}^\varepsilon(\delta + X_{s-}, Y_{s-})\mathrm{d}[Y_s^c, Y_s^c]\\
I_3 &= \sum_{0<s\leq t}\Big[U^\varepsilon(\delta + X_s, Y_s) - U^\varepsilon(\delta + X_{s-}, Y_{s-})\\
&\qquad - U_x^\varepsilon(\delta + X_{s-}, Y_{s-})\triangle X_s - U_y^\varepsilon(\delta + X_{s-}, Y_{s-})\triangle Y_s\Big].
\end{aligned}
\tag{3.5}
$$

Now we deal with each of the terms I_i separately. By the differential subordination of Y to X, we have $|Y_0| = |\triangle Y_0| \leq |\triangle X_0| = X_0$ (see Lemma 3.1). Since $\delta > \varepsilon\sqrt{2}$, we have $|Y_0 - v| \leq \delta + X_0 - u$ for (u, v) lying in the support of g^ε and the inequality (2.7) implies $I_0 \leq L(K, \alpha)$. Let $X = X_0 + M + A$, $Y = Y_0 + N + B$ be the Doob-Meyer decomposition guaranteed by the α-differential subordination. We have

$$
\begin{aligned}
&\int_{0+}^t U_x^\varepsilon(\delta + X_{s-}, Y_{s-})\mathrm{d}A_s + \int_{0+}^t U_y^\varepsilon(\delta + X_{s-}, Y_{s-})\mathrm{d}B_s\\
&\qquad \leq \int_{0+}^t U_x^\varepsilon(\delta + X_{s-}, Y_{s-})\mathrm{d}A_s + \int_{0+}^t |U_y^\varepsilon(\delta + X_{s-}, Y_{s-})|\mathrm{d}|B|_s\\
&\qquad \leq \int_{0+}^t U_x^\varepsilon(\delta + X_{s-}, Y_{s-})\mathrm{d}A_s + \int_{0+}^t \alpha|U_y^\varepsilon(\delta + X_{s-}, Y_{s-})|\mathrm{d}A_s \leq 0.
\end{aligned}
$$

Here in the second passage we have exploited α-subordination (since X is a submartingale, we have $|A|_t = A_t$ for all t) and in the third one we have used (3.3). Furthermore, we have

$$
\mathbb{E}\left[\int_{0+}^t U_x^\varepsilon(\delta + X_{s-}, Y_{s-})\mathrm{d}M_s + \int_{0+}^t U_y^\varepsilon(\delta + X_{s-}, Y_{s-})\mathrm{d}N_s\right] = 0,
$$

as the stochastic integrals are martingales. In consequence, the term I_1 has nonpositive expectation. Moreover, I_2 is nonpositive due to (3.2) and the differential subordination of Y^c to X^c (simply approximate the integrals by discrete sums – see e.g. page 533 of [16] for details). Finally, I_3 is nonpositive in virtue of Lemma 3.1 and the fact that Lemma 2.3 is valid for U^ε. Plugging the above estimates for I_i into (3.4) gives $\mathbb{E}U^\varepsilon(\delta + X_t, Y_t) \leq L(K, \alpha)$. The majorization (2.6) carries over to the functions U^ε, V^ε, so

$$
\mathbb{E}V^\varepsilon(\delta + X_t, Y_t) \leq L(K, \alpha). \tag{3.6}
$$

Since $V(x, y) \geq -Kx\log^+ x$, we see that for $x \geq 0$,

$$
V^\varepsilon(\delta + x, y) \geq -K(x + \delta + \varepsilon)\log^+(x + \delta + \varepsilon) \geq -\kappa_1 x\log^+ x - \kappa_2, \tag{3.7}
$$

for some absolute constants κ_1, κ_2. Therefore, by (3.1), we may let $\varepsilon \to 0$ in (3.6) and use Fatou's lemma to obtain

$$
\mathbb{E}V(\delta + X_t, Y_t) = \mathbb{E}|Y_t| - K\mathbb{E}(\delta + X_t)\log^+(\delta + X_t) \leq L(K, \alpha).
$$

Finally, let $\delta \to 0$ and use Lebesgue's dominated convergence theorem (together with (3.1) and (3.7)) to get the claim. $\square$

Proof of (1.11). We assume that

$$\sup_t \mathbb{E}|X_t| \log^+ |X_t| < \infty \tag{3.8}$$

and prove that for any $t \geq 0$, $\mathbb{E}|Y_t| \leq K\mathbb{E}|X_t| \log^+ |X_t| + L(K, 0)$. Introduce the function $U : \mathbb{R}^2 \to \mathbb{R}$ by

$$U(x, y) = U_{K,0}(|x|, y) \tag{3.9}$$

and let V_K be defined by (2.3). In virtue of (2.6), we have

$$U_{K,0}(x, y) \geq V_K(|x|, y).$$

Furthermore, since $(U_{K,0})_x(0+, y) = 0$, we see that for any $y \in \mathbb{R}$ and $|\gamma| \leq 1$ the function $t \mapsto U(t, y + \gamma t)$ is concave on whole $\mathbb{R}$. The remaining part of the proof is just the repetition of the arguments used previously. $\square$

Proof of (1.12). Let $\varepsilon > 0$ and assume g^ε, used above, satisfies the additional symmetry condition $g^\varepsilon(u, v) = g^\varepsilon(u, -v)$ for all u, v. Let U, U^ε, $V^\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ be given by (3.9) and

$$\begin{aligned} U^\varepsilon(x, y) &= \int_{\mathbb{R}^2} U(x - u, y - v) g^\varepsilon(u, v) \mathrm{d}u\mathrm{d}v, \\ V^\varepsilon(x, y) &= \int_{\mathbb{R}^2} V(|x - u|, y - v) g^\varepsilon(u, v) \mathrm{d}u\mathrm{d}v. \end{aligned} \tag{3.10}$$

Introduce $W^\varepsilon : \mathbb{R}^3 \to \mathbb{R}$ by

$$W^\varepsilon(x, y, z) = y + U^\varepsilon(x, y - z). \tag{3.11}$$

This function is of class C^∞. By the symmetry of g^ε and U with respect to the second variable, we have that for all x, y, $U^\varepsilon(x, y) = U^\varepsilon(x, -y)$. Consequently,

$$W^\varepsilon_z(x, y, y) = 0 \quad \text{ for any } x,\ y \in \mathbb{R}. \tag{3.12}$$

Furthermore, by (2.6), we have

$$W^\varepsilon(x, y, z) \geq y + V^\varepsilon(x, y - z). \tag{3.13}$$

Moreover, for any $y \in \mathbb{R}$ and $|\gamma| \leq 1$, the function $t \mapsto U^\varepsilon(t, y + \gamma t)$ is concave (see the previous proof), and hence so is the function $t \mapsto W^\varepsilon(t, y + \gamma t, z)$. This gives

$$W^\varepsilon_{xx} h^2 + 2W^\varepsilon_{xy} hk + W^\varepsilon_{yy} k^2 \leq 0, \qquad |k| \leq |h|. \tag{3.14}$$

Apply Itô's formula to obtain

$$W^\varepsilon(X_t, Y_t, Y_t^*) = I_0 + I_1 + I_2 + I_3/2 + I_4,$$

where

$$
\begin{aligned}
I_0 &= W^\varepsilon(X_0, Y_0, Y_0^*),\\
I_1 &= \int_{0+}^t W_x^\varepsilon(X_{s-}, Y_{s-}, Y_{s-}^*)\mathrm{d}X_s + \int_{0+}^t W_y^\varepsilon(X_{s-}, Y_{s-}, Y_{s-}^*)\mathrm{d}Y_s,\\
I_2 &= \int_{0+}^t W_z^\varepsilon(X_{s-}, Y_{s-}, Y_{s-}^*)\mathrm{d}Y_s^* - \sum_{0<s\leq t} W_z^\varepsilon(X_{s-}, Y_{s-}, Y_{s-}^*)\triangle Y_s^*,\\
I_3 &= \int_{0+}^t W_{xx}^\varepsilon(X_{s-}, Y_{s-}, Y_{s-}^*)\mathrm{d}[X_s^c, X_s^c] + 2\int_{0+}^t W_{xy}^\varepsilon(X_{s-}, Y_{s-}, Y_{s-}^*)\mathrm{d}[X_s^c, Y_s^c]\\
&\quad + \int_{0+}^t W_{yy}^\varepsilon(X_{s-}, Y_{s-}, Y_{s-}^*)\mathrm{d}[Y_s^c, Y_s^c],\\
I_4 &= \sum_{0<s\leq t}\Big[W^\varepsilon(X_s, Y_s, Y_s^*) - W^\varepsilon(X_{s-}, Y_{s-}, Y_{s-}^*)\\
&\qquad\qquad - W_x^\varepsilon(X_{s-}, Y_{s-}, Y_{s-}^*)\triangle X_s - W_y^\varepsilon(X_{s-}, Y_{s-}, Y_{s-}^*)\triangle Y_s\Big].
\end{aligned}
$$

Observe that I_1 has zero expectation. Moreover, $I_2 \leq 0$ by (3.12): indeed, the contribution coming from the jump part of Y^* vanishes, so only the continuous part matters; however, the support of $\mathrm{d}(Y^*)^c$ is precisely the set $\{s : Y_{s-} = Y_{s-}^*\}$ and $W_z^\varepsilon(X_{s-}, Y_{s-}, Y_{s-}^*)$ is equal to 0 there. Furthermore, $I_3 \leq 0$ by (3.14) and $I_4 \leq 0$ due to the concavity of the function W^ε. Hence, by (3.13),

$$
\mathbb{E}Y_t + \mathbb{E}V^\varepsilon(X_t, Y_t - Y_t^*) \leq \mathbb{E}W^\varepsilon(X_t, Y_t, Y_t^*) \leq \mathbb{E}W^\varepsilon(X_0, Y_0, Y_0^*).
$$

Both U and V are continuous, so letting $\varepsilon \to 0$ we obtain

$$
\begin{aligned}
\mathbb{E}Y_t + \mathbb{E}(Y_t^* - Y_t) - K\mathbb{E}|X_t|\log^+|X_t| &= \mathbb{E}Y_t^* - K\mathbb{E}|X_t|\log^+|X_t|\\
&\leq \mathbb{E}[Y_0 + U(X_0, 0)]\\
&\leq \mathbb{E}[|X_0| + U(X_0, 0)].
\end{aligned}
\tag{3.15}
$$

Now it suffices to use Lemma 2.5 to complete the proof. □

4. Sharpness

Throughout this section we deal with the discrete-time case.

4.1. **On the method.** We will use two techniques of showing that the constants are the best possible. One method is just to construct appropriate examples. However, sometimes it is more convenient to take a different approach. Namely, the validity of a given estimate for (sub-)martingales implies the existence of a certain special function and then one exploits its properties to obtain the lower bound for the constant appearing in the inequality. This second approach has been successful in a number of papers (see e.g. [5], [6], [11], [12]) and we will describe it now.

We assume that the underlying probability space is the interval $[0, 1]$ equipped with its Borel subsets and Lebesgue's measure. Recall that a discrete-time real-valued process $f = (f_n)$ is *simple* if for any n the random variable f_n takes only a finite number of values and there is N such that $\mathbb{P}(f_N = f_{N+1} = f_{N+2} = \ldots) = 1$. Clearly, for such processes there is an almost sure limit f_∞.

For a fixed $x \geq 0$, $y \in \mathbb{R}$, let $S_\alpha(x, y)$ denote the class of all pairs (f, g) of simple processes starting from (x, y) such that f is a nonnegative submartingale and g satisfies $|dg_n| \leq |df_n|$ and $|\mathbb{E}(dg_n|\mathcal{F}_{n-1})| \leq \alpha\mathbb{E}(df_n|\mathcal{F}_{n-1})$ almost surely for any $n \geq 1$ (so this is "almost" α-subordination - possibly violated only for $n = 0$). Here $(\mathcal{F}_n)$ is the natural filtration of (f, g). Similarly, for x, $y \in \mathbb{R}$, let $M(x, y)$ denote the set of all pairs (f, g) of simple martingales (with respect to the natural filtration) starting from (x, y) such that $|dg_n| \leq |df_n|$ for any $n \geq 1$.

Let $V : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a fixed function, not necessarily measurable, and introduce, for $\alpha \in [0, 1]$, the function $U = U_{V,\alpha} : [0, \infty) \times \mathbb{R} \to (-\infty, \infty]$ by the formula

$$U(x, y) = \sup_{S_\alpha(x,y)} \mathbb{E}V(f_\infty, g_\infty).$$

We have the following general fact (consult Lemma 2.1 in [6], see also Theorem 4.2 below).

Theorem 4.1. *Suppose $\alpha \in [0, 1]$, $\beta \in \mathbb{R}$ and assume that the inequality*

$$\mathbb{E}V(f_\infty, g_\infty) \leq \beta \tag{4.1}$$

holds for all simple f, g such that f is a nonnegative submartingale and g is α-subordinate to f. Then $U_{V,\alpha}$ satisfies the following:

(i) We have $U(x, y) \geq V(x, y)$,

(ii) If $|y| \leq x$, then $U(x, y) \leq \beta$,

(iii) For any $y \in \mathbb{R}$, the functions $t \mapsto U(t, y+t)$ and $t \mapsto U(t, y-t)$ are concave,

(iv) For any $y \in \mathbb{R}$, the functions $t \mapsto U(y, y + \alpha t)$ and $t \mapsto U(y, y - \alpha t)$ are nonincreasing.

(v) The function U is finite.

Proof. (i) This follows from the fact that $S_\alpha(x, y)$ contains the pair (x, y) of constant processes.

(ii) This is a consequence of (4.1): for any $(f, g) \in S_\alpha(x, y)$, where $|y| \leq x$, we have that g is α-subordinate to f and hence $\mathbb{E}V(f_\infty, g_\infty) \leq \beta$. It remains to take supremum over (f, g).

(iii) Let $t > 0$, $a \in (0, t]$, $y \in \mathbb{R}$ and $\varepsilon = \pm 1$. Take $(f^1, g^1) \in S_\alpha(t+a, y+t+\varepsilon a)$, $(f^2, g^2) \in S_\alpha(t - a, y + t - \varepsilon a)$ and define (f, g) by $(f_0, g_0) \equiv (t, y + t)$, $(f_1, g_1) = (t + a, y + t + \varepsilon a)\chi_{[0,1/2]} + (t - a, y + t - \varepsilon a)\chi_{(1/2,1]}$ and, for $n \geq 2$,

$$f_n(\omega) = f^1_{n-1}(2\omega)\chi_{[0,1/2]}(\omega) + f^2_{n-1}(2\omega - 1)\chi_{(1/2,1]}(\omega),$$
$$g_n(\omega) = g^1_{n-1}(2\omega)\chi_{[0,1/2]}(\omega) + g^2_{n-1}(2\omega - 1)\chi_{(1/2,1]}(\omega).$$

It can be easily seen that (f, g) belongs to $S_\alpha(t, y + t)$ and

$$(f_\infty(\omega), g_\infty(\omega)) = (f^1_\infty(2\omega), g^1_\infty(2\omega))\chi_{[0,1/2]}(\omega) + (f^2_\infty(2\omega-1), g^2_\infty(2\omega-1))\chi_{(1/2,1]}(\omega).$$

Hence

$$U(t, y + t) \geq \mathbb{E}V(f_\infty, g_\infty) = \mathbb{E}V(f^1_\infty, g^1_\infty)/2 + \mathbb{E}V(f^2_\infty, g^2_\infty)/2.$$

Now take supremum over (f^1, g^1) and (f^2, g^2) to obtain the concavity.

(iv) Let $t \geq 0$, $a > 0$, $y \in \mathbb{R}$ and $\varepsilon = \pm 1$. Take $(f, g) \in S_\alpha(t+a, y+\varepsilon\alpha(t+a))$ and let $(\overline{f}, \overline{g})$ be given by $(\overline{f}_0, \overline{g}_0) = (t, y + \varepsilon\alpha t)$ and, for $n \geq 1$, $(\overline{f}_n, \overline{g}_n) = (f_{n-1}, g_{n-1})$. Then it can be verified readily that $(\overline{f}, \overline{g})$ lies in $S_\alpha(t, y + \varepsilon\alpha t)$ and $(\overline{f}_\infty, \overline{g}_\infty) = (f_\infty, g_\infty)$. Consequently,

$$U(t, y + \alpha\varepsilon t) \geq \mathbb{E}V(\overline{f}_\infty, \overline{g}_\infty) = \mathbb{E}V(f_\infty, g_\infty)$$

and it suffices to take supremum over (f, g) to get the claim.

(v) This follows immediately from (ii) and (iii). For any (x, y) there is $\varepsilon \in \{-1, 1\}$ such that the halfline $\{(x + t, y + \varepsilon t) : t \geq -x\}$ intersects with the set

$D = \{(x,y) : |y| \leq x\}$ at infinitely many points. It suffices to use the concavity of U along this halfline and finiteness of U on D. $\square$

A similar argumentation leads to the martingale version of Theorem 4.1 (consult e.g. [2], [3] and Theorem 2.1 in [5]). It will be needed in the proof of the sharpness of (1.6). Let $V : \mathbb{R}^2 \to \mathbb{R}$ be a function and let $U = U_V : \mathbb{R}^2 \to (-\infty, \infty]$ be given by

$$U(x,y) = \sup_{M_\alpha(x,y)} \mathbb{E}V(f_\infty, g_\infty).$$

Theorem 4.2. *Suppose $\beta \in \mathbb{R}$ and assume that the inequality*

$$\mathbb{E}V(f_\infty, g_\infty) \leq \beta$$

holds for all simple martingales f, g such that g is differentially subordinate to f. Then U_V satisfies the following:

(i) We have $U(x,y) \geq V(x,y)$.

(ii) If $|y| \leq x$, then $U(x,y) \leq \beta$.

(iii) For any $y \in \mathbb{R}$, the functions $t \mapsto U(t, y+t)$ and $t \mapsto U(t, y-t)$ are concave.

(iv) For any $(x,y) \in [0,\infty)$ we have $U(x,y) < \infty$.

This result can be further extended to study the inequalities involving maximal functions. Let $V : \mathbb{R}^3 \to \mathbb{R}$ be a fixed function satisfying

$$V(x,y,z) = V(x,y,y \vee z) \quad \text{ for all } x,\ y,\ z.$$

Let $U = U_V : \mathbb{R}^3 \to (-\infty, \infty]$ be given by

$$U(x,y,z) = \sup_{M_\alpha(x,y)} \mathbb{E}V(f_\infty, g_\infty, g^*_\infty \vee z).$$

Theorem 4.3. *Suppose $\beta \in \mathbb{R}$ and assume that the inequality*

$$\mathbb{E}V(f_\infty, g_\infty, g^*_\infty) \leq \beta$$

holds for all simple martingales f, g such that g is differentially subordinate to f. Then U_V satisfies the following:

(i) We have $U(x,y,z) \geq V(x,y,z)$,

(ii) For any x, y, z, $U(x,y,z) = U(x,y,y \vee z)$.

(iii) If $|y| \leq x$, then $U(x,y,y \vee z) \leq \beta$,

(iv) For any $x \in \mathbb{R}$, $y \leq z$, $\varepsilon \in \{-1,1\}$, $\alpha \in (0,1)$ and t_1, t_2 such that $\alpha t_1 + (1-\alpha)t_2 = 0$, we have

$$\alpha U(x+t_1, y+\varepsilon t_1, z) + (1-\alpha)U(x+t_2, y+\varepsilon t_2, z) \leq U(x,y,z).$$

(v) For any $(x,y) \in [0,\infty)$ we have $U(x,y,z) < \infty$.

For the proof and discussion, we refer the reader to [11] and [12].

4.2. **Sharpness of** (1.4)**,** $K \geq K_0$**.** Let $\varepsilon > 0$. The paper [10] contains the construction of the pair (f,g) which enjoys the following properties: f is a submartingale taking values in $[0,1]$, g is α-subordinate to f and $||g||_1 \geq L(K,\alpha) - \varepsilon$. For such a pair, we have $||g||_1 - K \sup_n \mathbb{E} f_n \log^+ f_n = ||g||_1 \geq L(K,\alpha) - \varepsilon$ and hence the constant $L(K,\alpha)$ cannot be replaced in (1.4) by a smaller one.

4.3. **Sharpness of** (1.4)**,** $1 < K < K_0$**.** Suppose that the inequality (1.4) holds with some universal constant $\beta(K,\alpha)$ and let V be given by (2.3). By Theorem 4.1, there is $U : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ satisfying the properties (i)–(v). For $x \geq 0$, $y \in \mathbb{R}$, let $A(y) = U(0,y)$, $B(x) = U(x,0)$ and $C(y) = U(\alpha/(2\alpha+1), y-\alpha/(2\alpha+1))$. Clearly, A is even.

It is convenient to divide the remaining part of the proof into a few intermediate steps.

Step 1. We start with the observation that for any $x \geq 0$ and $y \in \mathbb{R}$,

$$U(x,y+\delta) \leq U(x,y) + \delta. \tag{4.2}$$

This follows immediately from the definition of U. Indeed, if $(f,g) \in S_\alpha(x,y+\delta)$, then $(f,g-\delta) \in S_\alpha(x,y)$, so, by the triangle inequality,

$$\mathbb{E}\left[|g_\infty| - Kf_\infty \log^+ f_\infty\right] \leq \mathbb{E}\left[|(g-\delta)_\infty| - Kf_\infty \log^+ f_\infty\right] + \delta \leq U(x,y) + \delta$$

and it suffices to take supremum over (f,g).

Step 2. We will establish the bound

$$B\left(K/(K-1)\right) \geq -K/(K-1). \tag{4.3}$$

Let us introduce the function W, given on $[1,\infty) \times \mathbb{R} \to \mathbb{R}$ by the formula

$$W(x,y) = \inf_{\lambda > 1/x} [U(\lambda x, \lambda y)/\lambda + Kx \log \lambda].$$

This function enjoys (i), (iii) (with obvious restriction to the domain of W) and is finite. Indeed, the first property is a consequence of the fact that for any $x \geq 1$ and $\lambda > 1/x$,

$$U(\lambda x, \lambda y)/\lambda + Kx\log\lambda \geq |y| - K\lambda x \log(\lambda x) + Kx \log \lambda = V(x,y).$$

To prove (iii), one shows that for any $(x,y) \in (1,\infty) \times \mathbb{R}$ and any $\varepsilon > 0$ there is $\delta > 0$ such that if $a \in (0,\delta)$ and $x - a \geq 1$, then

$$W(x,y) \geq (W(x+a, y \pm a) + W(x-a, y \mp a))/2 - \varepsilon.$$

Furthermore, W has the following homogeneity-type property: for any $x \geq 1$, $y \in \mathbb{R}$ and $\mu \geq 1/x$,

$$W(\mu x, \mu y) = \mu W(x,y) - K\mu x \log \mu. \tag{4.4}$$

By properties (i) and (iii), we have, for $x = K/(K-1)$,

$$\begin{aligned} W(x,0) &\geq \frac{K\delta}{x+K\delta} W(1, x-1) + \frac{x}{x+K\delta} W(x+\delta, -\delta) \\ &\geq K\delta(x-1)/(x+K\delta) + xW(x+\delta,-\delta)/(x+K\delta) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} W(x+\delta,-\delta) \geq& \frac{K\delta}{x+2\delta} W\left(\frac{(K-1)(x+2\delta)}{K}, \frac{x+2\delta}{K}\right) + \frac{x+2\delta-K\delta}{x+2\delta} W(x+2\delta, 0) \\ \geq& \frac{K\delta}{x+2\delta}\left[\frac{x+2\delta}{K} - (K-1)(x+2\delta)\log\left(\frac{(K-1)(x+2\delta)}{K}\right)\right] \\ &+ (x+2\delta-K\delta)W(x+2\delta,0)/(x+2\delta) \\ =& \frac{K\delta}{x+2\delta}\left[\frac{x+2\delta}{K} - (K-1)(x+2\delta)\log\left(\frac{(K-1)(x+2\delta)}{K}\right)\right] \\ &+ \frac{x+2\delta-K\delta}{x+2\delta}\left[\frac{x+2\delta}{x}W(x,0) - K(x+2\delta)\log\left(1+\frac{2\delta}{x}\right)\right], \end{aligned}$$

where in the last passage we have exploited (4.4). Insert this into (4.5), subtract $W(x,0)$ from both sides, divide throughout by δ and let $\delta \to 0$. As the result, we obtain $W(x,0) \geq -K/(K-1)$. The final observation is that $W(x,0) \leq U(x,0)$, so (4.3) follows.

Step 3. Now we will prove that

$$B(c+1) \geq -c\alpha - (\alpha+1)/(2\alpha+1). \tag{4.6}$$

To this end, note that by the properties (i) and (iii), for any $x \geq 1$,

$$\begin{aligned} B(x) &\geq \frac{\delta}{x-1+\delta}U(1,x-1) + \frac{x-1}{x-1+\delta}U(x+\delta,-\delta) \\ &\geq \delta(x-1)/(x-1+\delta) + (x-1)U(x+\delta,-\delta)/(x-1+\delta) \end{aligned}$$

and

$$\begin{aligned} U(x+\delta,-\delta) &\geq \frac{\delta}{x-1+2\delta}U(1,-(x-1+2\delta)) + \frac{x-1+\delta}{x-1+2\delta}B(x+2\delta) \\ &\geq \delta + (x-1)B(x+2\delta)/(x-1+\delta). \end{aligned}$$

Combining these two estimates we obtain, after some manipulations,

$$\frac{B(x)}{x-1} \geq \frac{B(x+2\delta)}{x+2\delta-1} + \frac{2\delta}{x-1} - \frac{\delta^2 x}{(x-1+\delta)(x-1)},$$

so, by induction,

$$\frac{B(x)}{x-1} \geq \frac{B(x+2N\delta)}{x+2N\delta-1} + \sum_{k=0}^{N-1}\left[\frac{2\delta}{x+2k\delta-1} - \frac{\delta^2(x+2k\delta)}{(x+(2k+1)\delta-1)(x+2k\delta-1)}\right].$$

Now set $x=c+1$, $\delta = (\frac{K}{K-1}-c-1)/(2N)$ and let $N\to\infty$ to obtain

$$B(c+1) \geq c(K-1)B(K/(K-1)) - c\log(c(K-1)),$$

as the sum converges to the integral $\int_{c+1}^{K/(K-1)}(x-1)^{-1}\mathrm{d}x = \log(c(K-1))$. Now (4.6) follows from (4.3) and (2.2).

Step 4. The next step is to establish the bound

$$\begin{aligned} B(c-(2\alpha+1)^{-1}) \geq &\frac{\alpha+1}{c(2\alpha+1)+\alpha}A(c-(2\alpha+1)^{-1}) \\ &+\frac{c(2\alpha+1)-1}{c(2\alpha+1)+\alpha}\left[-c\alpha + \frac{\alpha(\alpha+1)}{2\alpha+1} + \frac{(\alpha+1)^2}{c(2\alpha+1)^2}\right]. \end{aligned} \tag{4.7}$$

We proceed as previously. Using the concavity of $t\mapsto U(t,c-(2\alpha+1)^{-1}-t)$ and $t\mapsto U(t,c+1-t)$ we can bound $B(c-(2\alpha+1)^{-1})$ from below by a convex combination of $A(c-(2\alpha+1)^{-1})$, $B(c+1)$ and $U(1,c)$. It suffices to use (4.6) and $U(1,c)\geq V(1,c)=c$ to obtain the desired inequality.

Step 5. Now we will deal with the estimate

$$A(c-(2\alpha+1)^{-1}) \geq c + \frac{2\alpha^2+2\alpha-1}{2\alpha+1} - \frac{2\alpha(\alpha+1)}{c(2\alpha+1)^2} + \frac{B(c-(2\alpha+1)^{-1})}{c(2\alpha+1)}. \tag{4.8}$$

This is the most elaborate part; we will show the inequality only for $\alpha>0$; the case $\alpha=0$ can be treated similarly. Use (iv) and then (iii) to obtain, for any $y\geq c-(2\alpha+1)^{-1}$ and $0<\delta<\alpha$,

$$\begin{aligned} A(y) \geq U(\delta, y+\alpha\delta) &\geq (2\alpha+1)\delta C(y+(\alpha+1)\delta)/\alpha \\ &+ (\alpha-(2\alpha+1)\delta)A(y+(\alpha+1)\delta)/\alpha \end{aligned} \tag{4.9}$$

and, similarly, using (iv) and (i), one gets

$$\begin{aligned} C(y+(\alpha+1)\delta) \geq{}& (2\alpha+1)\delta y/(2+(2\alpha+1)\delta)+\delta/2 \\ &+\frac{(2\alpha+1)(\alpha+1)\delta}{\alpha(2+(2\alpha+1)\delta)}A(y)+\frac{2\alpha-(2\alpha+1)(\alpha+1)\delta}{\alpha(2+(2\alpha+1)\delta)}C(y). \end{aligned} \tag{4.10}$$

Multiply both sides of (4.9) by

$$\lambda=\big(2\alpha+3+\sqrt{(2\alpha+1)^2-4\delta(2\alpha+1)}\big)/(4+2\delta(2\alpha+1)).$$

and add it to (4.10) to obtain, after some manipulations,

$$\begin{aligned} (A(y)-y)\gamma_1-(C(y)-y)\gamma_2 \geq{}& r[(A(y+(\alpha+1)\delta)-(y+(\alpha+1)\delta))\gamma_1 \\ &-(C(y+(\alpha+1)\delta)-(y+(\alpha+1)\delta))\gamma_2] \\ &+(\lambda-1)(\alpha+1)\delta+\delta/2, \end{aligned} \tag{4.11}$$

where

$$\gamma_1=\lambda-\delta\frac{(\alpha+1)(2\alpha+1)}{\alpha(2+\delta(2\alpha+1))},\ \gamma_2=\frac{2\alpha-(\alpha+1)(2\alpha+1)\delta}{\alpha(2+\delta(2\alpha+1))}$$

and

$$r=\frac{(2+\delta(2\alpha+1))(\alpha-\lambda(2\alpha+1)\delta)}{2\alpha-(\alpha+1)(2\alpha+1)\delta}=1-\delta\frac{(2\alpha+1)(2\lambda-(2\alpha+1))}{2\alpha}+o(\delta).$$

By induction, (4.11) gives, for any integer $N\geq 1$,

$$\begin{aligned} (A(y)-y)\gamma_1-(C(y)-y)\gamma_2 \geq{}& r^N[(A(y+N(\alpha+1)\delta)-(y+N(\alpha+1)\delta))\gamma_1 \\ &-(C(y+N(\alpha+1)\delta)-(y+N(\alpha+1)\delta))\gamma_2] \\ &+[(\lambda-1)(\alpha+1)\delta+\delta/2]\,(r^N-1)/(r-1). \end{aligned}$$

Now fix $z>y$, set $\delta=(z-y)/N$ (here N is sufficiently large, so that $\delta<\alpha/(2\alpha+1)$) and let $N\to\infty$. Then $\gamma_1\to\alpha+1$, $\gamma_2\to 1$ and $r^N\to\exp[(y-z)\frac{2\alpha+1}{2\alpha(\alpha+1)}]$, so we obtain, after some computations,

$$\begin{aligned} (A(y)-y)(\alpha+1)-(C(y)-y) \geq{}& \exp\left[(y-z)\frac{2\alpha+1}{2\alpha(\alpha+1)}\right]\big[(A(z)-z)(\alpha+1) \\ &-(C(z)-z)-\alpha(2\alpha(\alpha+1)+1)/(2\alpha+1)\big] \\ &+\alpha(2\alpha(\alpha+1)+1)/(2\alpha+1). \end{aligned}$$

Now let $z\to\infty$. Since $A(z)=U(0,z)\geq V(0,z)=z$ and, by (4.2), $C(z)\leq C(0)+z$, we obtain

$$(A(y)-y)(\alpha+1)-(C(y)-y)\geq\alpha(2\alpha(\alpha+1)+1)/(2\alpha+1).$$

Take $y=c-(2\alpha+1)^{-1}$ and combine it with the following consequence of (iii):

$$C(c-(2\alpha+1)^{-1})\geq\frac{\alpha B(c-(2\alpha+1)^{-1})}{c(2\alpha+1)-1}+\frac{c(2\alpha+1)-(\alpha+1)}{c(2\alpha+1)-1}A(c-(2\alpha+1)^{-1}).$$

As a result, we obtain (4.8).

Step 6. The last inequality we need is

$$(\alpha+2)U(0,0)\geq(\alpha+1)A(c-(2\alpha+1)^{-1})+B(c-(2\alpha+1)^{-1}). \tag{4.12}$$

Fix a positive integer N and let $\delta=(c-(2\alpha+1)^{-1})/N$, $k<N$. Arguing as above, one can establish the inequalities

$$A(k(\alpha+1)\delta)\geq\frac{k(\alpha+1)+\alpha}{(k+1)(\alpha+1)}A((k+1)(\alpha+1)\delta)+\frac{B((k+1)(\alpha+1)\delta)}{(k+1)(\alpha+1)} \tag{4.13}$$

and

$$B(k(\alpha+1)\delta) \geq \frac{2k(\alpha+1)+\alpha}{(2k+1)(k+1)(\alpha+1)} A((k+1)(\alpha+1)\delta) \tag{4.14}$$
$$+ \left[\frac{1}{(2k+1)(k+1)(\alpha+1)} + \frac{k}{k+1}\right] B((k+1)(\alpha+1)\delta).$$

Multiply (4.13) throughout by $\alpha+1-(2k+1)^{-1}$ and add it to (4.14). After some manipulations, one obtains

$$(\alpha+1)[A(k(\alpha+1)\delta) - A((k+1)(\alpha+1)\delta)] + B(k(\alpha+1)\delta) - B((k+1)(\alpha+1)\delta)$$
$$\geq \big(A(k(\alpha+1)\delta) - A((k+1)(\alpha+1)\delta)\big)/(2k+1) \geq (\alpha+1)\delta/(2k+1),$$

where the latter inequality follows from (4.2). Write the above estimate for $k = 0, 1, 2, \ldots, N-1$ and add the obtained inequalities to get

$$(\alpha+1)A(0) + B(0) \geq (\alpha+1)A(c-(2\alpha+1)^{-1}) + B(c-(2\alpha+1)^{-1})$$
$$+(\alpha+1)\delta \sum_{k=0}^{N-1} (2k+1)^{-1}.$$

It suffices to let $N \to \infty$ to obtain (4.12); the last term in the estimate above tends to 0, as it is of order $N^{-1}\log N$.

Step 7. Combine (4.7) and (4.8) to get

$$A(c-(2\alpha+1)^{-1}) \geq c-(2\alpha+1)^{-1} + \alpha + (\alpha+1)/(c(2\alpha+1)^2),$$
$$B(c-(2\alpha+1)^{-1}) \geq \alpha+1-c\alpha.$$

Plugging this into (4.12) yields $U(0,0) \geq L(K,\alpha)$. Now use (ii) with $x = y = 0$ to complete the proof.

4.4. **Sharpness of** (1.6)**,** $K \geq K_0$**.** This is very simple: a pair $(f,g) \equiv (1,1)$ of constant martingales gives equality in (1.6).

4.5. **Sharpness of** (1.6)**,** $1 < K < K_0$**.** Suppose the inequality holds with some constant $\beta(K)$ and use Theorem 4.2 with $V(x,y) = |y| - K|x|\log^+|x|$, $(x,y) \in \mathbb{R}^2$, to obtain the existence of a function U satisfying (i)–(iv). By (iii),

$$U\left((c+1)/2, (c+1)/2\right) \geq (c-1)U(c+1,0)/(2c) + (c+1)U(1,c)/(2c).$$

We may repeat word by word the argumentation from Steps 2 and 3 in Section 4.3, and obtain $U(c+1,0) \geq -1$. Furthermore, by (i), $U(1,c) \geq c$. Plugging this above yields $U\left((c+1)/2, (c+1)/2\right) \geq L(K,0)$, and it suffices to use (ii) with $x = y = (c+1)/2$ to obtain $\beta(K) \geq L(K,0)$.

4.6. **Sharpness of** (1.7)**,** $K > 1$**.** Suppose that the inequality holds with some $\beta^*(K)$. Apply Theorem 4.3, with $V(x,y,z) = y \vee z - K|x|\log^+|x|$, to get a function U satisfying the properties (i)–(v) listed in the statement. It follows from the very definition of U and the special form of V that

$$U(x,y,z) = z + U(x, y-z, 0) \tag{4.15}$$

and

$$U(x,y,z) = U(-x,y,z) \tag{4.16}$$

for all x, y, z. The key ingredient in the proof is the following estimate

$$U(x,x,x) \geq U_{K,0}(x,0) + x \qquad \text{for all } x \geq 0,$$

from which the claim is deduced immmediately, in virtue of Lemma 2.5 and property (iii). By (4.15), this can be rewritten in the form

$$U(x,0,0) \geq U_{K,0}(x,0) \qquad \text{for all } x \geq 0. \tag{4.17}$$

We will only deal with the case $K > K_1$; the remaining ones can be studied in a similar manner.

Step 1. To begin, note that by (iv), for any $x \geq 0$ and $\delta > 0$ we have

$$U(x,0,0) \geq \frac{1-x}{1-x+\delta}U(x-\delta,\delta,0) + \frac{\delta}{1-x+\delta}U(1,x-1,0).$$

Now use (ii) and (4.15) to obtain $U(x-\delta,\delta,0) = \delta + U(x-\delta,0,0)$. Furthermore, by (i), $U(1,x-1,0) \geq V(1,x-1,0) = 0$, so we get

$$U(x,0,0) \geq (1-x)\delta/(1-x+\delta) + (1-x)U(x-\delta,0,0)/(1-x+\delta). \tag{4.18}$$

Use this twice, with $x = 0$ and then with $x = \delta$. Combining these estimates with (4.16), we obtain

$$U(0,0,0) \geq 1, \tag{4.19}$$

that is, (4.17) for $x = 0$.

Step 2. Now fix $x \in (0, c(K,0)]$, a positive integer N, and set $\delta = x/N$. The inequality (4.18) implies that for any $k = 1, 2, \ldots, N$,

$$U(k\delta,0,0)/(1-k\delta) \geq U((k-1)\delta,0,0)/(1-(k-1)\delta) + \delta/(1-(k-1)\delta),$$

so, by induction,

$$\frac{U(x,0,0)}{1-x} \geq U(0,0,0) + \delta\sum_{k=1}^{N}\frac{1}{1-(k-1)\delta}.$$

Use (4.19) and let $\delta \to 0$ to obtain

$$U(x,0,0) \geq 1 - x + (1-x)\int_0^x \frac{1}{1-s}\mathrm{d}s = U_{K,0}(x,0).$$

Step 3. Now we show (4.17) for $x \geq K/(K-1)$. To this end, consider $W : \mathbb{R}^3 \to \mathbb{R}$ given by

$$W(x,y,z) = \inf_{\lambda \geq 1/x}[U(\lambda x, \lambda y, \lambda z)/\lambda + Kx\log\lambda],$$

and observe W satisfies (i), (ii), (iv) and the homogeneity property

$$W(\mu x, \mu y, \mu z) = \mu W(x,y,z) - K\mu x \log \mu. \tag{4.20}$$

Applying (iv),

$$W(x,0,0) \geq \frac{K\delta}{x+K\delta}\cdot W(x-x/K, -x/K, 0) + \frac{x}{x+K\delta}W(x+\delta,\delta,0),$$

which, by (i), (ii) and (4.20), is not smaller than

$$-\frac{K\delta}{x+K\delta}\cdot K(x-x/K)\log(x-x/K) + \frac{x\delta}{x+K\delta} + \frac{x+\delta}{x+K\delta}W(x,0,0).$$

Subtracting $W(0,0,0)$ from both sides, dividing by δ and letting $\delta \to 0$ yields $W(x,0,0) \geq U_{K,0}(x,0)$. It suffices to use $U(x,0,0) \geq W(x,0,0)$ to obtain (4.17).

Step 4. Now we establish (4.17) for $x \in [1+c, K/(K-1))$. By (iv),

$$U(x,0,0) \geq \frac{\delta}{x-1+\delta}U(1,1-x,0) + \frac{x-1}{x-1+\delta}(U(x+\delta,0,0)+\delta).$$

Arguing as in Step 2, this leads to

$$\frac{U(x,0,0)}{x-1} \geq \frac{U(K/(K-1),0,0)}{K/(K-1)-1} + \int_x^{K/(K-1)} (s-1)^{-1}\mathrm{d}s,$$

which is (4.17).

Step 5. It remains to consider the inequality (4.17) on interval $(1-c, 1+c)$. By (iv),

$$U(x,0,0) \geq \frac{2\delta}{x-1+c+2\delta} U\left(\frac{x+1-c}{2}, \frac{1-c-x}{2}, 0\right) + \frac{(x-1+c)U(x+\delta,\delta,\delta)}{x-1+c+2\delta}.$$

Furthermore, again by (iv), and then by (i) and Step 2,

$$\begin{aligned} U\left(\frac{x+1-c}{2}, \frac{1-c-x}{2}, 0\right) &\geq \frac{x-1+c}{2c} U(1,-c,0) + \frac{c+1-x}{2c} U(1-c,0,0) \\ &\geq (c+1-x)(1-\log c)/(2c). \end{aligned}$$

Plugging this into the previous estimate and using (4.15), this yields, after some manipulations,

$$\frac{U(x,0,0)}{(x-1+c)^2} \geq \frac{U(x+\delta,0,0)}{(x+\delta-1+c)^2} + \frac{\delta(c+1-x)}{(x-1+c+\delta)^3}(1-\log c) + \frac{\delta}{(x-1+c)^2} + r,$$

where

$$r \geq -\delta^2 \cdot \frac{K(x+\delta)\log^+(x+\delta)}{[(x-1+c)^2 + 2\delta(x-1+c)](x-1+c+\delta)^2}.$$

Arguing as in Step 2, this gives

$$\frac{U(x,0,0)}{(x-1+c)^2} \geq \frac{U(1+c,0,0)}{(2c)^2} + \int_x^{c+1} \frac{c+1-s}{(s-1+c)^3}(1-\log c) + \frac{1}{(c-1+s)^2}\mathrm{d}s,$$

which is equivalent to (4.17). The proof is complete.

4.7. **The case $K \leq 1$.** That none of (1.4), (1.6) and (1.7) hold, follows from the fact that the corresponding constants $L(K,\alpha)$, $L(K,0)$ and $L^*(K)$ tend to ∞ as K tends to 1. For example, if (1.4) were valid with some $K \leq 1$ and $L(K,\alpha) < \infty$, then, for any $K' > 1$,

$$||g||_1 \leq K' \sup_n \mathbb{E} f_n \log^+ f_n + L(K,\alpha),$$

so $L(K',\alpha) \leq L(K,\alpha)$. This would contradict (1.5) for K' sufficiently close to 1. The argumentation for (1.6) and (1.7) is the same.

5. Inequalities for smooth functions

As an application of Theorems 1.4 and 1.5, we present logarithmic estimates for differentially subordinate smooth functions on Euclidean domains. Let us introduce the necessary background. Suppose that Ω is an open subset of $\mathbb{R}^n$, n being a positive integer. Let D be a bounded subdomain of Ω with $0 \in D$ and $\partial D \subset \Omega$. Denote by μ the harmonic measure on ∂D with respect to 0. Consider two real-valued C^2 functions u, v on Ω. Following [4], we say that v is differentially subordinate to u if

$$|\nabla v(x)| \leq |\nabla u(x)| \quad \text{for } x \in \Omega.$$

Furthermore, for $\alpha \geq 0$, the function v is α-subordinate to u if it is differentially subordinate to u and, in addition,

$$|\Delta v(x)| \leq \alpha |\Delta u(x)| \quad \text{for } x \in \Omega$$

(see [6] and [8]).

Theorem 5.1. *Let $\alpha \in [0,1]$ and suppose that u is subharmonic and nonnegative, v is α-subordinate to u and $|v(0)| \leq u(0)$. Then, for $K > 1$,*

$$\int_{\partial D} |v(x)| d\mu(x) \leq K \int_{\partial D} u(x) \log^+ u(x) d\mu(x) + L(K, \alpha). \tag{5.1}$$

Theorem 5.2. *Suppose that u, v are harmonic, v is differentially subordinate to u and $|v(0)| \leq |u(0)|$. Then, for $K > 1$,*

$$\int_{\partial D} |v(x)| d\mu(x) \leq K \int_{\partial D} |u(x)| \log^+ |u(x)| d\mu(x) + L(K, 0). \tag{5.2}$$

We will only provide the proof of (5.1), the inequality (5.2) can be established in the same manner.

Proof of Theorem 5.1. This is standard. Consider n-dimensional Brownian motion W starting from 0 and let τ denote the exit time of D: $\tau = \inf\{t : W_t \notin D\}$. Consider the processes

$$X = (X_t)_{t \geq 0} = (u(W_{\tau \wedge t}))_{t \geq 0}, \quad Y = (Y_t)_{t \geq 0} = (v(W_{\tau \wedge t}))_{t \geq 0}$$

and write Itô's formula: for any $t \geq 0$,

$$X_t = u(0) + \int_0^t \nabla u(W_{\tau \wedge s}) dW_s + \frac{1}{2} \int_0^t \triangle u(W_{\tau \wedge s}) ds = X_0 + M_t + A_t,$$

$$Y_t = v(0) + \int_0^t \nabla v(W_{\tau \wedge s}) dW_s + \frac{1}{2} \int_0^t \triangle v(W_{\tau \wedge s}) ds = Y_0 + N_t + B_t.$$

Since

$$[M, M]_t - [N, N]_t = |u(0)|^2 - |v(0)|^2 + \int_0^t \left(|\nabla u(W_{\tau \wedge s})|^2 - |\nabla v(W_{\tau \wedge s})|^2 \right) ds$$

and

$$\alpha |A|_t - |B|_t = \frac{1}{2} \int_0^t \left(\alpha |\triangle u(W_{\tau \wedge s})| - |\triangle v(W_{\tau \wedge s})| \right) ds,$$

we see that α-subordination of the functions u and v implies that Y is α-subordinate to X. Since $||Y||_1 = \int_{\partial D} |v(x)| d\mu(x)$, $\sup_t \mathbb{E} X_t \log^+ X_t = \int_{\partial D} u(x) \log^+ u(x) d\mu(x)$, it suffices to use (1.4) to complete the proof. $\square$

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