SHARP LOGARITHMIC INEQUALITIES FOR RIESZ TRANSFORMS

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Abstract. Let d be a given positive integer and let $\{R_j\}_{j=1}^d$ denote the collection of Riesz transforms on $\mathbb{R}^d$. For any $K > 2/\pi$ we determine the optimal constant L such that the following holds. For any locally integrable Borel function f on $\mathbb{R}^d$, any Borel subset A of $\mathbb{R}^d$ and any $j = 1, 2, \ldots, d$ we have

$$\int_A |R_j f(x)|\,\mathrm{d}x \leq K \int_{\mathbb{R}^d} \Psi(|f(x)|)\,\mathrm{d}x + |A| \cdot L.$$

Here $\Psi(t) = (t+1)\log(t+1) - t$ for $t \geq 0$. The proof is based on probabilistic techniques and the existence of certain special harmonic functions. As a by-product, we obtain related sharp estimates for the so-called re-expansion operator, an important object in some problems of mathematical physics.

1. Introduction

One of the most basic examples of Calderón-Zygmund singular integrals in $\mathbb{R}^d$ is the collection of Riesz transforms [20]:

$$R_j f(x) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{(d+1)/2}} \int_{\mathbb{R}^d} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)\mathrm{d}y, \qquad j = 1, 2, \ldots, d,$$

where the integrals are supposed to exist in the sense of Cauchy principal values. In the particular case $d = 1$, the family consists of only one element, the Hilbert transform $\mathcal{H}$ on $\mathbb{R}$. Alternatively, R_j can be defined as the Fourier multiplier with the symbol $-i\xi_j/|\xi|$, $\xi \in \mathbb{R}^d \setminus \{0\}$; that is, we have the following relation between the Fourier transforms of f and $R_j f$:

$$\widehat{R_j f}(\xi) = -i\frac{\xi_j}{|\xi|}\hat{f}(\xi), \qquad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}. \tag{1.1}$$

It has been long of interest to study various norms of these operators. The classical result of M. Riesz [19] states that $\mathcal{H}$ is a bounded operator on $L^p(\mathbb{R})$ if and only if $1 < p < \infty$. Gokhberg and Krupnik [7] derived the precise value of the norm $||\mathcal{H}||_{L^p(\mathbb{R})\to L^p(\mathbb{R})}$ for $p = 2^k$, $k = 1, 2, \ldots$, and Pichorides [18] determined the norms for the remaining p: we have

$$||\mathcal{H}||_{L^p(\mathbb{R})\to L^p(\mathbb{R})} = C_p := \begin{cases} \tan\left(\frac{\pi}{2p}\right) & \text{if } 1 < p \leq 2, \\ \cot\left(\frac{\pi}{2p}\right) & \text{if } p \geq 2. \end{cases} \tag{1.2}$$

Using the so-called method of rotations, Iwaniec and Martin [14] extended this result to the d-dimensional setting: they proved that for $1 < p < \infty$ and any

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function $f \in L^p(\mathbb{R}^d)$,

$$||R_j f||_{L^p(\mathbb{R}^d)} \leq C_p ||f||_{L^p(\mathbb{R}^d)}, \qquad j = 1, 2, \ldots, d, \tag{1.3}$$

and the constant C_p cannot be decreased. In other words, they showed that the norms $||R_j||_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)}$ and $||\mathcal{H}||_{L^p(\mathbb{R})\to L^p(\mathbb{R})}$ coincide. An alternative, probabilistic proof of the estimate (1.3), based on a sharp estimate for orthogonal martingales, was given by Bañuelos and Wang in [1].

Our motivation comes from the question about the limit case $p = 1$. Riesz transforms are not bounded on L^1, but there are several important substitutes for (1.3). Kolmogorov [16] proved the weak-type $(1, 1)$ estimate

$$|\{x \in \mathbb{R} : |\mathcal{H}f(x)| \geq 1\}| \leq c_1 ||f||_{L^1(\mathbb{R})}$$

for some universal constant $c_1 < \infty$. The optimal value of c_1 was found by Davis [5] to be equal to

$$\frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots}{1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots} \simeq 1.34\ldots.$$

This result was further extended by Janakiraman [15], who established the weak-type (p, p) bound

$$|\{x \in \mathbb{R} : |\mathcal{H}f(x)| \geq 1\}| \leq \left(\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\left|\frac{2}{\pi}\log|t|\right|^p}{t^2+1}\mathrm{d}t\right)^{-1/p} ||f||^p_{L^p(\mathbb{R})}, \qquad 1 \leq p \leq 2,$$

and proved that the constant is the best possible. The question about the sharp version of this result in the range $p > 2$ seems to be open, to the best of the author's knowledge. Another open problem concerns the analogues of the above estimates for Riesz transforms.

The purpose of this paper is to study a certain logarithmic inequality, which can be regarded as another natural extension of (1.3) to the case $p = 1$. Throughout the paper, the Young functions Φ, $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ are given by

$$\Phi(t) = e^t - 1 - t \qquad \text{and} \qquad \Psi(t) = (t + 1)\log(t + 1) - t.$$

For any $K > 2/\pi$, define

$$L(K) = \frac{K}{\pi}\int_{\mathbb{R}} \frac{\Phi\left(\left|\frac{2}{\pi K}\log|t|\right|\right)}{t^2+1}\mathrm{d}t. \tag{1.4}$$

The following statement is one of our main results.

Theorem 1.1. *Let d be a positive integer and let $K > 2/\pi$. Then for any Borel function f on $\mathbb{R}^d$ and any Borel subset A of $\mathbb{R}^d$ we have*

$$\int_A |R_j f(x)|dx \leq K \int_{\mathbb{R}^d} \Psi(|f(x)|)dx + |A| \cdot L(K), \qquad j = 1, 2, \ldots, d. \tag{1.5}$$

For each K the constant $L(K)$ is the best possible.

It is easy to see that $L(K)$ tends to infinity as K decreases to $2/\pi$ and hence the logarithmic estimate does not hold with any finite $L(K)$ when $K \leq 2/\pi$. A few remarks concerning the form of (1.5) are in order. First, we cannot replace the function Ψ by the more familiar function $x \mapsto |x|\log|x|$ or $x \mapsto |x|\log^+|x|$. Indeed, if (1.5) held after such a modification, we would apply it to a function bounded by 1 and obtain that R_j sends bounded functions to bounded functions, a contradiction. The next observation is that we are not allowed to replace the integral on the right of (1.5) by its local version $\int_A \Psi(|f(x)|)\,\mathrm{d}x$. This can be seen, for example, by taking $A = [0,1]^d$, $f = 1_{[1,2]\times[0,1]^{d-1}}$ and using the fact that $R_j f$ is unbounded on $[0,1]^d$. These two observations explain why we have chosen the form (1.5) for the investigation.

A few words about the proof and the organization of the paper. Our approach will exploit the probabilistic techniques of Bañuelos and Wang [1]: the key role in the proof will be played by a certain appropriate martingale inequality. However, it should be stressed here that the arguments from [1] do not lead directly to the inequality (1.5), due to its local form. To overcome this difficulty, we shall first establish a certain exponential estimate, which can be regarded as a dual version of (1.5). The corresponding exponential inequality for martingales is established in the next section, and in Section 3 we apply the duality arguments to prove (1.5). Then, in Section 4, we deal with the optimality of the constant $L(K)$. In the final part of the paper we apply the results to the study of the so-called re-expansion operator, an object arising in the problems of mathematical physics.

2. A martingale inequality

As mentioned in the Introduction, the results of this paper depend heavily on an appropriate martingale inequality. Let us start with introducing the necessary probabilistic background and notation. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, equipped with $(\mathcal{F}_t)_{t\geq 0}$, a nondecreasing family of sub-σ-fields of $\mathcal{F}$, such that $\mathcal{F}_0$ contains all the events of probability 0. Let X, Y be two adapted real-valued martingales with right-continuous trajectories that have limits from the left. The symbol $[X,Y]$ will stand for the quadratic covariance process of X and Y, see e.g. Dellacherie and Meyer [6] for details. The martingales X, Y are said to be orthogonal if the process $[X,Y]$ is constant with probability 1. Following Bañuelos and Wang [1] and Wang [21], we say that Y is differentially subordinate to X, if the process $([X,X]_t - [Y,Y]_t)_{t\geq 0}$ is nonnegative and nondecreasing as a function of t.

The differential subordination implies many interesting inequalities comparing the sizes of X and Y. The literature on this subject is quite extensive, we refer the interested reader to the survey [4] by Burkholder and the paper of Wang [21]. Here we only mention one result, due to Bañuelos and Wang [21], which will be needed in our further considerations. We use the notation $||X||_p = \sup_{t\geq 0} ||X_t||_p$ for $1 \leq p \leq \infty$.

Theorem 2.1. *Suppose that X, Y are orthogonal martingales such that Y is differentially subordinate to X. Then for any $1 < p < \infty$,*

$$||Y||_p \leq C_p ||X||_p,$$

where C_p is given in (1.2). The constant is the best possible.

The main result of this section is the following.

Theorem 2.2. *Suppose that X, Y are orthogonal martingales such that $||X||_\infty \leq 1$, Y is differentially subordinate to X and $Y_0 \equiv 0$. Then for any $K > 2/\pi$ we have*

$$\sup_{t\geq 0} \mathbb{E}\Phi\left(|Y_t|/K\right) \leq \frac{L(K)||X||_1}{K}. \tag{2.1}$$

The inequality is sharp.

The proof of this statement will be based on the existence of a certain special harmonic function. Let $H=\mathbb{R}\times(0,\infty)$ denote the upper half-space and let $S=(-1,1)\times\mathbb{R}$ stand for the vertical strip in $\mathbb{R}^2$. Fix $K>2/\pi$ and define $\mathcal{U}:H\to\mathbb{R}$ by the Poisson integral

$$\mathcal{U}(\alpha,\beta)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\beta\Phi\left(\left|\frac{2}{\pi K}\log|t|\right|\right)}{(\alpha-t)^2+\beta^2}\mathrm{d}t-L(K)K^{-1}. \tag{2.2}$$

Then $\mathcal{U}$ is a harmonic function on H and, for $t\neq 0$,

$$\lim_{(\alpha,\beta)\to(t,0)}\mathcal{U}(\alpha,\beta)=\Phi\left(\left|\frac{2}{\pi K}\log|t|\right|\right)-L(K)K^{-1}. \tag{2.3}$$

Let $\phi:S\to H$ be a conformal mapping given by $\phi(z)=i\exp(-i\pi z/2)$ and introduce the function U, defined on the closure $\overline{S}$ of S by the formula

$$U(x,y)=\begin{cases}\Phi(|y|/K)-L(K)/K & \text{if } |x|=1,\\ \mathcal{U}(\phi(x,y)) & \text{if } |x|<1.\end{cases}$$

It is not difficult to check that for $(x,y)\in S$ we have

$$U(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos\left(\frac{\pi}{2}x\right)\Phi\left(\left|\frac{2}{\pi K}\log|s|+\frac{y}{K}\right|\right)}{s^2-2s\sin\left(\frac{\pi}{2}x\right)+1}\mathrm{d}s-L(K)K^{-1}. \tag{2.4}$$

As the composition of a harmonic function and a conformal mapping, we see that U is harmonic on S. In addition, in view of (2.3), U is continuous on its domain. Furthermore, it is easy to see that U satisfies the symmetry condition

$$U(x,y)=U(x,-y)=U(-x,y)\qquad \text{for all } (x,y)\in\overline{S}. \tag{2.5}$$

Indeed, this is equivalent to

$$\mathcal{U}(\alpha,\beta)=\mathcal{U}(-\alpha,\beta)=\mathcal{U}\left(\frac{\alpha}{\alpha^2+\beta^2},\frac{\beta}{\alpha^2+\beta^2}\right)\qquad \text{for all } (\alpha,\beta)\in H,$$

which can be verified by the substitutions $t:=-t$ and $t:=1/t$ in (2.2).

We shall need the following further properties of U.

Lemma 2.3. *(i) We have $U_{xx}\leq 0$ on S.*

(ii) We have $U(x,0)\leq 0$ for $x\in[-1,1]$.

(iii) For any $(x,y)\in S$ we have

$$U(x,y)\geq\Phi(|y|/K)-L(K)K^{-1}|x|. \tag{2.6}$$

Proof. (i) Since the function $x\mapsto\Phi(|x|)$ is convex, (2.4) implies that for a fixed $x\in[-1,1]$, the function $U(x,\cdot)$ is also convex. It suffices to use the harmonicity of U on S.

(ii) From (i) and (2.5), we infer that $U(x,0)\leq U(0,0)=0$.

(iii) By (i) and (2.5), it suffices to establish the majorization for $x\in\{0,1\}$. If $x=1$, then both sides of (2.6) are equal. To deal with the case $x=0$, observe that for any $k=2,3,\ldots$, $s\neq 0$ and $y\in\mathbb{R}$ we have

$$\left|\frac{2}{\pi}\log|s|+y\right|^k+\left|\frac{2}{\pi}\log|s|-y\right|^k\geq 2\left|\frac{2}{\pi}\log|s|\right|^k+2|y|^k.$$

Dividing throughout by $k!\cdot K^k$ and summing all the obtained estimates yields

$$\Phi\left(\left|\frac{2}{\pi K}\log|s|+\frac{y}{K}\right|\right)+\Phi\left(\left|\frac{2}{\pi K}\log|s|-\frac{y}{K}\right|\right)\geq 2\Phi\left(\left|\frac{2}{\pi K}\log|s|\right|\right)+2\Phi\left(\frac{|y|}{K}\right).$$

Multiply both sides by $(\pi(s^2+1))^{-1}$ and integrate over $\mathbb{R}$ with respect to the variable s to obtain

$$\left[U(0,y)+\frac{L(K)}{K}\right]+\left[U(0,-y)+\frac{L(K)}{K}\right]\geq 2\left[U(0,0)+\frac{L(K)}{K}\right]+2\Phi\left(\frac{y}{K}\right).$$

Combining this with (2.5) and the equality $U(0,0)=0$, we get the desired majorization on the y-axis. □

We shall require the following technical fact, which follows immediately from Corollary 1 of Bañuelos and Wang [2].

Lemma 2.4. *Suppose that X, Y are real-valued orthogonal martingales such that Y is differentially subordinate to X. Then Y has continuous paths and is orthogonal and differentially subordinate to X^c, the continuous part of X.*

We are ready to establish the exponential estimate.

Proof of (2.1). Fox $t\geq 0$. Since U is of class C^2, we may apply Itô's formula to obtain

$$U(X_t,Y_t)=U(X_0,Y_0)+I_1+I_2/2+I_3, \tag{2.7}$$

where

$$\begin{aligned}
I_1&=\int_{0+}^t U_x(X_{s-},Y_s)\,\mathrm{d}X_s+\int_{0+}^t U_y(X_{s-},Y_s)\,\mathrm{d}Y_s,\\
I_2&=\int_{0+}^t U_{xx}(X_{s-},Y_s)\,\mathrm{d}[X^c,X^c]_s\\
&\quad+2\int_{0+}^t U_{xy}(X_{s-},Y_s)\,\mathrm{d}[X^c,Y]_s+\int_{0+}^t U_{yy}(X_{s-},Y_s)\,\mathrm{d}[Y,Y]_s,\\
I_3&=\sum_{0<s\leq t}\big[U(X_s,Y_s)-U(X_{s-},Y_s)-U_x(X_{s-},Y_s)\Delta X_s\big].
\end{aligned}$$

Here $\Delta X_s=X_s-X_{s-}$ denotes the jump of X at time s. Observe that $U(X_0,Y_0)=U(X_0,0)\leq 0$, because of the assumption $Y_0\equiv 0$ and the part (ii) of Lemma 2.3. Next, we have $\mathbb{E}I_1=0$, by the properties of stochastic integrals. Using the orthogonality of X^c and Y, we see that the middle term in I_2 vanishes. Combining this with Lemma 2.3 (i) and the differential subordination of Y to X^c, we obtain

$$I_2\leq\int_{0+}^t U_{xx}(X_{s-},Y_s)\mathrm{d}[Y,Y]_s+\int_{0+}^t U_{yy}(X_{s-},Y_s)\mathrm{d}[Y,Y]_s=0,$$

because U is harmonic. Finally, I_3 is also nonpositive, because of Lemma 2.3 (i). Plugging all these facts into (2.7) and integrating both sides gives $\mathbb{E}U(X_t,Y_t)\leq 0$ and hence, by (2.6),

$$\mathbb{E}\Phi(|Y_t|/K)\leq L(K)K^{-1}\mathbb{E}|X_t|\leq L(K)K^{-1}||X||_1.$$

It remains to take supremum over $t\geq 0$ to complete the proof. □

Sharpness. This will follow once we have established the optimality of $L(K)$ in (1.5); see Remark 4.1 below. □

3. Inequalities for Riesz transforms in $\mathbb{R}^d$

There is a well-known representation of Riesz transforms in terms of the so-called background radiation process, introduced by Gundy and Varopoulos in [10]. Let us briefly describe this connection. Throughout this section, d is a fixed positive integer. Suppose that X is a Brownian motion in $\mathbb{R}^d$ and let Y be an independent Brownian motion in $\mathbb{R}$ (both processes start from the appropriate origins). For any $y > 0$, introduce the stopping time $\tau(y) = \inf\{t \geq 0 : Y_t \in \{-y\}\}$. If f belongs to $\mathcal{S}(\mathbb{R}^d)$, the class of rapidly decreasing functions on $\mathbb{R}^d$, let $U_f : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ stand for the Poisson extension of f to the upper half-space. That is,

$$U_f(x,y) := \mathbb{E} f\left(x + X_{\tau(y)}\right).$$

For any $(d+1)\times(d+1)$ matrix A we define the martingale transform $A{*}f$ by

$$A{*}f(x,y) = \int_{0+}^{\tau(y)} A\nabla U_f(x+X_s, y+Y_s)\cdot \mathrm{d}(X_s,Y_s).$$

Note that $A * f(x,y)$ is a random variable for each x, y. Now, for any $f \in C_0^\infty$, any $y > 0$ and any matrix A as above, define $\mathcal{T}_A^y f : \mathbb{R}^d \to \mathbb{R}$ through the bilinear form

$$\int_{\mathbb{R}^d} \mathcal{T}_A^y f(x) g(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathbb{E}\big[A{*}f(x,y) g(x + X_{\tau(y)})\big]\mathrm{d}x, \tag{3.1}$$

where g runs over $C_0^\infty(\mathbb{R}^d)$. Less formally, $\mathcal{T}^y f$ is given as the following conditional expectation with respect to the measure $\tilde{\mathbb{P}} = \mathbb{P}\otimes \mathrm{d}x$ ($\mathrm{d}x$ denotes Lebesgue's measure on $\mathbb{R}^d$): for any $z \in \mathbb{R}^d$,

$$\mathcal{T}_A^y f(z) = \tilde{E}\left[A{*}f(x,y) | x + X_{\tau(y)} = z\right].$$

See Gundy and Varopoulos [10] for the rigorous statement of this equality. The interplay between the operators $\mathcal{T}_A^y$ and Riesz transforms is explained in the following theorem, consult [10] or Gundy and Silverstein [9].

Theorem 3.1. *Let $A^j = [a^j_{\ell m}]$, $j = 1, 2, \ldots, d$ be the $(d+1)\times(d+1)$ matrices given by*

$$a^j_{\ell m} = \begin{cases} 1 & \text{if } \ell = d+1,\ m = j, \\ -1 & \text{if } \ell = j,\ m = d+1, \\ 0 & \text{otherwise.}\end{cases}$$

Then $\mathcal{T}^y_{A^j} f \to R_j f$ almost everywhere as $y \to \infty$.

We shall require the following auxiliary fact.

Lemma 3.2. *Let $f \in C_0^\infty(\mathbb{R}^d)$ and $A = A^j$ for some j. Then (3.1) holds for all $g \in L^q(\mathbb{R}^d)$, $1 < q < \infty$.*

Proof. Fix $x \in \mathbb{R}$ and $y > 0$. Consider the pair $\xi = (\xi_t)_{t\geq 0}$, $\zeta = (\zeta_t)_{t \geq 0}$ of martingales given by

$$\begin{aligned}\xi_t &= U_f(x + X_{\tau(y)\wedge t}, y + Y_{\tau(y)\wedge t}) \\ &= U_f(x,y) + \int_{0+}^{\tau(y)\wedge t} \nabla U_f(x+X_s, y+Y_s)\cdot \mathrm{d}(X_s,Y_s)\end{aligned}$$

and

$$\zeta_t = \int_{0+}^{\tau(y)\wedge t} A^j \nabla U_f(x+X_s, y+Y_s)\cdot \mathrm{d}(X_s,Y_s),$$

for $t \geq 0$. Then the martingale ζ is differentially subordinate to ξ, since

$$[\xi,\xi]_t - [\zeta,\zeta]_t = |U_f(x,y)|^2 + \sum_{k\notin\{j,d+1\}} \int_{0+}^{\tau(y)\wedge t} \left|\frac{\partial U_f}{\partial x_k}(x+X_s, y+Y_s)\right|^2 \mathrm{d}s$$

is nonnegative and nondecreasing as a function of t. Furthermore, ξ and ζ are orthogonal, which is a direct consequence of the equality $\langle A^j x, x\rangle = 0$, valid for all $x \in \mathbb{R}^d$. Indeed,

$$[\xi,\zeta]_t = \int_{0+}^{\tau(y)\wedge t} \langle A^j \nabla U_f(x+X_s, y+Y_s), \nabla U_f(x+X_s, y+Y_s)\rangle \mathrm{d}s = 0.$$

Therefore, by Theorem 2.1,

$$||\zeta_{\tau(y)}||_p^p = ||\zeta||_p^p \leq C_p^p ||\xi||_p^p = C_p^p ||\xi_{\tau(y)}||_p^p, \qquad 1 < p < \infty.$$

Integrating both sides with respect to $x \in \mathbb{R}^d$ gives

$$\int_{\mathbb{R}^d} \mathbb{E}|A{*}f(x,y)|^p \mathrm{d}x \leq C_p^p \int_{\mathbb{R}^d} \mathbb{E}|f(x+X_{\tau(y)})|^p \mathrm{d}x = C_p^p ||f||_{L^p(\mathbb{R}^d)}^p,$$

by virtue of Fubini's theorem. In addition, for any $g \in C_0^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \mathbb{E}|g(x+X_{\tau(y)})|^q \mathrm{d}x = ||g||_{L^q(\mathbb{R}^d)}^q.$$

Combining these estimates with (3.1) and Hölder's inequality yields

$$\begin{aligned} \left|\int_{\mathbb{R}^d} \mathcal{T}_A^y f(x) g(x)\, \mathrm{d}x\right| &= \left|\int_{\mathbb{R}^d} \mathbb{E}\big[A{*}f(x,y) g(x+X_{\tau(y)})\big] \mathrm{d}x\right| \\ &\leq C_p ||f||_{L^p(\mathbb{R}^d)} ||g||_{L^q(\mathbb{R}^d)}, \end{aligned} \tag{3.2}$$

which is the claim, since $C_0^\infty(\mathbb{R}^d)$ is dense in $L^q(\mathbb{R}^d)$. □

We turn to the dual version of Theorem 1.1.

Theorem 3.3. *For any Borel function f on $\mathbb{R}^d$ bounded in absolute value by 1 we have the sharp estimate*

$$\int_{\mathbb{R}^d} \Phi\left(|R_j f(x)|/K\right)\, dx \leq \frac{L(K)||f||_{L^1(\mathbb{R}^d)}}{K}, \qquad j = 1, 2, \ldots, d. \tag{3.3}$$

Proof. Fix $j \in \{1, 2, \ldots, d\}$, $x \in \mathbb{R}$ and $y > 0$. By a standard density argument, it suffices to establish the estimate (3.3) for $f \in C_0^\infty(\mathbb{R}^d)$. Consider the martingales ξ and ζ introduced in the proof of the previous lemma. These processes are orthogonal, ζ is differentially subordinate to ξ and $\zeta_0 \equiv 0$, so by (2.1), we have

$$\mathbb{E}\Phi(|\zeta_{\tau(y)}|/K) = \sup_{t\geq 0} \mathbb{E}\Phi(|\zeta_t|/K) \leq L(K)K^{-1}||\xi||_1.$$

Integrating this estimate with respect to $x \in \mathbb{R}^d$ and using Fubini's theorem yields

$$\int_{\mathbb{R}^d} \mathbb{E}\Phi\left(|A{*}f(x,y)|/K\right) \mathrm{d}x \leq L(K)K^{-1}||f||_{L^1(\mathbb{R}^d)}.$$

Pick $q \in (1,\infty)$ and $g \in L^q(\mathbb{R}^d)$. Since Ψ is the Legendre transform of Φ (i.e., Ψ' and Φ' are the inverses of each other), we obtain, by Young's inequality,

$$\begin{aligned}\left|\int_{\mathbb{R}^d} \mathbb{E}\left[\frac{A{*}f(x,y)}{K}\, g(x+X_{\tau(y)})\right] \mathrm{d}x\right| \\ &\leq \int_{\mathbb{R}^d} \mathbb{E}\Phi\left(|A{*}f(x,y)/K|\right) \mathrm{d}x + \int_{\mathbb{R}^d} \mathbb{E}\Psi\big(|g(x+X_{\tau(y)})|\big)\,\mathrm{d}x \\ &\leq L(K)K^{-1}||f||_{L^1(\mathbb{R}^d)} + \int_{\mathbb{R}^d} \Psi(|g(x)|)\,\mathrm{d}x.\end{aligned}$$

Combining this with (3.1) gives

$$\int_{\mathbb{R}^d} \left[\frac{\mathcal{T}^y_{A^j} f(x)}{K} g(x) - \Psi(|g(x)|)\right] \mathrm{d}x \leq L(K)||f||_{L^1(\mathbb{R}^d)}/K.$$

Now fix $M > 0$ and put

$$g(x) = \frac{\mathcal{T}^y_{A^j} f(x)}{|\mathcal{T}^y_{A^j} f(x)|}\left[\exp\left(\min\left\{\frac{|\mathcal{T}^y_{A^j} f(x)|}{K}, M\right\}\right) - 1\right]$$

(if $\mathcal{T}^y_{A^j} f(x) = 0$, set $g(x) = 0$). It is easy to see that $|g| \leq c|\mathcal{T}^y_{A^j} f|$ for some positive $c = c(M,K)$ and hence $g \in L^q(\mathbb{R}^d)$, since the same is true for $T^y_{A^j} f$ (use (3.2) and the fact that $f \in C_0^\infty(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$). We get

$$\begin{aligned}&\int_{\mathbb{R}^d} \Phi\left(\left|\frac{\mathcal{T}^y_{A^j} f(x)}{K}\right|\right) 1_{\{|\mathcal{T}^y_{A^j} f(x)|\leq MK\}} \mathrm{d}x \\ &+ \int_{\mathbb{R}^d} \left[\frac{|\mathcal{T}^y_{A^j} f(x)|(e^M - 1)}{K} - \Psi(e^M-1)\right] 1_{\{|\mathcal{T}^y_{A^j} f(x)|> MK\}} \mathrm{d}x \leq L(K)K^{-1}||f||_{L^1(\mathbb{R}^d)}.\end{aligned}$$

The expressions under both above integrals are nonnegative, so letting $M \to \infty$ yields, by Fatou's lemma,

$$\int_{\mathbb{R}^d} \Phi\left(\left|\frac{\mathcal{T}^y_{A^j} f(x)}{K}\right|\right) \mathrm{d}x \leq L(K)K^{-1}||f||_{L^1(\mathbb{R}^d)}.$$

It suffices to let $y \to \infty$ and apply the assertion of Theorem 3.1 and Fatou's lemma again. The sharpness of the estimate will follow from the optimality of the constant $L(K)$ in (1.5). See Remark 4.1 below. □

Proof of (1.5). Fix f satisfying $\int_{\mathbb{R}^d}\Psi(|f|)<\infty$ and put $g=1_A R_jf/|R_jf|$ ($g=0$ if the denominator is zero). By Parseval's identity and (1.1), we get

$$\begin{aligned}\int_A |R_jf(x)|\mathrm{d}x &= \int_{\mathbb{R}^d} R_jf(x)g(x)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^d} \widehat{R_jf}(x)\,\widehat{g}(x)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^d} \widehat{f}(x)\,\widehat{R_jg}(x)\,\mathrm{d}x \qquad (3.4)\\ &= \int_{\mathbb{R}^d} f(x)\,R_jg(x)\,\mathrm{d}x\\ &\leq K\int_{\mathbb{R}^d}\Psi(|f(x)|)\,\mathrm{d}x + K\int_{\mathbb{R}^d}\Phi(|R_jg(x)|/K)\,\mathrm{d}x\\ &\leq K\int_{\mathbb{R}^d}\Psi(|f(x)|)\,\mathrm{d}x + L(K)||g||_{L^1(\mathbb{R}^d)}.\end{aligned}$$

Here in the fifth line we have exploited Young's inequality and in the latter passage we have used (3.3) and the fact that g takes values in $[-1,1]$. It suffices to note that $||g||_{L^1(\mathbb{R}^d)}\leq |A|$ to complete the proof. □

4. Sharpness

The purpose of this section is to show that for each $K>2/\pi$ and $d\geq 1$, the constant $L(K)$ is the best possible in (1.5). This will be accomplished by providing appropriate examples of f and A. For the sake of convenience, we consider the cases $d=1$ and $d>1$ separately.

4.1. **Sharpness, the case $d=1$.** Let D denote the open unit disc of $\mathbb{C}$ and let $G: D\cap H\to H$ be defined by $G(z)=-(1-z)^2/(4z)$ (recall that H stands for the upper half-plane). It is not difficult to verify that G is conformal and hence so is its inverse L. Let us extend L to the continuous function on $\overline{H}=\{z\in\mathbb{C}:\mathrm{Im}z\geq 0\}$. Consider another conformal map $F:\overline{D}\to\overline{S}$ (recall that S is the strip $\{z\in\mathbb{C}: |\mathrm{Re}z|<1\}$), given by

$$F(z)=\frac{2i}{\pi}\log\left[\frac{iz-1}{z-i}\right]-1.$$

The following properties of L and F will be needed below. First, observe that L maps $[0,1]$ onto $\{e^{i\theta}:0\leq\theta\leq\pi\}$. Specifically, for $x\in[0,1]$ we have

$$L(x)=e^{i\theta},\ \text{where } \theta\in[0,\pi] \text{ is uniquely determined by } x=\sin^2(\theta/2). \tag{4.1}$$

Moreover, L maps $\mathbb{R}\setminus[0,1]$ onto $(-1,1)$; precisely, we have

$$L(x)=\begin{cases}1-2x-2\sqrt{x^2-x} & \text{if } x<0,\\ 1-2x+2\sqrt{x^2-x} & \text{if } x>1.\end{cases} \tag{4.2}$$

Concerning F, we have that

$$F \text{ maps the unit circle onto the boundary of } S \tag{4.3}$$

and

$$F \text{ maps } [-1,1] \text{ onto itself.} \tag{4.4}$$

For any positive integer n, let $V_n:\overline{H}\to\overline{S}$ be given by $V_n(z)=F(L^{2n}(z))$, and define $\varphi_n:\mathbb{R}\to\mathbb{R}$ by the formula $\varphi_n(x)=\mathrm{Re}V_n(x)$. Since V_n is conformal and

$\lim_{z\to\infty} V_n(z) = 0$, we have $\mathcal{H}\varphi_n =$ImV_n. Using (4.1), we compute that for any $K > 2/\pi$ we have

$$\begin{aligned}\int_{\mathbb{R}} \Phi\left(|\mathcal{H}\varphi_n(x)|/K\right)\mathrm{d}x &\geq \int_0^1 \Phi\left(\left|\mathrm{Im}F(L^{2n}(x))\right|/K\right)\mathrm{d}x\\ &= \frac{1}{2}\int_0^\pi \Phi\left(\left|\mathrm{Im}F(e^{2in\theta})\right|/K\right)\sin\theta\mathrm{d}\theta\\ &= \frac{1}{2}\int_0^{2n\pi} \Phi\left(\left|\mathrm{Im}F(e^{i\theta})\right|/K\right)\sin\left(\frac{\theta}{2n}\right)\frac{\mathrm{d}\theta}{2n}\\ &= \frac{1}{2}\int_0^{2\pi} \Phi\left(\left|\mathrm{Im}F(e^{i\theta})\right|/K\right)\sum_{k=0}^{n-1}\sin\left(\frac{k\pi}{n}+\frac{\theta}{2n}\right)\frac{\mathrm{d}\theta}{2n}\\ &= \frac{1}{2}\int_0^{2\pi} \Phi\left(\left|\mathrm{Im}F(e^{i\theta})\right|/K\right)\frac{\cos\left(\frac{\theta-\pi}{n}\right)}{2n\sin\left(\frac{\pi}{2n}\right)}\mathrm{d}\theta\\ &\xrightarrow{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \Phi\left(\left|\mathrm{Im}F(e^{i\theta})\right|/K\right)\mathrm{d}\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi} \Phi\left(\left|\frac{2}{\pi}\log\left(\frac{\sin\theta}{1-\cos\theta}\right)\right|/K\right)\mathrm{d}\theta\\ &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{\Phi\left(\left|\frac{2}{K\pi}\log t\right|\right)}{t^2+1}\mathrm{d}t\\ &= \frac{L(K)}{K}.\end{aligned}$$

We turn to the optimality of $L(K)$ in (1.5). Fix $\varepsilon > 0$, $n \geq 1$ and define the function $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \big(\exp\left(|\mathcal{H}\varphi_n(x)|/K\right) - 1\big)\,\mathrm{sgn}\left(\mathcal{H}\varphi_n(x)\right).$$

Using the above calculation, we derive that

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{H}f_n(x)\varphi_n(x)\mathrm{d}x &= \int_{\mathbb{R}} f_n(x)\mathcal{H}\varphi_n(x)\,\mathrm{d}x\\ &= \int_{\mathbb{R}} \big(\exp\left(|\mathcal{H}\varphi_n(x)|/K\right)-1\big)\,|\mathcal{H}\varphi_n(x)|\,\mathrm{d}x\\ &= K\int_{\mathbb{R}}\Psi\left(|f_n(x)|\right)\mathrm{d}x + K\int_{\mathbb{R}}\Phi\left(|\mathcal{H}\varphi_n(x)|\right)\mathrm{d}x\\ &\geq K\int_{\mathbb{R}}\Psi\left(|f_n(x)|\right)\mathrm{d}x + L(K) - \varepsilon,\end{aligned}$$

provided n is sufficiently large. Next, observe that $|\mathrm{Re}F| \leq 1$, so $|\varphi_n| \leq 1$ and thus for a fixed $\delta > 0$ we have

$$\begin{aligned}\int_{-\delta}^{1+\delta} |\mathcal{H}f_n(x)|\mathrm{d}x &\geq \int_{-\delta}^{1+\delta} \mathcal{H}f_n(x)\varphi_n(x)\mathrm{d}x\\ &= \int_{\mathbb{R}} \mathcal{H}f_n(x)\varphi_n(x)\mathrm{d}x - \int_{\mathbb{R}\setminus[-\delta,1+\delta]} \mathcal{H}f_n(x)\varphi_n(x)\mathrm{d}x\\ &\geq K\int_{\mathbb{R}}\Psi\left(|f_n(x)|\right)\mathrm{d}x + L(K) - \varepsilon - \int_{\mathbb{R}\setminus[-\delta,1+\delta]} \mathcal{H}f_n(x)\varphi_n(x)\mathrm{d}x.\end{aligned}$$

Now we shall prove that the last integral is smaller than ε for sufficiently large n. To do this, pick $\eta \in (1, \pi K/2)$ and note that the function f_n belongs to $L^\eta(\mathbb{R})$; in addition, its norm can be bounded from above by a constant depending only on η and K (and not on n). This can be seen by combining (3.3) with the elementary bound $(e^t-1)^\eta \leq 2\Phi(t\eta) + (e^3-1)^\eta$, valid for $t \geq 0$. Therefore, if $\eta' = \eta/(\eta-1)$ denotes the harmonic conjugate to η, then

$$\int_{\mathbb{R}\setminus[-\delta,1+\delta]} \mathcal{H}f_n(x)\varphi_n(x)\mathrm{d}x \leq ||\mathcal{H}f||_{L^\eta(\mathbb{R})} \left(\int_{\mathbb{R}\setminus[-\delta,1+\delta]} |\varphi_n(x)|^{\eta'}\mathrm{d}x \right)^{1/\eta'}.$$

By (1.2), the first factor on the right can be bounded from above by the constant depending only on η and K. Furthermore, combining (4.2), (4.4) and the equality $F(0) = 0$, we easily check that if $n \to \infty$, then the last integral converges to 0: φ_n decays sufficiently fast outside $[-\delta, 1+\delta]$. Putting all the above things together, we have shown that if we take $A = [-\delta, 1+\delta]$ and pick n large enough, then

$$\int_A |\mathcal{H}f_n(x)|\,\mathrm{d}x > K\int_{\mathbb{R}} \Psi\left(|f_n(x)|\right)\mathrm{d}x + \frac{L(K)-2\varepsilon}{1+2\delta}\cdot|A|.$$

Since ε and δ were arbitrary, the constant $L(K)$ is indeed the best possible in (1.5).

4.2. **Sharpness, the case $d > 1$.** Of course, it suffices to focus on Riesz transform R_1 only. Suppose that for a fixed $K > 2/\pi$ we have

$$\int_A |R_1 f(x)|\mathrm{d}x \leq K\int_{\mathbb{R}^d} \Psi(|f(x)|)\mathrm{d}x + L\cdot|A| \tag{4.5}$$

for all Borel subsets A of $\mathbb{R}^d$ and all Borel functions $f : \mathbb{R}^d \to \mathbb{R}$. For $t > 0$, define the dilation operator δ_t as follows: for any function $g : \mathbb{R}\times\mathbb{R}^{d-1} \to \mathbb{R}$, we let $\delta_t g(\xi,\zeta) = g(\xi, t\zeta)$; for any $A \subset \mathbb{R}\times\mathbb{R}^{d-1}$, let $\delta_t A = \{(\xi, t\zeta) : (\xi,\zeta) \in A\}$. By (4.5), the operator $T_t := \delta_t^{-1}\circ R_1 \circ \delta_t$ satisfies

$$\begin{aligned}\int_A |T_t f(x)|\mathrm{d}x &= t^{d-1}\int_{\delta_t^{-1}A} |R_1\circ\delta_t f(x)|\mathrm{d}x \\ &\leq t^{d-1}\left[K\int_{\mathbb{R}^d}\Psi(|\delta_t f(x)|)\mathrm{d}x + L\cdot|\delta_t^{-1}A|\right] \\ &= K\int_{\mathbb{R}^d}\Psi(|f(x)|)\mathrm{d}x + L\cdot|A|.\end{aligned} \tag{4.6}$$

Now fix $f \in L^2(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}} \Psi(|f|) < \infty$. It is not difficult to check that the Fourier transform $\mathcal{F}$ satisfies the identity $\mathcal{F} = t^{d-1}\delta_t\circ\mathcal{F}\circ\delta_t$ and hence the operator T_t has the property that

$$\widehat{T_t f}(\xi,\zeta) = -i\frac{\xi}{(\xi^2+t^2|\zeta|^2)^{1/2}}\widehat{f}(\xi,\zeta), \qquad (\xi,\zeta)\in\mathbb{R}\times\mathbb{R}^{d-1}.$$

By Lebesgue's dominated convergence theorem, we have

$$\lim_{t\to 0}\widehat{T_t f}(\xi,\zeta) = \widehat{T_0 f}(\xi,\zeta)$$

in $L^2(\mathbb{R}^d)$, where $\widehat{T_0 f}(\xi,\zeta) = -i\,\mathrm{sgn}\,(\xi)\widehat{f}$. Combining this with Plancherel's theorem, we conclude that there is a sequence $(t_n)_{n\geq 1}$ decreasing to 0 such that $T_{t_n}f$ converges to $T_0 f$ almost everywhere. Using Fatou's lemma and (4.6), we obtain

$$\int_A |T_0 f(x)|\mathrm{d}x \leq K\int_{\mathbb{R}^d}\Psi(|f(x)|)\mathrm{d}x + L\cdot|A|. \tag{4.7}$$

Note that T_t are bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$ (in fact, $||T_t||_p = ||R_1||_p$), so is T_0 and thus the above estimate holds true for all $f \in L^p(\mathbb{R}^d)$. Next, fix $\varepsilon > 0$ and $\eta \in (1, \pi K/2)$. By the reasoning from the previous subsection, there is a Borel subset B of $\mathbb{R}$ and $h \in L^\eta(\mathbb{R})$ such that

$$\int_B |\mathcal{H}h(x)|\mathrm{d}x > K \int_{\mathbb{R}} \Psi(|h(x)|)\,\mathrm{d}x + (L(K) - \varepsilon)|B|. \tag{4.8}$$

Define $f : \mathbb{R} \times \mathbb{R}^{d-1} \to \mathbb{R}$ by $f(\xi, \zeta) = h(\xi)1_{[0,1]^{d-1}}(\zeta)$. We have $f \in L^\eta(\mathbb{R}^d)$ and $T_0 f(\xi, \zeta) = \mathcal{H}h(\xi)1_{[0,1]^{d-1}}(\zeta)$, which is due to the identity

$$\widehat{T_0 f}(\xi, \zeta) = -i\mathrm{sgn}\,(\xi)\,\widehat{h}(\xi)\widehat{1_{[0,1]^{d-1}}}(\zeta).$$

Plug this into (4.7) with the choice $A = B \times [0,1]^{d-1}$ to obtain

$$\int_B |\mathcal{H}h(\xi)|\mathrm{d}\xi \leq K \int_{\mathbb{R}} \Psi(|h(\xi)|)\mathrm{d}\xi + L \cdot |B|.$$

This implies $L > L(K)$ by virtue of (4.8) and the fact that $\varepsilon > 0$ was arbitrary. The proof is complete.

Remark 4.1. The optimality of the constant $L(K)$ immediately implies the sharpness of (2.1) and (3.3) for each $K > 2/\pi$. Indeed, if any of these estimates could be sharpened, this would yield an improvement of $L(K)$ in (1.5): see the last passage in (3.4).

5. Logarithmic estimates for the re-expansion operator

Let $\mathcal{F}_c$ and $\mathcal{F}_s$ be the cosine and sine Fourier transforms on $\mathbb{R}_+$, respectively. That is, for $x > 0$ and any Borel function f on $\mathbb{R}_+$,

$$\mathcal{F}_c f(x) = \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}_+} f(t) \cos tx \;\mathrm{d}t, \qquad \mathcal{F}_s f(x) = \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}_+} f(t) \sin tx \;\mathrm{d}t.$$

Both $\mathcal{F}_c$ and $\mathcal{F}_s$ are unitary and self-adjoint operators on $L^2(\mathbb{R}_+)$. We define the re-expansion operator Π on $\mathbb{R}_+$ by the identity $\Pi = \mathcal{F}_s\mathcal{F}_c$. This operator is interesting from the analytical point of view, as the object of spectral analysis and also appears naturally in the scattering theory. To be more specific, let T, T_0 be two self-adjoint operators on a Hilbert space. The wave operators $W_\pm = W_\pm(T, T_0)$ are defined by

$$W_\pm(T, T_0) = \lim_{t \to \pm\infty} e^{itT} e^{-itT_0}$$

(the limit is understood in the sense of strong operator convergence). One expands a given function with respect to the eigenfunctions of T_0 and then takes the inverse transform using the eigenfunctions of T. If we put T, T_0 to be the operator $-\frac{d^2}{dx^2}$ on $L^2(\mathbb{R}_+)$ with the boundary conditions $f(0) = 0$ and $f'(0) = 0$, respectively, then $W_\pm(T_0, T) = \pm\Pi$ (see Birman [3]). The re-expansion operator appears also in the polar decomposition of $-i\frac{d}{dx}$ on $L^2(\mathbb{R}_+)$ with the domain defined by $f(0) = 0$ (again, see [3]) and arises in other problems of mathematical physics (see [8], [12] and [13]).

The next observation is that Π can be represented as singular integral operator:

$$\Pi f(x) = \frac{1}{\pi}\mathrm{p.v.}\int_{\mathbb{R}_+} \frac{2xf(t)}{x^2 - t^2}\mathrm{d}t, \qquad x > 0.$$

This formula relates Π to $\mathcal{H}_{+}^{\mathbb{R}}$, the Hilbert transform on $\mathbb{R}_+$, and $\mathcal{L}$, the Laplace transform, which are given by

$$\mathcal{H}^{\mathbb{R}_+}f(x) = \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}_+}\frac{f(t)}{x-t}\mathrm{d}t, \qquad \mathcal{L}f(x) = \sqrt{\frac{2}{\pi}}\int_{\mathbb{R}_+} f(t)e^{-tx}\mathrm{d}t, \qquad x>0.$$

The connection is given by the identity

$$\Pi = \mathcal{H}^{\mathbb{R}_+} + \mathcal{H}_1, \tag{5.1}$$

where

$$\mathcal{H}_1 f(x) = \frac{1}{2}\mathcal{L}^2 f(x) = \frac{1}{\pi}\int_{\mathbb{R}_+}\frac{f(t)}{x+t}\mathrm{d}t, \quad x>0.$$

The question about various norms of Π has gathered some interest in the literature. Hollenbeck, Kalton and Verbitsky [11] proved that the re-expansion operator has the same p-th norm as the Hilbert transform: $||\Pi||_{L^p(\mathbb{R}_+)\to L^p(\mathbb{R}_+)} = ||\mathcal{H}||_{L^p(\mathbb{R})\to L^p(\mathbb{R})}$ for $1\le p\le\infty$. Then it was shown by the author in [17] that the weak p-th norms of Π and $\mathcal{H}$ coincide for $1\le p\le 2$: $||\Pi||_{L^p(\mathbb{R}_+)\to L^{p,\infty}(\mathbb{R}_+)} = ||\mathcal{H}||_{L^p(\mathbb{R})\to L^{p,\infty}(\mathbb{R})}$. We shall establish the following further result in this direction, using the special function U invented in Section 2.

Theorem 5.1. *Let $K>2/\pi$ be a fixed constant.*

(i) If $||f||_{L^\infty(\mathbb{R}_+)}\le 1$, then

$$\int_{\mathbb{R}_+}\Phi(|\Pi f(x)|/K)\,dx \le L(K)K^{-1}||f||_{L^1(\mathbb{R}_+)}. \tag{5.2}$$

The constant $L(K)K^{-1}$ is the best possible.

(ii) For any Borel subset A of $\mathbb{R}_+$ and any Borel function $f:\mathbb{R}_+\to\mathbb{R}$ we have

$$\int_A |\Pi f(x)|dx \le K\int_{\mathbb{R}_+}\Psi(|f(x)|)dx + L(K)\cdot|A|. \tag{5.3}$$

The constant $L(K)$ is the best possible.

Since $L(K)$ explodes as K approaches $2/\pi$, we conclude that the above estimates do not hold with any finite constant $L(K)$ when $K\le 2/\pi$.

Proof of (5.2). By standard density arguments, it suffices to show the estimate for $f\in C_0^\infty(\mathbb{R}_+)$. Consider the complex Fourier transform $\mathcal{F}$ on the upper half-plane H, defined by

$$\mathcal{F}f(x,y) = \mathcal{F}f(z) = \sqrt{\frac{2}{\pi}}\int_{\mathbb{R}_+} f(t)e^{izt}\mathrm{d}t.$$

Of course, $\mathcal{F}f$ is an analytic on H and can be extended to a continuous function of $\overline{H}$ by $\mathcal{F}f(x) = f(x)1_{\{x\ge 0\}}$, $x\in\mathbb{R}$. Since $\mathcal{F}_c$ is unitary and self-adjoint, the substitution $g=\mathcal{F}_c f$ transforms (5.2) into

$$\int_{\mathbb{R}_+}\Phi(|\mathcal{F}_s g(x)|/K)\,\mathrm{d}x \le L(K)K^{-1}||\mathcal{F}_c g||_{L^1(\mathbb{R}_+)}. \tag{5.4}$$

The smoothness of f guarantees the bounds

$$|\mathcal{F}g(z)| \le \frac{c}{1+|z|}, \qquad |(\mathcal{F}g)'(z)| \le \frac{c}{1+|z|^2} \tag{5.5}$$

for all $z \in H$ and some absolute constant c depending only on g. Consider the functions u, v on $\overline{H}$ given by

$$u(x,y) = u_g(x,y) = \mathrm{Re}\mathcal{F}g(x,y) = \sqrt{\frac{2}{\pi}}\int_{\mathbb{R}_+} g(t)\cos tx\, e^{-yt}\mathrm{d}t,$$
$$v(x,y) = v_g(x,y) = \mathrm{Im}\mathcal{F}g(x,y) = \sqrt{\frac{2}{\pi}}\int_{\mathbb{R}_+} g(t)\sin tx\, e^{-yt}\mathrm{d}t.$$

These functions are harmonic and satisfy Cauchy-Riemann equations on H. Furthermore, for all x, $y > 0$ we have

$$\begin{aligned} u(x,0) &= \mathcal{F}_c g(x), & v(x,0) &= \mathcal{F}_s g(x),\\ u(0,y) &= \mathcal{L}g(y), & v(0,y) &= 0,\\ u_x(0,y) &= 0, & v_x(0,y) &= -(\mathcal{L}g)'(y). \end{aligned} \tag{5.6}$$

Recall the function U from Section 2 and define $G(x,y) = U(u(x,y), v(x,y))$ for $(x,y) \in \overline{H}$. The definition makes sense, since $|u(x,y)| \leq 1$ for all $(x,y) \in H$ (here we use the assumption $||f||_{L^\infty(\mathbb{R}_+)} \leq 1$). Clearly, G is continuous; furthermore, it is harmonic on H, since u and v satisfy Cauchy-Riemann equations. Thus, applying Green's formula for the region $D_R = \{(x,y) \in \mathbb{R}^2 : x > 0,\, y > 0,\, x^2 + y^2 < R^2\}$, $R > 1$, we obtain

$$0 = \iint_{D_R} y\Delta G(x,y)\mathrm{d}x\mathrm{d}y = \oint_{\partial D_R} G(x,y) - yG_y(x,y)\mathrm{d}x + yG_x(x,y)\mathrm{d}y.$$

Let $R \to \infty$. Then (5.5) implies that the line integral over the arc

$$\{(x,y) : x > 0,\, y > 0,\, x^2 + y^2 = R^2\}$$

converges to zero and we get

$$0 = \int_{\mathbb{R}_+} G(x,0)\mathrm{d}x - \int_{\mathbb{R}_+} yG_x(0,y)\mathrm{d}y,$$

or, equivalently, using (5.6),

$$\int_{\mathbb{R}_+} U\big(\mathcal{F}_c g(x), \mathcal{F}_s g(x)\big)\mathrm{d}x = -\int_{\mathbb{R}_+} yU_y\big(\mathcal{L}g(y), 0\big)(\mathcal{L}g)'(y)\mathrm{d}y.$$

However, $U_y(x,0) = 0$ for all $x \in (-1,1)$, due to (2.5); it remains to apply (2.6) to complete the proof of (5.4). □

Remark 5.2. By the similar reasoning, one can establish an analogous bound for the adjoint operator $\Pi^* = \mathcal{F}_c\mathcal{F}_s$:

$$\int_{\mathbb{R}_+} \Phi(|\Pi^* f(x)|/K) \leq L(K)K^{-1}||f||_{L^1(\mathbb{R}_+)}. \tag{5.7}$$

Indeed, one uses $g = \mathcal{F}_s f$ and $G(x,y) = U(v(x,y), -u(x,y))$ instead of g and G appearing above. The remaining arguments are essentially the same.

Proof of (5.3). Fix a Borel function $f:\mathbb{R}_+\to\mathbb{R}$, a Borel subset A of $\mathbb{R}_+$ and put $g=1_A\Pi f/|\Pi f|$ ($g=0$ when the denominator is zero). Note that g takes values in $[-1,1]$ and $||g||_{L^1(\mathbb{R}_+)}\leq|A|$. Therefore, using Young's inequality and (5.7), we get

$$\begin{aligned}\int_A|\Pi f(x)|\mathrm{d}x&=\int_{\mathbb{R}_+}\Pi f(x)g(x)\mathrm{d}x\\&=\int_{\mathbb{R}_+}f(x)\Pi^*g(x)\mathrm{d}x\\&\leq K\int_{\mathbb{R}_+}\Psi(|f(x)|)\mathrm{d}x+K\int_{\mathbb{R}_+}\Phi(|\Pi^*g(x)|/K)\mathrm{d}x\\&\leq K\int_{\mathbb{R}_+}\Psi(|f(x)|)\mathrm{d}x+L(K)\cdot||g||_{L^1(\mathbb{R}_+)}\\&\leq K\int_{\mathbb{R}_+}\Psi(|f(x)|)\mathrm{d}x+L(K)\cdot|A|.\end{aligned}$$

This finishes the proof. □

Sharpness of (5.2) *and* (5.3). Of course, it suffices to focus on the logarithmic estimate (5.3). We shall exploit (5.1). We have proved in the previous section that for any $K>2/\pi$ and $\varepsilon>0$ there is a bounded Borel subset A of $\mathbb{R}$ and a Borel function $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$\int_A|\mathcal{H}f(x)|\mathrm{d}x>K\int_{\mathbb{R}}\Psi(|f(x)|)\mathrm{d}x+(L(K)-\varepsilon)\cdot|A|.$$

For a fixed $s>-\inf A$ and $x\in\mathbb{R}$, define $f^{(s)}(x)=f(x-s)1_{\{x\geq 0\}}$. If $x>0$, then

$$\mathcal{H}^{\mathbb{R}_+}f^{(s)}(x)=\frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}_+}\frac{f(t-s)}{x-t}\mathrm{d}t=\mathcal{H}f(x-s)-\frac{1}{\pi}\int_{-\infty}^{-s}\frac{f(t)}{x-s-t}\mathrm{d}t.$$

However,

$$\int_{s+A}\left|\int_{-\infty}^{-s}\frac{f(t)}{x-s-t}\mathrm{d}t\right|\mathrm{d}x\leq|A|\int_{-\infty}^{-s}\frac{|f(t)|}{\inf A-t}\mathrm{d}t\to 0$$

as $s\to\infty$. Furthermore,

$$\int_{s+A}|\mathcal{H}_1f^{(s)}(x)|\mathrm{d}x=\int_{s+A}\left|\int_0^\infty\frac{f(t)}{x+s+t}\mathrm{d}t\right|\mathrm{d}x\leq|A|\int_0^\infty\frac{|f(t)|}{s+t}\mathrm{d}t\to 0$$

and

$$\int_{\mathbb{R}_+}\Psi(|f^{(s)}(x)|)\mathrm{d}x=\int_{-s}^\infty\Psi(|f(x)|)\mathrm{d}x\to\int_{\mathbb{R}}\Psi(|f(x)|)\mathrm{d}x$$

as $s\to\infty$. Consequently, for sufficiently large s,

$$\begin{aligned}\int_{s+A}|\Pi f^{(s)}(x)|\mathrm{d}x&\geq\int_A|\mathcal{H}f(x)|\mathrm{d}x-2\varepsilon|A|\\&\geq K\int_{\mathbb{R}}\Psi(|f(x)|)\mathrm{d}x+(L(K)-3\varepsilon)\cdot|s+A|\\&\geq K\int_{\mathbb{R}_+}\Psi(|f^{(s)}(x)|)\mathrm{d}x+(L(K)-4\varepsilon)\cdot|s+A|.\end{aligned}$$

This proves the desired optimality of the constant $L(K)$. □

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