Logarithmic estimates for the Hilbert transform and the Riesz projection

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Abstract. We study sharp LlogL inequalities for the Hilbert transform and Riesz projection acting on vector-valued functions defined on the unit circle.

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1. Introduction

Assume that $f(\zeta) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\zeta^n$ is a complex-valued integrable function on the unit circle $\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. Here $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}d\theta$ stands for the n-th Fourier coefficient of f. For $p \geq 1$, let $H^p(\mathbb{T},\mathbb{C})$ consist of all f satisfying $\hat{f}(n) = 0$ for $n < 0$. Then $H^p(\mathbb{T},\mathbb{C})$ is a closed subspace of $L^p(\mathbb{T},\mathbb{C})$ and can be identified with the space of analytic functions on the unit disc $\mathbb{D}$. The Riesz projection (or analytic projection) $P_+ : L^p(\mathbb{T},\mathbb{C}) \to H^p(\mathbb{T},\mathbb{C})$, is the operator given by

$$P_+f(\zeta) = \sum_{n\geq 0} \hat{f}(n)\zeta^n, \qquad \zeta \in \mathbb{T}.$$

We introduce P_-, the co-analytic projection on $\mathbb{T}$, by $P_- = I - P_+$. These two projections are closely related to another classical operator, the Hilbert transform (conjugate function) on $\mathbb{T}$, which is defined by

$$\mathcal{H}f(\zeta) = -i\sum_{n\in\mathbb{Z}} \operatorname{sgn}(n)\hat{f}(n)\zeta^n, \qquad \zeta \in \mathbb{T}.$$

A classical theorem of M. Riesz states that the operator P_+ (equivalently, the Hilbert transform $\mathcal{H}$) is bounded on $L^p(\mathbb{T},\mathbb{C})$ for $1 < p < \infty$. The question about the precise value of the norms of these operators has gathered some

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interest in the literature. For $p = 2^k$, $k = 1, 2, \ldots$, the exact values of the norms of $\mathcal{H}$ were determined by Gohberg and Krupnik [5], who showed that

$$||\mathcal{H}||_{L^p(\mathbb{T},\mathbb{C})\to L^p(\mathbb{T},\mathbb{C})} = \cot(\pi/(2p)).$$

For the remaining values of $1 < p < \infty$, the norms of the operator $\mathcal{H}$ acting on real L^p spaces were found by Pichorides [10] and, independently, by Cole (unpublished work, see Gamelin [4]):

$$||\mathcal{H}||_{L^p(\mathbb{T},\mathbb{R})\to L^p(\mathbb{T},\mathbb{R})} = \cot(\pi/(2p^*)),$$

where $p^* = \max\{p, p/(p-1)\}$. Consult also Essén [1] and Verbitsky [12]. These norms do not change while passing to the complex L^p spaces (see e.g. Pełczyński [9]):

$$||\mathcal{H}||_{L^p(\mathbb{T},\mathbb{C})\to L^p(\mathbb{T},\mathbb{C})} = \cot(\pi/(2p^*)), \qquad 1 < p < \infty.$$

For the Riesz projection, Hollenbeck and Verbitsky [7], [8] proved that

$$||P_\pm||_{L^p(\mathbb{T},\mathbb{C})\to L^p(\mathbb{T},\mathbb{C})} = \csc(\pi/p), \qquad 1 < p < \infty.$$

The Hilbert transform is not bounded on $L^1(\mathbb{T},\mathbb{C})$, but, as shown by Zygmund [13], there are absolute K, $L < \infty$ such that

$$||\mathcal{H}f||_1 \leq K||f||_{L\log L} + L. \tag{1.1}$$

Here $||f||_{L\log L} = \frac{1}{2\pi}\int_{\mathbb{T}} |f| \log^+ |f| \mathrm{d}\zeta$. There is a natural question about the optimal values of K and L, which was partially answered by Pichorides [10] under the additional assumption that f is real-valued. Namely, he proved that for a fixed $K > 2/\pi$ there is a universal $L = L(K) < \infty$ such that (1.1) is valid; on the other hand, when $K \leq 2/\pi$, then such L does not exist. See also Essén, Shea and Stanton [2], [3] for related results in this direction.

The purpose of this paper is to study the vector-valued analogues of Pichorides' result, both for the Hilbert transform and the Riesz projection. In addition we shall derive, for each K, the best value of the constant $L(K)$ in a certain version of (1.1). Consider the Hilbert space $\ell^2_{\mathbb{C}}$ with norm $|\cdot|$ and scalar product $\cdot$. Let

$$L\log L = \left\{ f : \mathbb{T} \to \ell^2_{\mathbb{C}} : ||f||_{L\log L} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})| \log^+ |f(e^{i\theta})| \mathrm{d}\theta < \infty \right\}$$

be the $L\log L$ class for $\ell^2_{\mathbb{C}}$-valued functions on the unit circle. It is easy to see that $P_\pm$ and $\mathcal{H}$ can be extended to the operators acting on this class, either by defining them coordinatewise, or simply by noting that the previous definitions make sense in this new setting.

Let us state our main result. Introduce the functions Φ, $\Psi : \mathbb{R} \to \mathbb{R}$ by

$$\Phi(x) = e^{|x|} - 1 - |x|, \qquad \Psi(x) = (|x|+1)\log(|x|+1) - |x|. \tag{1.2}$$

Theorem 1.1. *If $K > 2/\pi$ and $f \in L\log L$, then*

$$||\mathcal{H}f||_1 \leq \frac{K}{2\pi}\int_{-\pi}^{\pi} \Psi(|f(e^{i\theta})|)\,d\theta + L(K), \tag{1.3}$$

where

$$L(K) = \frac{K}{\pi}\int_{-\infty}^{\infty} \frac{\Phi\left(\frac{2}{\pi K}\log|s|\right)}{s^2+1}\,ds < \infty.$$

The constant $L(K)$ is the best possible.

Since $\Psi(x) \leq |x|\log^+|x| + a$ for some absolute a, we get the following.

Corollary 1.2. *Let K be a fixed positive number. There is an absolute $L < \infty$ such that for any $f \in L\log L$,*

$$||\mathcal{H}f||_1 \leq K||f||_{L\log L} + L,$$

if and only if $K > 2/\pi$.

Since $P_+f = \frac{1}{2}(f + i\mathcal{H}f) + \frac{1}{2}\hat{f}(0)$ and $P_-f = \frac{1}{2}(f - i\mathcal{H}f) - \frac{1}{2}\hat{f}(0)$, we get the following result for the Riesz and co-analytic projections.

Corollary 1.3. *Let K be a fixed positive number.*

(i) There is an absolute $L < \infty$ such that for any $f \in L\log L$,

$$||P_+f||_1 \leq K||f||_{L\log L} + L,$$

if and only if $K > 1/\pi$.

(ii) There is an absolute $L = L < \infty$ such that for any $f \in L\log L$,

$$||P_-f||_1 \leq K||f||_{L\log L} + L,$$

if and only if $K > 1/\pi$.

A few words about our approach and the organization of the paper. Pichorides' proof rests on a construction of a certain special superharmonic function; this gives the LlogL result for nonnegative f and then the general statement follows from a decomposition of an arbitrary function to its positive and negative parts. This argument does not work in the vector-valued setting described above and hence a new method is needed. We shall use duality and deduce (1.3) from a certain sharp exponential inequality: see the next section. In Section 3 we exhibit examples which give the optimality of $L(K)$.

2. Proof of Theorem 1.1

First let us state a well-known fact from complex analysis (see e.g. Theorem 4.13 in [11]). For $z = (z_1, z_2, \ldots) \in \ell^2_{\mathbb{C}}$, we define the conjugation by $\overline{z} = (\overline{z_1}, \overline{z_2}, \ldots)$ and then, for w, $z \in \ell^2_{\mathbb{C}}$, we have $w \cdot \overline{z} = \sum_{j=1}^{\infty} w_j z_j$.

Theorem 2.1. *Suppose that D is a given subdomain of $\mathbb{C}$ and let $D' = \{(w,z) \in \ell^2_{\mathbb{C}} \times \ell^2_{\mathbb{C}} : w \cdot \overline{z} \in D\}$. If $\phi : D \to \mathbb{R}$ is harmonic, then $U : D' \to \mathbb{R}$ given by $U(w,z) = \phi(w \cdot \overline{z})$ is pluriharmonic.*

Let $H = \{(x,y) : y > 0\}$ denote the upper half-space and let $S = \{(x,y) \in \mathbb{R}^2 : |x| < 1\}$ stand for the vertical strip in $\mathbb{R}^2$. Fix $K > 2/\pi$ and define $\mathcal{V} : H \to \mathbb{R}$ by the Poisson integral

$$\mathcal{V}(\alpha,\beta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\beta\Phi\left(\frac{2}{\pi K}\log|t|\right)}{(\alpha-t)^2+\beta^2}\mathrm{d}t. \tag{2.1}$$

Then $\mathcal{V}$ is a harmonic function on H and

$$\lim_{(\alpha,\beta)\to(t,0)} \mathcal{V}(\alpha,\beta) = \Phi\left(\frac{2}{\pi K}\log|t|\right). \tag{2.2}$$

Consider a conformal map $\phi(z) = i\exp(\pi z/2)$, which maps S onto H, and introduce $V : \overline{S} \to \mathbb{R}$ by

$$V(x,y) = \begin{cases} \Phi(y/K) & \text{if } |x| = 1, \\ \mathcal{V}(\phi(x,y)) & \text{if } |x| < 1. \end{cases}$$

We easily check that for $(x,y) \in S$ we have

$$V(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos\left(\frac{\pi}{2}x\right)\Phi\left(\frac{2}{\pi K}\log|s| + \frac{y}{K}\right)}{s^2 - 2s\sin\left(\frac{\pi}{2}x\right) + 1}\mathrm{d}s. \tag{2.3}$$

The function V is harmonic on S, because it is a real part of a certain holomorphic function G:

$$V = \mathrm{Re}G. \tag{2.4}$$

In addition, in view of (2.2), V is a continuous function on the closure of S. It is not difficult to see that V satisfies the condition

$$V(x,y) = V(x,-y) = V(-x,y) \qquad \text{for all } (x,y) \in S. \tag{2.5}$$

Indeed, this can be verified by the substitutions $s := -s$ and $s := 1/s$ in (2.3).

We shall need the following further properties of V.

Lemma 2.2. *(i) We have $V(x,0) \le V(0,0)$ for all $x \in [-1,1]$.*

(ii) If $x \in (-1,1)$ and $y \ge 0$, then $V_{yyy}(x,y) \ge 0$.

(iii) If $x \in [0,1)$ and $y \ge 0$, then $yV_x(x,y) + xV_y(x,y) \le 0$.

(iv) There are a_0, a_1, a_2, $\ldots \in \mathbb{C}$ such that the holomorphic function G given by (2.4) satisfies $G(z) = \sum_{n=0}^{\infty} a_n z^{2n}$ for all $z \in S$.

Proof. (i) Since Φ is convex, (2.3) implies that for a fixed $x \in [-1,1]$, the function $V(x,\cdot)$ is also convex. Hence, by the harmonicity of V, we have $V_{xx} \le 0$ on S and it remains to apply (2.5) to get the estimate.

(ii) The function Φ is of class C^2, so Fubini's theorem yields

$$\begin{aligned} V_{yyy}(x,y) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos\left(\frac{\pi}{2}x\right)\Phi'''\left(\frac{2}{\pi}\log|s| + y\right)}{s^2 - 2s\sin\left(\frac{\pi}{2}x\right) + 1}\mathrm{d}s \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\cos\left(\frac{\pi}{2}x\right)\left[\Phi'''\left(\frac{2}{\pi}\log|s| + y\right) + \Phi'''\left(-\frac{2}{\pi}\log|s| + y\right)\right]}{s^2 - 2s\sin\left(\frac{\pi}{2}x\right) + 1}\mathrm{d}s, \end{aligned}$$

where the second passage can be justified using the substitution $s := 1/s$. However, $\Phi'''(s) = e^{|s|}\operatorname{sgn} s$ for $s \neq 0$, so the sum in the square brackets under the last integral is nonnegative for almost all s.

(iii) First note that

$$V_{xy} \le 0 \qquad \text{for } x \in [0,1),\ y \ge 0. \tag{2.6}$$

Indeed, by (2.5), we have $V_x(0,y) = 0$ for any $y \in \mathbb{R}$; this implies $V_{xy}(0,y) = 0$ for all y. Furthermore, by (ii) and the fact that V is harmonic, we have

$V_{xyx} = -V_{yyy} \leq 0$ on $[0,1) \times [0,\infty)$ and hence (2.6) follows. Next, fix $y \geq 0$ and let $F(x) = yV_x(x,y) + xV_y(x,y)$, $x \in [0,1)$. Since $F(0) = 0$, we will be done if we show that F is nonincreasing. Using the harmonicity of V, we get, for $x \in (0,1)$,

$$\begin{aligned} F'(x) &= yV_{xx}(x,y) + V_y(x,y) + xV_{xy}(x,y) \\ &= (-yV_{yy}(x,y) + V_y(x,y)) + xV_{xy}(x,y). \end{aligned}$$

This is nonpositive: indeed, (2.5) gives $V_y(x,0) = 0$, so by (ii) and the mean-value property,

$$\begin{aligned} -yV_{yy}(x,y) + V_y(x,y) &= -yV_{yy}(x,y) + V_y(x,y) - V_y(x,0) \\ &= y(-V_{yy}(x,y) + V_{yy}(x,y')) \leq 0, \end{aligned}$$

for some $y' \in [0,y]$; in addition, $xV_{xy}(x,y) \leq 0$ in view of (2.6). Thus $F' \leq 0$, as desired.

(iv) By (2.5), the partial derivatives of V of odd order vanish at $(0,0)$ and hence so do those of ImG, by Cauchy-Riemann equations. This implies $G^{(2n+1)}(0) = 0$ and the claim follows. □

Consider the region $D = \{z \in \mathbb{C} : |2\mathrm{Re} z^{1/2}| \leq 1\}$.

Lemma 2.3. *The function $z \mapsto V(2z^{1/2})$, $z \in D$, is harmonic.*

Proof. First notice that the function is well defined: in view of (2.5) it does not matter which square root of z we take. The assertion is an immediate consequence of Lemma 2.2 (iv): the function $z \mapsto G(2z^{1/2})$ is holomorphic and hence its real part is harmonic. □

Let $W : \{(w,z) \in \ell^2_{\mathbb{C}} \times \ell^2_{\mathbb{C}} : |w + \overline{z}| \leq 1\} \to \mathbb{R}$ be defined by the formula $W(w,z) = V(2(w\cdot\overline{z})^{1/2})$. The definition makes sense, in view of the following.

Lemma 2.4. *For any w, $z \in \ell^2_{\mathbb{C}}$ we have*

$$2|Re(w \cdot \overline{z})^{1/2}| \leq |w + \overline{z}| \quad and \quad 2|Im(w \cdot \overline{z})^{1/2}| \leq |w - \overline{z}|. \tag{2.7}$$

Proof. It suffices to establish the first estimate; the second follows by the substitution $-z$ in the place of z. We have

$$\begin{aligned} |w + \overline{z}|^2 &= (w + \overline{z}) \cdot (w + \overline{z}) \\ &= |w|^2 + |z|^2 + 2\,\mathrm{Re}(w \cdot \overline{z}) \\ &\geq 2|w \cdot z| + 2\,\mathrm{Re}(w \cdot \overline{z}) \\ &= \left[(w \cdot \overline{z})^{1/2} + \overline{(w \cdot \overline{z})^{1/2}}\right]^2 \\ &= (2\mathrm{Re}(w \cdot \overline{z})^{1/2})^2. \end{aligned}$$

The proof is complete. □

Lemma 2.5. *For any w, $z \in \ell^2_{\mathbb{C}}$ such that $|w + \overline{z}| \leq 1$, we have*

$$W(w,z) \geq \Phi(|w - \overline{z}|/K). \tag{2.8}$$

Proof. Fix $s\in\mathbb{R}$ and consider the function $F_s(x)=V(\sqrt{x^2+s},x)$, defined for nonnegative x satisfying $x^2+s\geq 0$. By Lemma 2.2 (iii), this function is nonincreasing: this is due to

$$F_s'(x)=\frac{x}{\sqrt{x^2+s}}V_x(\sqrt{x^2+s},x)+V_y(\sqrt{x^2+s},x)\leq 0.$$

Next, note that $V(x,y)\geq\Phi(y/K)$ on $\overline{S}$: indeed, both sides are equal when $|x|=1$ and we have $V_{xx}\leq 0$ on S (see the proof of Lemma 2.2 (i)). Therefore, using (2.7),

$$\begin{aligned}\Phi(|w-\overline{z}|/K)&\leq V(|w+\overline{z}|,|w-\overline{z}|)=F_{|w+\overline{z}|^2-|w-\overline{z}|^2}(|w-\overline{z}|)\\&\leq F_{|w+\overline{z}|^2-|w-\overline{z}|^2}(2|\mathrm{Im}(w\cdot\overline{z})^{1/2}|)=V(2(w\cdot\overline{z})^{1/2}),\end{aligned}$$

where the latter follows from the definition of $F_{|w+\overline{z}|^2-|w-\overline{z}|^2}$ and the identity

$$(2|\mathrm{Re}(w\cdot\overline{z})^{1/2}|)^2+|w-\overline{z}|^2-|w+\overline{z}|^2=(2|\mathrm{Im}(w\cdot\overline{z})^{1/2}|)^2.\quad\square$$

Proof of (1.3). Let g be a Borel function on $\mathbb{T}$ taking values in the unit ball of $\ell^2_{\mathbb{C}}$ and let g_+, g_- denote the harmonic extensions of $P_+g-\frac{1}{2}\hat{g}(0)$ and $\overline{P_-g}+\frac{1}{2}\overline{\hat{g}(0)}$ to the unit disc $\mathbb{D}$. By Theorem 2.1, the function W is pluriharmonic and thus $W(g_+,g_-)$ is harmonic on $\mathbb{D}$ (we need the bound $|g|\leq 1$ to guarantee that $|g_++\overline{g_-}|\leq 1$, so $W(g_+,g_-)$ is well defined). Apply the mean-value property and Lemma 2.2 (i) to get

$$\frac{1}{2\pi}\int_{\mathbb{T}}W\big(g_+(\zeta),g_-(\zeta)\big)\mathrm{d}\zeta=W\left(\frac{1}{2}\hat{g}(0),\frac{1}{2}\overline{\hat{g}(0)}\right)=V\left(|\hat{g}(0)|,0\right)\leq V(0,0).$$

Combine this with (2.8) to obtain the following dual of (1.3):

$$\frac{1}{2\pi}\int_{\mathbb{T}}\Phi(|\mathcal{H}g(\zeta)|/K)\mathrm{d}\zeta\leq V(0,0)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Phi\left(\frac{2}{\pi K}\log|s|\right)}{s^2+1}\mathrm{d}s.\tag{2.9}$$

Now we are ready to establish (1.3). Take $f\in L\log L$, let g be as above and observe that the function Ψ is the Legendre transform of Φ (that is, $\Psi'|_{\mathbb{R}_+}$ is the inverse to $\Phi'|_{\mathbb{R}_+}$). Consequently, by Young's inequality,

$$\begin{aligned}\left|\frac{1}{2\pi}\int_{\mathbb{T}}\mathcal{H}f(\zeta)\cdot g(\zeta)\mathrm{d}\zeta\right|&=\left|\frac{1}{2\pi}\int_{\mathbb{T}}f(\zeta)\cdot\mathcal{H}g(\zeta)\mathrm{d}\zeta\right|\\&\leq\frac{K}{2\pi}\int_{\mathbb{T}}\Psi(|f(\zeta)|)\mathrm{d}\zeta+\frac{K}{2\pi}\int_{\mathbb{T}}\Phi(|\mathcal{H}g(\zeta)|/K)\mathrm{d}\zeta\\&\leq\frac{K}{2\pi}\int_{\mathbb{T}}\Psi(|f(\zeta)|)\mathrm{d}\zeta+\frac{K}{\pi}\int_{-\infty}^{\infty}\frac{\Phi\left(\frac{2}{\pi K}\log|s|\right)}{s^2+1}\mathrm{d}s,\end{aligned}$$

in view of (2.9). Taking supremum over all g as above gives

$$||\mathcal{H}f||_1\leq\frac{K}{2\pi}\int_{\mathbb{T}}\Psi(|f(\zeta)|)\mathrm{d}\zeta+\frac{K}{\pi}\int_{-\infty}^{\infty}\frac{\Phi\left(\frac{2}{\pi K}\log|s|\right)}{s^2+1}\mathrm{d}s,$$

which is the claim. $\square$

3. Sharpness of (1.3) and (2.9)

We shall prove now that for each $K > 2/\pi$, the constants

$$\frac{K}{\pi}\int_{-\infty}^{\infty}\frac{\Phi\left(\frac{2}{\pi K}\log|s|\right)}{s^2+1}\mathrm{d}s \qquad \text{and} \qquad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Phi\left(\frac{2}{\pi K}\log|s|\right)}{s^2+1}\mathrm{d}s$$

are optimal in (1.3) and (2.9), respectively, even if f is assumed to be real-valued. Of course, it suffices to focus on the logarithmic estimate. Consider the function $w:\mathbb{T}\to\mathbb{R}$ given by

$$w(e^{i\phi}) = \frac{2}{\pi}\log\left|\frac{1+\sin\phi}{\cos\phi}\right|.$$

It is easy to check that $\mathcal{H}w(e^{i\phi}) = 1_{\{|\phi|\leq \pi/2\}} - 1_{\{|\phi|>\pi/2\}}$ for $|\phi|\leq\pi$. Indeed, consider a conformal mapping $F:\mathbb{D}\to S$, given by the formula $F(z) = (2/\pi)\log[(iz-1)/(z-i)] - i$ and observe that $\operatorname{Re}F = w$ and $\operatorname{Im}F = \mathcal{H}w$ on $\mathbb{T}$. Next, introduce the function u on the unit circle by

$$u(e^{i\phi}) = \Phi'\left(\left|\frac{2}{K\pi}\log\left|\frac{1+\sin\phi}{\cos\phi}\right|\right|\right)\operatorname{sgn}\left(\log\left|\frac{1+\sin\phi}{\cos\phi}\right|\right), \qquad |\phi|\leq\pi.$$

We have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi(u(e^{i\phi}))\mathrm{d}\phi &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi\left(\Phi'\left(\left|\frac{2}{K\pi}\log\left|\frac{1+\sin\phi}{\cos\phi}\right|\right|\right)\right)\mathrm{d}\phi\\ &= \frac{2}{\pi}\int_{0}^{\pi/2}\Psi\left(\Phi'\left(\frac{2}{K\pi}\log\frac{1+\sin\phi}{\cos\phi}\right)\right)\mathrm{d}\phi,\end{aligned}$$

which, after substitution $t=\frac{2}{\pi}\log\left(\frac{1+\sin\phi}{\cos\phi}\right)$, becomes

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi(u(e^{i\phi}))\mathrm{d}\phi = \int_0^\infty\frac{\Psi(\Phi'(t/K))}{\cosh(\pi t/2)}\mathrm{d}t.$$

On the other hand, since $||\mathcal{H}w||_\infty = 1$, we have

$$\begin{aligned}||\mathcal{H}u||_1 &\geq \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{H}u(e^{i\phi})\mathcal{H}w(e^{i\phi})\mathrm{d}\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi}u(e^{i\phi})w(e^{i\phi})\mathrm{d}\phi\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi'\left(\left|\frac{2}{K\pi}\log\left|\frac{1+\sin\phi}{\cos\phi}\right|\right|\right)\left|\frac{2}{\pi}\log\left|\frac{1+\sin\phi}{\cos\phi}\right|\right|\mathrm{d}\phi\\ &= \int_0^\infty\frac{\Phi'(t/K)t}{\cosh(\pi t/2)}\mathrm{d}t.\end{aligned}$$

It suffices to use the identities

$$\Phi'(t)t = \Psi(\Phi'(t)) + \Phi(t), \qquad t\geq 0,$$

and

$$K\int_0^\infty\frac{\Phi(t/K)}{\cosh(\pi t/2)}\mathrm{d}t = \frac{K}{\pi}\int_{-\infty}^{\infty}\frac{\Phi\left(\frac{2}{\pi K}\log|s|\right)}{s^2+1}\mathrm{d}s$$

to complete the proof.

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