A LOGARITHMIC BOUND FOR A STOPPED BROWNIAN MOTION

Abstract. We study an estimate which can be regarded as a logarithmic version of Burkholder-Davis-Gundy inequality. Namely, for any $K > 0$ we determine the best constant $L(K) \in (0, \infty]$ for which the following holds. If B is a standard Brownian motion and τ is an adapted stopping time, then

$$\mathbb{E}\sqrt{\tau} \leq K\mathbb{E}\Psi(|B_\tau|) + L(K),$$

where $\Psi(x) = (x+1)\log(x+1) - x$. Using standard embedding theorems, we obtain a related logarithmic bound involving a continuous-path martingale and its square bracket.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathcal{F}_0$ contains all the events of measure 0. Let $B = (B_t)_{t\geq 0}$ be an adapted standard one-dimensional Brownian motion. Celebrated inequalities of Burkholder, Davis and Gundy state that for any $p \in (0, \infty)$ there are finite universal constants c_p, C_p such that

$$c_p||\tau^{1/2}||_p \leq \left|\left| \sup_{0\leq t\leq \tau} |B_t| \right|\right|_p \leq C_p||\tau^{1/2}||_p \tag{1.1}$$

for any adapted stopping time τ. These inequalities are of fundamental importance to the theory of stochastic integration and have numerous applications and extensions. Very little is known about the optimal values of the constants c_p and C_p; to the best of the author's knowledge, the only result in this direction is that for $p = 2$, the optimal choices for c_p and C_p are 1 and 2, respectively.

2010 *Mathematics Subject Classification.* 60G40, 60G42, 35R35, 45G10.
Key words and phrases. Brownian motion, optimal stopping problem, free-boundary problem, Burkholder-Davis-Gundy inequality.

Consider the version of (1.1) in which the term $\sup_{0\leq t\leq\tau}|B_t|$ is replaced by $|B_\tau|$. Then there are finite absolute constants a_p and A_p for which

$$a_p||\tau^{1/2}||_p \leq ||B_\tau||_p, \qquad \text{if } 1<p<\infty \quad \text{and} \quad ||\tau^{1/2}||_p<\infty, \tag{1.2}$$

and

$$||B_\tau||_p \leq A_p||\tau^{1/2}||_p, \qquad \text{if } 0<p<\infty. \tag{1.3}$$

These estimates follow immediately from (1.1) and Doob's maximal inequality for martingales [4]. In contrast with (1.1), the optimal values of the constants a_p and A_p are known (cf. Davis [3]). For $p=2n$, n a positive integer, they are respectively the smallest and the largest positive zeros of the Hermite polynomial of order $2n$. (For $p=4$, this has been proved by Novikov [11]). For the remaining values of p, the description is a little more complicated. Let ν_p be the smallest positive zero of the confluent hypergeometric function and let μ_p be the largest positive zero of the parabolic cylinder function of parameter p (see Abramovitz and Stegun [1]). Then the best value of a_p is μ_p when $1<p<2$ and ν_p for $2\leq p<\infty$; in addition, the best choice for A_p is ν_p when $0<p<2$ and μ_p for $2\leq p<\infty$.

The above results have been extended in many directions and the literature on this subject is very extensive. We only mention here the works [5], [6], [7], [8], [9], [10] and [14]. We also refer the reader to the monograph [12] for an overview of the inequalities which can be obtained with the use of the optimal stopping techniques.

There is a natural question about the limit case $p=1$ in (1.2). The constant a_p goes to 0 as p approaches 1, so the moment inequality fails to hold for general stopping times τ. However, combining (1.1) with the well-known logarithmic estimate for the maximal function (see e.g. [4]) yields the existence of finite constants K and L such that

$$\mathbb{E}\sqrt{\tau} \leq K\mathbb{E}|B_\tau|\log^+|B_\tau| + L, \tag{1.4}$$

whenever τ is an adapted stopping time satisfying $\tau \in L^{p/2}$ for some $p > 1$. We will be interested in providing the sharp version of this estimate. The following two natural questions arise:

(I) For which $K > 0$ there is a finite $L = L(K)$ for which (1.4) holds?

(II) If K is as in (I), what is the optimal choice for $L(K)$?

Actually, we will study a slightly different form of (1.4), in which the term $|B_\tau| \log^+ |B_\tau|$ is replaced by $\Psi(|B_\tau|)$. Here $\Psi : [0, \infty) \to [0, \infty)$ is a Young function given by the formula $\Psi(x) = (x + 1) \log(x + 1) - x$. This choice of "LlogL" term has some advantages over the classical function $x \mapsto x \log^+ x$. First, we have that $\Psi(x) - x \log x \to -\infty$ as $x \to \infty$, so, in a sense,

$$\mathbb{E}\sqrt{\tau} \leq K \mathbb{E}\Psi(|B_\tau|) + L(K) \tag{1.5}$$

is more tight than (1.4). Actually, for any $\alpha < 1$ there is $\beta > 0$ such that $\Psi(x) \geq \alpha x \log^+ x - \beta$ for all x; this implies that the answer to (I) is the same, no matter which function we choose. Secondly, and most importantly, Ψ is smooth and strictly increasing. This allowed us to overcome all the technical difficulties and obtain relatively compact formulas.

From now on, we focus on (1.5) and our purpose is to give the full answers to both questions above. The problem (I) has a very simple solution: the finite L exists if and only if $K > \sqrt{2/\pi}$. However, the answer to (II) turns out to be more complicated; the description of the optimal value of $L(K)$ involves an auxiliary parameter, and we postpone it to the Section 3 below.

Our approach rests on solving an optimal stopping problem closely related to (1.5). This will be presented and explained in detail in the next two sections.

Let us conclude this section by giving an application of the above result. By the embedding theorem of Dambis, Dubins and Schwarz (see [2], [6]; [13] is also a convenient reference), the inequality (1.5) implies the following sharp inequality:

$$\mathbb{E}\sqrt{\langle X, X\rangle_\infty} \leq K \mathbb{E}\Psi(|X_\infty|) + L(K), \qquad K > \sqrt{2/\pi}. \tag{1.6}$$

Here $X = (X_t)_{t\geq 0}$ is an arbitrary continuous-path local martingale starting from 0 whose quadratic covariance process $\langle X, X\rangle$ satisfies the integrability condition $\langle X, X\rangle_\infty \in L^{p/2}$ for some $p > 1$. This inequality can be further applied to obtain, for example, interesting sharp bounds for Bessel processes, stochastic integrals and stopped geometric Brownian motion. We omit the details and leave them to the interested reader.

2. An optimal stopping problem

Clearly, the questions posed in the previous section lead to the following optimal stopping problem:

$$U = \sup_\tau \mathbb{E}\, G(B_\tau, \tau), \tag{2.1}$$

where the gain function G is given by $G(x,t) = \sqrt{t} - K\Psi(|x|)$ for $(x,t) \in \mathbb{R}\times[0,\infty)$. Here the supremum is taken over all stopping times τ of B such that τ belongs to $L^{p/2}$ for some $p > 1$. In order to treat the problem successfully (i.e., enable the use of Markovian arguments), we extend it so that the process $((B_t, t))_{t\geq 0}$ can start at the arbitrary points of $\mathbb{R}\times[0,\infty)$. This is straightforward: for any $(x,t) \in \mathbb{R}\times[0,\infty)$ define the underlying value function by

$$U(x,t) = \sup_\tau \mathbb{E} G(x + B_\tau, t + \tau), \tag{2.2}$$

where the supremum is taken over the same class of stopping times as previously. Of course, as soon as we manage to find the value function U, we are done: the optimal constant $L(K)$ in (1.5) equals $U(0,0)$.

To solve the optimal stopping problem (2.2), we use an approach which has proved to be very efficient in this type of settings. The procedure consists of two parts. The first step, presented in this section, exploits more or less heuristic a priori considerations concerning the structure of U and yields *a candidate* U_0 for the value function. The second step of the analysis, which is the contents of Section 3, is the rigorous verification that both functions U and U_0 coincide on $\mathbb{R}\times[0,\infty)$.

We start with introducing the continuation set

$$C = \{(x,t) \in \mathbb{R} \times [0,\infty) : U(x,t) > G(x,t)\}$$

and the stopping set

$$D = \{(x,t) \in \mathbb{R} \times [0,\infty) : U(x,t) = G(x,t)\}.$$

Clearly, all we need is to identify C and the restriction of U to this set. The lemma below provides the initial insight into the shape of the continuation region.

Lemma 2.1. *If $(x,t) \in C$ and $s < t$, then $(x,s) \in C$.*

Proof. Since $(x,t) \in C$, there is a stopping time τ belonging to $L^{p/2}$ for some $p > 1$, such that $\mathbb{E}\, G(x + B_\tau, t + \tau) > G(x,t)$, i.e.,

$$\mathbb{E}\big(\sqrt{t+\tau} - K\Psi(|x + B_\tau|)\big) > \sqrt{t} - K\Psi(|x|).$$

On the other hand, for any fixed $s \geq 0$, the function $r \mapsto \sqrt{r+s} - \sqrt{r}$ is nonincreasing, which combined with the previous bound gives

$$\begin{aligned}\mathbb{E}\big(\sqrt{s+\tau} - K\Psi(|x + B_\tau|)\big) &\geq \mathbb{E}\big(\sqrt{t+\tau} - K\Psi(|x + B_\tau|)\big) + \sqrt{s} - \sqrt{t}\\ &> \sqrt{s} - K\Psi(|x|).\end{aligned}$$

Consequently, we get $\mathbb{E}\, G(x + B_\tau, s + \tau) > G(x,s)$ and hence $(x,s) \in C$. $\square$

In the next lemma we establish the key two properties of the function U.

Lemma 2.2. *(i) The function U enjoys the symmetry condition*

$$U(x,t) = U(-x,t) \qquad \text{for all } x \in \mathbb{R} \text{ and } t \geq 0. \tag{2.3}$$

(ii) If $(0,s) \in D$, then the function U satisfies the homogeneity property

$$U(\lambda(x+1) - 1, \lambda^2 t) = \lambda U(x,t) - K\lambda(x+1)\log\lambda + K(\lambda - 1) \tag{2.4}$$

for all $x \geq 0$, $t \geq s$ and all λ such that $\lambda(x+1) - 1 \geq 0$ and $\lambda^2 t \geq s$.

Proof. (i) This follows at once from the analogous symmetry of G and the fact that the process $(-B_t)_{t\geq 0}$ is also a Brownian motion.

(ii) We start from observing that by Lemma 2.1, the whole halfline $\{0\}\times[s,\infty)$ is contained in the stopping set D. Consequently, by the strong Markov property, in the derivation of $U(\lambda(x+1)-1,\lambda^2 t)$ and $U(x,t)$ we may restrict ourselves to those τ, for which the processes $(\lambda(x+1)-1+B_{\tau\wedge u})_{u\geq 0}$ (respectively, $(x+B_{\tau\wedge u})_{u\geq 0}$) are nonnegative. Pick such a stopping time. We apply Brownian scaling and note that $\tilde{\tau}=\tau/\lambda^2$ is a stopping time for the Brownian motion $(\tilde{B}_t)_{t\geq 0}=(\lambda^{-1}B_{\lambda^2 t})_{t\geq 0}$. Now, rewrite the definition of $U(\lambda(x+1)-1,\lambda^2 t)$ in the form

$$\begin{aligned}\sup_\tau \mathbb{E}\Big\{\lambda\sqrt{t+\tau/\lambda^2}-K\lambda\left(x+1+\lambda^{-1}B_{\lambda^2(\tau/\lambda^2)}\right)\log\left[\lambda\left(x+1+\lambda^{-1}B_{\lambda^2(\tau/\lambda^2)}\right)\right]&\\ +K\lambda\left(x+1+\lambda^{-1}B_{\lambda^2(\tau/\lambda^2)}\right)-K\Big\}&\\ =\lambda\sup_{\tilde{\tau}}\mathbb{E}\Big\{\sqrt{t+\tilde{\tau}}-K\Psi(x+\tilde{B}_{\tilde{\tau}})-K(x+1+\tilde{B}_{\tilde{\tau}})\log\lambda\Big\}+K(\lambda-1).&\end{aligned}$$

Since $\tau\in L^p$ for some $p>1$, we have $\mathbb{E}(x+1+\tilde{B}_{\tilde{\tau}})=x+1$ and (2.4) follows. □

The above lemma gives the following information on the stopping region D.

Lemma 2.3. *Suppose that the set $\{t:(0,t)\in D\}$ is nonempty and let t_0 denote its infimum. Then*

$$\Big\{(x,t):\sqrt{t}\geq\sqrt{t_0}(|x|+1)\Big\}\subseteq D. \tag{2.5}$$

Proof. First, note that D is symmetric with respect to t-axis; this follows immediately from (2.3). Therefore, we will be done if we show that

$$\{(x,t)\in[0,\infty)^2:\sqrt{t}\geq\sqrt{t_0}(x+1)\}\subset D.$$

By Lemma 2.1, we see that the whole halfline $\{0\}\times(t_0,\infty)$ is contained inside D. Actually, we will show that $(0,t_0)$ also belongs to D. To do this, take an arbitrary stopping time τ belonging to $L^{p/2}$ for some $p>1$. By Lebesgue's dominated convergence theorem, we have $\mathbb{E}\sqrt{t_0+n^{-1}+\tau}\to\mathbb{E}\sqrt{t_0+\tau}$ as $n\to\infty$ and hence

$$\liminf_{n\to\infty} U(0,t_0+n^{-1})\geq \mathbb{E}G(B_\tau,t_0+\tau).$$

Taking the supremum over all τ as above gives

$$U(0,t_0)\leq \liminf_{n\to\infty} U(0,t_0+n^{-1})=\liminf_{n\to\infty} G(0,t_0+n^{-1})=G(0,t_0),$$

so $U(0,t_0)=G(0,t_0)$ and $(0,t_0)\in D$, as claimed. Now, pick arbitrary nonnegative y, s satisfying $\sqrt{s}\geq\sqrt{t_0}(y+1)$ and apply the identity (2.4) to $x=0$, $\lambda=y+1$ and $t=\lambda^{-2}s$. The application is allowed, since $\lambda\geq 1$, $t\geq\lambda^{-2}(y+1)^2t_0=t_0$ and $\lambda^2t\geq\lambda^2t_0\geq t_0$. We obtain

$$U(y,s)=\lambda U(0,t)-K(y+1)\log(y+1)+Ky=G(y,s)$$

so $(y,s)\in D$. This proves the claim. $\square$

The remaining part of the analysis rests on the additional three assumptions. First, we impose the condition that U is a continuous function on its whole domain. The second requirement is that we have equality in (2.5); that is, the continuation and stopping regions are given by

$$C=\{(x,t):\sqrt{t}<\sqrt{t_0}(|x|+1)\},\qquad D=\{(x,t):\sqrt{t}\geq\sqrt{t_0}(|x|+1)\}.$$

From the general theory of optimal stopping for the Markov processes (see Chapter I in [12]) we infer that the stopping time which gives equality in (2.2), is defined by

$$\tau_D=\inf\{s\geq 0:(x+B_s,t+s)\in D\}.$$

Thus, standard arguments based on the strong Markov property and classical results on PDEs show that U is of class $C^{2,1}$ on C and satisfies the heat equation

$$U_t+\frac{1}{2}U_{xx}=0\qquad\text{on } C. \tag{2.6}$$

The final condition we impose concerns the principle of smooth-fit. Namely, we assume that

$$U_x(x+,t) = G_x(x+,t) \qquad \text{for } x \geq 0 \text{ and } \sqrt{t} = \sqrt{t_0}(x+1). \tag{2.7}$$

Exploiting the above properties, we will identify the explicit formula for the candidate for U. If $t \geq t_0$ and $\sqrt{t} < \sqrt{t_0}(|x|+1)$, then, by (2.4), we have

$$U(x,t) = \sqrt{\frac{t}{t_0}}\gamma\left(\sqrt{\frac{t_0}{t}}(|x|+1)\right) + \frac{K(|x|+1)}{2}\log\frac{t_0}{t} - K,$$

where $\gamma(x) = U(x-1,t_0) + K$ for $x \geq 1$. Applying (2.6), we check that γ satisfies the differential equation

$$t_0\gamma''(s) - s\gamma'(s) + \gamma(s) = Ks \qquad \text{for } s > 1. \tag{2.8}$$

It is not difficult to find the full class of solutions. It is given by

$$\begin{aligned}\gamma(s) = -Ks\log s - K\sqrt{2\pi}s\int_1^s \frac{\sqrt{t_0}}{u^2}\exp\left(\frac{u^2}{2t_0}\right)\Phi\left(-\frac{u}{\sqrt{t_0}}\right)\mathrm{d}u\\ + \alpha s\int_1^s \exp\left(\frac{u^2}{2t_0}\right)\frac{\mathrm{d}u}{u^2} + \beta s,\end{aligned}$$

where α, β are arbitrary real parameters and

$$\Phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^t \exp(-u^2/2)\mathrm{d}u$$

denotes the distribution function of the standard normal variable. To determine α and β which correspond to the function we search for, observe first that α cannot be negative. Indeed, otherwise the inequality $U(x,t_0) \geq G(x,t_0)$ would not be satisfied for large $|x|$. Similarly, α cannot be positive. To show this, note that for any $x > 0$ and any stopping time τ satisfying the usual integrability we have

$\sqrt{t_0+\tau} \le \sqrt{t_0(x+1)^2+\tau}$ and thus

$$\begin{aligned}\mathbb{E}G(x+B_\tau, t_0+\tau) &\le \mathbb{E}G(x+B_\tau, t_0(x+1)^2+\tau)\\ &\le U(x, t_0(x+1)^2)\\ &= G(x, t_0(x+1)^2)\\ &= (x+1)\sqrt{t_0} - K\Psi(x) \le (x+1)\sqrt{t_0}.\end{aligned}$$

However, if α were larger than 0, this inequality would be violated for large x (the integral $\int_1^s \exp(u^2/(2t_0))u^{-2}\mathrm{d}u$ grows very fast for big s). Hence, we must have $\alpha = 0$. To determine β, we note that

$$\sqrt{t_0} = G(0, t_0) = U(0, t_0) = \gamma(1) - K,$$

which implies $\beta = K + \sqrt{t_0}$. Thus, we have obtained that

$$\gamma(s) = (K+\sqrt{t_0})s - Ks\log s - K\sqrt{2\pi}s\int_1^s \frac{\sqrt{t_0}}{u^2}\exp\left(\frac{u^2}{2t_0}\right)\Phi\left(-\frac{u}{\sqrt{t_0}}\right)\mathrm{d}u. \tag{2.9}$$

The next observation we make here concerns the value of the parameter t_0. It follows from the smooth-fit property (2.7) that $\gamma'(1+) = 0$; a direct differentiation in (2.9) shows that this is equivalent to

$$K = \frac{\exp(-(2t_0)^{-1})}{\sqrt{2\pi}\Phi(-t_0{}^{-1/2})}. \tag{2.10}$$

We will show that such a $t_0 > 0$ exists if and only if $K \in (\sqrt{2/\pi}, \infty)$. This is related to the fact that (1.5) holds with some finite L if and only if K belongs to this interval. We postpone the proofs of both these facts to the next section and continue with the construction of the candidate for U. So, assume that there is t_0 satisfying (2.10). It remains to define it on the set $\mathbb{R}\times[0, t_0)$. This is straightforward: since this set is entirely contained in C, Itô's formula combined with (2.6) implies that U is given by

$$\begin{aligned}U(x,t) &= \mathbb{E}U(x+B_{t_0-t},t_0)\\ &= \mathbb{E}\gamma(x+B_{t_0-t}+1) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\gamma\big(x+y\sqrt{1-t}+1\big)e^{-y^2/2}\mathrm{d}y.\end{aligned} \tag{2.11}$$

Summarizing, we have obtained the following candidate $U_0 : \mathbb{R}\times[0,\infty)\to\mathbb{R}$ for the value function:

$$U_0(x,t) = \begin{cases}\sqrt{t} - K\Psi(|x|) & \text{if } \sqrt{t/t_0} \geq |x|+1,\\ \sqrt{\frac{t}{t_0}}\gamma\left(\sqrt{\frac{t_0}{t}}(|x|+1)\right) + \frac{K(|x|+1)}{2}\log\frac{t_0}{t} - K & \text{if } 1 \leq \sqrt{t/t_0} < |x|+1,\\ \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\gamma\big(x+y\sqrt{t_0-t}+1\big)e^{-y^2/2}\mathrm{d}y & \text{if } 0<t<t_0,\end{cases}$$

where γ is given by (2.9).

3. Verification

Now we will provide the full answer to the questions (I) and (II) formulated in the introductory section, by showing that U_0 is indeed the value function of the optimal stopping problem (2.2). We start with the following auxiliary fact.

Lemma 3.1. *For any $K > \sqrt{2/\pi}$ there is a unique $t_0 > 0$ satisfying (2.10). Furthermore, we have $K\sqrt{t_0} > 1$.*

Proof. Consider the function

$$F(s) = \frac{\exp(-s^2/2)}{\sqrt{2\pi}\Phi(-s)}, \qquad s \geq 0.$$

We have $F(0) = \sqrt{2/\pi}$ and $\lim_{s\to\infty} F(s) = \infty$, so the existence and the uniqueness of t_0 will follow when we show that F is strictly increasing. To do this, it suffices to note that for all $s > 0$,

$$\frac{1}{\sqrt{2\pi}}\exp(-s^2/2) - s\Phi(-s) = \int_s^\infty \Phi(-u)\mathrm{d}u > 0 \tag{3.1}$$

and, in consequence,

$$F'(s) = \frac{\exp(-s^2/2)}{\sqrt{2\pi}\Phi(-s)^2}\left[\frac{1}{\sqrt{2\pi}}\exp(-s^2/2) - s\Phi(-s)\right] > 0.$$

It remains to note that the inequality $K\sqrt{t_0} > 1$ follows directly from (3.1) (divide throughout by $s\Phi(-s)$ and substitute $s = 1/\sqrt{t_0}$). □

The next step in the analysis is the following majorization.

Lemma 3.2. *We have $U_0(x,t) \geq G(x,t)$ for all $x \in \mathbb{R}$ and $t \geq 0$.*

Proof. By symmetry, it suffices to prove the assertion for $x \geq 0$. If $\sqrt{t} \geq \sqrt{t_0}(x+1)$, then we have the equality. If $\sqrt{t_0} \leq \sqrt{t} \leq \sqrt{t_0}(x+1)$, then the majorization can be rewritten in the form

$$F(s) := \frac{\gamma(s)}{s} - \frac{\sqrt{t_0}}{s} + K\log s - K \geq 0,$$

where $s = \sqrt{t_0/t}(x+1) \in [1,\infty)$. We have $F(1) = 0$ and

$$F'(s) = \frac{\sqrt{t_0}}{s^2}\sqrt{2\pi}\exp\left(\frac{s^2}{2t_0}\right)\Phi\left(-\frac{s}{\sqrt{t_0}}\right)\left[\frac{\exp\left(-s^2/(2t_0)\right)}{\sqrt{2\pi}\Phi\left(-s/\sqrt{t_0}\right)} - K\right].$$

Thus, $F'(1+) = 0$ in view of the definition of t_0; furthermore, by the reasoning presented in the proof of Lemma 3.1, the expression in the square brackets is a nondecreasing function of s. This implies $F'(s) \geq 0$ for all $s > 1$ and hence F is nonnegative.

Finally, we will show the majorization for $t < t_0$. In the light of what we have just proved, it suffices to establish the bound

$$U_0(x,t) - G(x,t) \geq U_0(x,t_0) - G(x,t_0),$$

or, equivalently,

$$\sqrt{t_0} - \sqrt{t} \geq U_0(x,t_0) - U_0(x,t).$$

However, by Itô's formula,

$$
\begin{aligned}
U_0(x,t_0)-U_0(x,t) &= \mathbb{E}\left[U_0(x,t_0)-U_0(x+B_{t_0-t},t_0)\right] \\
&= -\frac{1}{2}\int_0^{t_0-t}\mathbb{E}\gamma''(|x+B_s|+1)\mathrm{d}s.
\end{aligned}
\tag{3.2}
$$

We derive that

$$
-\gamma''(s) = K\sqrt{\frac{2\pi}{t_0}}\exp\left(\frac{s^2}{2t_0}\right)\Phi\left(-\frac{s}{\sqrt{t_0}}\right), \tag{3.3}
$$

and the right-hand side is strictly decreasing, in view of the proof of the previous lemma. Consequently, we have

$$
-\gamma''(|x+B_s|+1) \leq K\sqrt{\frac{2\pi}{t_0}}\exp\left(\frac{1}{2t_0}\right)\Phi\left(-\frac{1}{\sqrt{t_0}}\right) = \frac{1}{\sqrt{t_0}},
$$

where the latter equality is just (2.10). Plugging this into (3.2) gives

$$
U_0(x,t_0)-U_0(x,t) \leq \frac{1}{2}\int_0^{t_0-t}\frac{1}{\sqrt{t_0}}\mathrm{d}s = (\sqrt{t_0}-\sqrt{t})\frac{\sqrt{t_0}+\sqrt{t}}{2\sqrt{t_0}} \leq \sqrt{t_0}-\sqrt{t},
$$

and we are done. □

Theorem 3.3. *If $K>\sqrt{2/\pi}$, then the functions U and U_0 coincide.*

We split the reasoning into two parts: first we establish the inequality $U \geq U_0$, and then show the reverse bound.

Proof of the inequality $U \leq U_0$. The idea is to apply Itô's formula (or rather its extension) to the function U_0, and exploit the majorization of Lemma 3.2. Formally, there is a problem with sufficient regularity of U_0 (it is not of class C^2). However, the function is of class C^1 on its domain and of class C^2 outside the curves $\{(x,t): t=t_0\}$ and $\mathcal{C}=\{(x,t):\sqrt{t/t_0}=|x|+1\}$. Thus, by the change-of-variable formula with local time on curves (see Section 3 in [12]), we obtain, for any $s > 0$,

$$\begin{aligned}
&U_0(x + B_{\tau\wedge s}, t + (\tau \wedge s)) \\
&= U_0(x,t) + \int_{0+}^{\tau\wedge s} \left(U_{0t}(x + B_u, t+u) + \frac{1}{2} U_{0xx}(x + B_u, t+u) \right) 1_{\{(x+B_u, t+u)\notin \mathcal{C}\}} \mathrm{d}u \\
&\qquad + \int_{0+}^{\tau\wedge s} U_{0x}(x + B_u, t+u) \mathrm{d}B_u
\end{aligned}$$

almost surely. By the construction, the function U_0 satisfies $U_{0t}(x,t) + \frac{1}{2}U_{0xx}(x,t) = 0$ for $\sqrt{t} < \sqrt{t_0}(|x|+1)$. On the other hand, if $\sqrt{t} > \sqrt{t_0}(|x|+1)$, then

$$U_{0t}(x,t) + \frac{1}{2}U_{0xx}(x,t) = \frac{1}{2\sqrt{t}} - \frac{K}{2(|x|+1)} \leq \frac{1 - K\sqrt{t_0}}{2(|x|+1)\sqrt{t_0}} < 0,$$

where the latter bound follows from the second part of Lemma 3.1. Therefore, by the preceding formula, we get

$$U_0(x + B_{\tau\wedge s}, t + (\tau \wedge s)) \leq U_0(x,t) + \int_{0+}^{\tau\wedge s} U_{0x}(x + B_u, t+u) \, \mathrm{d}B_u$$

with probability 1. The integral on the right defines a local martingale; thus, replacing s with $s \wedge \sigma_n$ (where $(\sigma_n)_{n\geq 1}$ is an appropriate localizing sequence), we may assume that the expectation of the integral is zero. Consequently, we obtain

$$\mathbb{E}U_0(x + B_{\tau\wedge\sigma_n\wedge s}, t + (\tau \wedge \sigma_n \wedge s)) \leq U_0(x,t).$$

However, $\tau \in L^{p/2}$ for some $p > 1$; thus, by Burkholder-Davis-Gundy inequality, we have $\sup_{0\leq s\leq\tau} |B_s| \in L^p$ and therefore, letting $n \to \infty$ and then $s \to \infty$, we get

$$\mathbb{E}U_0(x + B_\tau, t + \tau) \leq U_0(x,t),$$

by Lebesgue's dominated convergence theorem. Since $U_0 \geq G$, taking the supremum over all τ gives the desired bound $U(x,t) \leq U_0(x,t)$ (but only for $t > 0$ - we have assumed this at the beginning). To show this inequality for $t = 0$, we simply note that $U(x,0) \leq U(x,s) \leq U_0(x,s)$ for any $s > 0$, and let $s \downarrow 0$. □

Proof of $U \geq U_0$. This is simple: for any (x,t) we must exhibit a stopping time τ belonging to $L^{p/2}$ for some $p>1$, such that $\mathbb{E}G(x+B_\tau, t+\tau) \geq U_0(x,t)$. If $\sqrt{t} \geq \sqrt{t_0}(|x|+1)$, then we take $\tau \equiv 0$. Suppose then that $\sqrt{t} < \sqrt{t_0}(|x|+1)$ and consider the stopping time

$$\tau = \inf\{s \geq 0 : \sqrt{t+s} = \sqrt{t_0}(|x+B_s|+1)\}.$$

It follows immediately from the results of Davis [3] and Novikov [11] that τ is $p/2$-integrable for some $p>1$. Precisely, $\tau \in L^{p/2}$ if and only if $\sqrt{t_0} < \mu_p^{-1}$, where μ_p is the largest positive zero of parabolic cylinder function of parameter p (cf. [1]). However, the process $(x+B_{\tau\wedge s}, t+(\tau\wedge s))_{s\geq 0}$ moves along the continuation set (in which U_0 is of class C^2 and satisfies the heat equation) and terminates at the common boundary of C and D (on which U_0 and G coincide). This, by Itô's formula, implies

$$\mathbb{E}G(x+B_\tau, t+\tau) = \mathbb{E}U_0(x+B_\tau, t+\tau) = U_0(x,\tau).$$

The proof is complete. $\square$

We are ready to establish the main result of this paper.

Theorem 3.4. *(i) If $K > \sqrt{2/\pi}$, then the best constant in* (1.5) *equals*

$$L(K) = \frac{2K\sqrt{t_0}}{\sqrt{2\pi}} - 2K\sqrt{2\pi}\int_0^\infty \exp\left(\frac{1}{2}\left(x+\frac{1}{\sqrt{t_0}}\right)^2\right)\Phi\left(-x-\frac{1}{\sqrt{t_0}}\right)\Phi(-x)\,dx.$$

(ii) If $K \leq \sqrt{2/\pi}$, then the inequality (1.5) *does not hold in general with any finite constant $L(K)$.*

Proof. (i) From the preceding considerations, we have that $L(K) = U(0,0)$. Plugging $x=t=0$ into (3.2) and using the formula (3.3) for γ'', we obtain that

$$U(0,0) = U(0,t_0) - \frac{K}{2}\frac{\sqrt{2\pi}}{\sqrt{t_0}}\int_0^{t_0} \mathbb{E}\exp\left(\frac{(|B_s|+1)^2}{2t_0}\right)\Phi\left(-\frac{|x+B_s|+1}{\sqrt{t_0}}\right)\mathrm{d}s.$$

Computing the expectation and integrating by parts, we obtain

$$U(0,0) = \sqrt{t_0} + J_1 + J_2 + J_3, \tag{3.4}$$

where

$$\begin{aligned}
J_1 &= -2K \int_0^\infty \exp\left(\frac{2x+1}{2t_0}\right) \Phi\left(-\frac{x+1}{\sqrt{t_0}}\right) \mathrm{d}x, \\
J_2 &= \frac{2K\sqrt{2\pi}}{\sqrt{t_0}} \int_0^\infty (x+1) \exp\left(\frac{(x+1)^2}{2t_0}\right) \Phi\left(-\frac{x+1}{\sqrt{t_0}}\right) \Phi\left(-\frac{x}{\sqrt{t_0}}\right) \mathrm{d}x, \\
J_3 &= -\frac{2K\sqrt{2\pi}}{\sqrt{t_0}} \int_0^\infty \exp\left(\frac{(x+1)^2}{2t_0}\right) \Phi\left(-\frac{x+1}{\sqrt{t_0}}\right) \Phi\left(-\frac{x}{\sqrt{t_0}}\right) \mathrm{d}x.
\end{aligned}$$

Integrating by parts again, we get that

$$J_2 = -\sqrt{t_0} - J_1 + 2K \int_0^\infty \Phi\left(-\frac{x}{\sqrt{t_0}}\right) \mathrm{d}x = -\sqrt{t_0} - J_1 + \frac{2K\sqrt{t_0}}{\sqrt{2\pi}}.$$

Plugging this into (3.4), and substituting $x := x\sqrt{t_0}$ in the integral in J_3, we obtain

$$U(0,0) = \frac{2K\sqrt{t_0}}{\sqrt{2\pi}} - 2K\sqrt{2\pi} \int_0^\infty \exp\left(\frac{1}{2}\left(x + \frac{1}{\sqrt{t_0}}\right)^2\right) \Phi\left(-x - \frac{1}{\sqrt{t_0}}\right) \Phi(-x) \mathrm{d}x.$$

This is precisely the claim.

(ii) If the inequality (1.5) were true with some $K \leq \sqrt{2/\pi}$ and $L' < \infty$, then it would automatically hold for an arbitrary $K > \sqrt{2/\pi}$ and this particular L'. However, when $K \downarrow \sqrt{2/\pi}$, then the parameter t_0 coming from (2.10) converges to infinity and $L(K)$ explodes. Thus we have $L(K) > L'$ for K sufficiently close to $\sqrt{2/\pi}$, a contradiction. The proof is complete. □

Acknowledgement

The author would like to thank an anonymous referee for the careful reading of the paper and several helpful suggestions which greatly improved the presentation. The research was supported by NCN (Narodowe Centrum Nauki, Poland), grant DEC-2012/05/B/ST1/00412.

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