SHARP LOGARITHMIC INEQUALITIES FOR TWO HARDY-TYPE OPERATORS

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Abstract. For any locally integrable f on $\mathbb{R}^n$, we consider the operators S and T which average f over balls of radius $|x|$ and center 0 and x, respectively:

$$Sf(x) = \frac{1}{|B(0,|x|)|}\int_{B(0,|x|)} f(t)\mathrm{d}t, \quad Tf(x) = \frac{1}{|B(x,|x|)|}\int_{B(x,|x|)} f(t)\mathrm{d}t$$

for $x \in \mathbb{R}^n$. The purpose of the paper is to establish sharp localized LlogL estimates for S and T. The proof rests on a corresponding one-weight estimate for martingale maximal function, a result which is of independent interest.

1. Introduction

A classical result of Hardy and Littlewood (cf. [5]) asserts that if f belongs to $L^p(\mathbb{R}_+)$ for some $p > 1$, then

$$\left(\int_0^\infty \left|\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right|^p \mathrm{d}x\right)^{1/p} \leq \frac{p}{p-1}\left(\int_0^\infty |f(t)|^p \mathrm{d}t\right)^{1/p} \tag{1.1}$$

and the constant $p/(p-1)$ is the best possible. By considering two-sided averages of f instead of one-sided, the above statement can be reformulated as follows: if f belongs to $L^p(\mathbb{R})$ for some $p > 1$, then

$$\left(\int_{\mathbb{R}} \left|\frac{1}{2|x|}\int_{-|x|}^{|x|} f(t)\mathrm{d}t\right|^p \mathrm{d}x\right)^{1/p} \leq \frac{p}{p-1}\left(\int_{\mathbb{R}} |f(t)|^p \mathrm{d}t\right)^{1/p}. \tag{1.2}$$

This estimate has a very natural version in $\mathbb{R}^n$. For any locally integrable function f on $\mathbb{R}^n$ and any nonzero $x \in \mathbb{R}^n$, define

$$Sf(x) = ⨍_{B(0,|x|)} f(t)\mathrm{d}t = \frac{1}{|B(0,|x|)|}\int_{B(0,|x|)} f(t)\mathrm{d}t,$$

the average of f over the ball of center 0 and radius $|x|$. As shown by Christ and Grafakos in [1], the operator S is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, and its norm does not depend on the dimension:

$$||S||_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \frac{p}{p-1}. \tag{1.3}$$

The inequalities (1.1) and (1.2) can be also rewritten in the following, different form

$$\left(\int_{\mathbb{R}} \left|\frac{1}{2|x|}\int_{x-|x|}^{x+|x|} f(t)\mathrm{d}t\right|^p \mathrm{d}x\right)^{1/p} \leq \frac{p}{2^{1/p}(p-1)}\left(\int_{\mathbb{R}} |f(t)|^p \mathrm{d}t\right)^{1/p}. \tag{1.4}$$

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This reformulation suggests another extension to the higher-dimensional setting. For any locally integrable function f on $\mathbb{R}^n$ and any nonzero $x\in\mathbb{R}^n$, define

$$Tf(x)=⨍_{B(x,|x|)} f(t)\mathrm{d}t=\frac{1}{|B(x,|x|)|}\int_{B(x,|x|)} f(t)\mathrm{d}t,$$

the average of f over the ball of center x and radius $|x|$. The action of T on $L^p(\mathbb{R}^n)$ was also studied in the aforementioned paper of Christ and Grafakos. It turns out that, as previously, T is bounded on $L^p(\mathbb{R}^n)$ if and only if $1<p<\infty$, but the norm does depend on the dimension. Precisely, we have

$$||T||_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}=p'\frac{\omega_{n-2}}{\omega_{n-1}}2^{n/p'-1}\beta\left(\frac{1}{2}\left(\frac{n}{p'}+1\right),\frac{n-1}{2}\right), \tag{1.5}$$

where $p'=p/(p-1)$, ω_{n-1} denotes the area of the unit sphere S^{n-1} in $\mathbb{R}^n$ and $\beta(s,t)=\int_0^1 x^{t-1}(1-x)^{s-1}\mathrm{d}x$ stands for the usual beta function (Christ and Grafakos use a slightly different formula for β).

The purpose of the present note is to study the localized LlogL estimates for the operators S and T, which can be regarded as analogues of (1.3) and (1.5) for $p=1$. Namely, for any positive integer n and any $K>0$, we will find the best constants $L^S(K,n)$ and $L^T(K,n)$ such that

$$⨍_{B(0,R)}|Sf(x)|\mathrm{d}x\leq K⨍_{B(0,R)}|f(x)|\log|f(x)|\mathrm{d}x+L^S(K,n) \tag{1.6}$$

and

$$⨍_{B(0,R/2)}|Tf(x)|\mathrm{d}x\leq K⨍_{B(0,R)}|f(x)|\log|f(x)|\mathrm{d}x+L^T(K,n) \tag{1.7}$$

for all $R>0$ and all locally integrable functions f on $\mathbb{R}^n$. Note that in the second inequality, the average of the function $|Tf|$ is taken over a smaller ball $B(0,R/2)$; this correction is necessary, since the analysis of Tf over a ball $B(0,r)$ requires the knowledge of f on the ball $B(0,2r)$ of twice larger radius. It should be pointed out that in the one-dimensional setting, the values of $L^S(K,1)$ and $L^T(K,1)$ can be extracted from the work of Gilat [3] (see also Graversen and Peskir [4]): both constants are infinite when $K\leq 1$, and $L^S(K,1)=L^T(K,1)=K^2e^{-1}/(K-1)$ when $K>1$. In higher dimensions, the behavior of the constants is as follows.

Theorem 1.1. *Let $K>0$ be a given constant and let $n\geq 2$ be an integer. Then*

$$L^S(K,n)=\begin{cases}\infty & \text{if } K\leq 1,\\ \dfrac{K^2e^{-1}}{K-1} & \text{if } K>1\end{cases}$$

and

$$L^T(K,n)=\begin{cases}\infty & \text{if } K\leq 2^{n-1},\\ K\int_0^1\exp\left(\frac{2^n\omega_{n-2}}{K\omega_{n-1}}\int_{s^{1/n}}^1(1-r^2)^{(n-3)/2}\ln\frac{r^n}{s}dr-1\right)ds & \text{if } K>2^{n-1}.\end{cases}$$

Thus, as in the L^p case, the constants $L^S(K,n)$ do not change if we keep K fixed and vary n. The behavior of $L^T(K,n)$ is more interesting. For a fixed K, the constant $L^T(K,n)$ does depend on the dimension; furthermore, if we fix n, we see that the threshold for K, between the infinite and finite values of $L^T(K,n)$, is also a function of n.

The proofs of (1.3) and (1.5), presented in [1], are purely analytic. Our approach to the above LlogL estimates will be probabilistic and will exploit a certain one-weight estimate for the martingale maximal function, which is of independent interest and connections. This estimate is proved in the next section, and the final part of the paper is devoted to the proof of Theorem 1.1.

2. An auxiliary martingale inequality

This is the main probabilistic part of the paper, and we start with the necessary background and notation. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, filtered by $(\mathcal{F}_t)_{t\geq 0}$, a nondecreasing family of sub-σ-algebras of $\mathcal{F}$. Let $X=(X_t)_{t\geq 0}$ be an adapted, uniformly integrable martingale, whose trajectories are right-continuous and have limits from the left. Denote by X_∞ the terminal value (pointwise limit) of the martingale X and let $\mathcal{M}X=\sup_{t\geq 0}|X_t|$ stand for the corresponding maximal function. Furthermore, for any random variable ξ, we will denote by ξ^* the nondecreasing rearrangement of ξ, i.e., a nonincreasing, right-continuous function on $(0,1]$, given by the formula $\xi^*(t)=\inf\{s>0\,:\,\mathbb{P}(|\xi|>s)\leq t\}$.

We start with the following statement.

Lemma 2.1. *For any uniformly integrable martingale X and any $t\in(0,1]$ we have*

$$(\mathcal{M}X)^*(t)\leq \frac{1}{t}\int_0^t X_\infty^*(s)ds.$$

Proof. If X_∞^* is constant on $(0,t]$, say, $X_\infty^*=c$ there, then we have $|X_\infty|\leq c$ almost surely and hence also

$$\mathcal{M}X\leq c=\frac{1}{t}\int_0^t X_\infty^*(s)\mathrm{d}s$$

with probability 1. This gives the desired bound. Suppose then that X_∞^* is not constant on $(0,t]$. By the definition of a nondecreasing rearrangement, the claim is equivalent to saying that

$$\mathbb{P}\left(\mathcal{M}X\geq \frac{1}{t}\int_0^t X_\infty^*(s)\mathrm{d}s\right)\leq t. \tag{2.1}$$

If the left-hand side is zero, the bound is trivial. If the probability is strictly positive, apply Doob's weak-type estimate $\alpha\mathbb{P}(\mathcal{M}X\geq\alpha)\leq\mathbb{E}|X_\infty|1_{\{\mathcal{M}X\geq\alpha\}}$ (cf. [2]) with $\alpha=\frac{1}{t}\int_0^t X_\infty^*(s)\mathrm{d}s$ to obtain

$$\mathbb{P}\left(\mathcal{M}X\geq\frac{1}{t}\int_0^t X_\infty^*(s)\mathrm{d}s\right)\leq t\frac{\mathbb{E}|X_\infty|1_{\{\mathcal{M}X\geq t^{-1}\int_0^t X_\infty^*(s)ds\}}}{\int_0^t X_\infty^*(s)\mathrm{d}s}.$$

By Hardy-Littlewood inequality $\mathbb{E}\xi\eta\leq\int_0^1\xi^*(s)\eta^*(s)\mathrm{d}s$ (cf. [5]), we see that

$$\begin{aligned}\mathbb{E}|X_\infty|1_{\{\mathcal{M}X\geq t^{-1}\int_0^t X_\infty^*(s)ds\}}&\leq\int_0^1 X_\infty^*(r)(1_{\{\mathcal{M}X\geq t^{-1}\int_0^t X_\infty^*(r)dr\}})^*(r)\mathrm{d}r\\&=\int_0^{\mathbb{P}\left(\mathcal{M}X\geq t^{-1}\int_0^t X_\infty^*(s)ds\right)}X_\infty^*(r)\mathrm{d}r,\end{aligned}$$

which combined with the preceding estimate yields

$$\frac{1}{t}\int_0^t X_\infty^*(s)\mathrm{d}s\leq\frac{1}{\mathbb{P}\left(\mathcal{M}X\geq t^{-1}\int_0^t X_\infty^*(s)ds\right)}\int_0^{\mathbb{P}\left(\mathcal{M}X\geq t^{-1}\int_0^t X_\infty^*(s)ds\right)}X_\infty^*(r)\mathrm{d}r.$$

Here we have used the fact that $\mathbb{P}\left(\mathcal{M}X \geq t^{-1}\int_0^t X_\infty^*(s)ds\right) \neq 0$, which we assumed earlier. Since X_∞^* is nonincreasing on $(0,1]$ and not constant on $(0,t]$, (2.1) follows. □

Theorem 2.2. *Suppose that X is a uniformly integrable martingale and w is a weight (a nonnegative, integrable random variable). Then for any $K > 0$ we have*

$$\mathbb{E}(\mathcal{M}X)\,w \leq K\mathbb{E}|X_\infty|\log|X_\infty| + K\int_0^1 \exp\left(K^{-1}\int_s^1 \frac{w^*(t)}{t}dt - 1\right)ds \tag{2.2}$$

and the inequality is sharp.

Here we need to clarify what we mean by sharpness. Fix $K > 0$ and let w be a given weight for which the expression on the right of (2.2) is finite. We will prove that there is a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with some filtration $(\tilde{\mathcal{F}}_t)_{t\geq 0}$, a weight $\tilde{w}$ and a uniformly integrable martingale $\tilde{X} = (\tilde{X}_t)_{t\geq 0}$ such that $\tilde{w}^* = w^*$ and

$$\tilde{\mathbb{E}}(\mathcal{M}\tilde{X})\,\tilde{w} = K\tilde{\mathbb{E}}|\tilde{X}_\infty|\log|\tilde{X}_\infty| + K\int_0^1 \exp\left(K^{-1}\int_s^1 \frac{w^*(t)}{t}\mathrm{d}t - 1\right)\mathrm{d}s.$$

Proof. The starting point is the Hardy-Littlewood inequality $\mathbb{E}\xi\eta \leq \int_0^1 \xi^*(s)\eta^*(s)\mathrm{d}s$, already used above. Combining it with the previous lemma gives

$$\begin{aligned}\mathbb{E}(\mathcal{M}X)\,w &\leq \int_0^1 (\mathcal{M}X)^*(t)w^*(t)\mathrm{d}t \leq \int_0^1 \frac{1}{t}\int_0^t X_\infty^*(s)\mathrm{d}s\; w^*(t)\mathrm{d}t \\ &= \int_0^1 X_\infty^*(s)\int_s^1 \frac{w^*(t)}{t}\mathrm{d}t\,\mathrm{d}s.\end{aligned}$$

However, one easily verifies that for any x, $y \geq 0$ we have an elementary bound

$$xy \leq Kx\log x + Ke^{y/K-1}.$$

Consequently, we get

$$\mathbb{E}(\mathcal{M}X)\,w \leq K\int_0^1 X^*(s)\log X^*(s)\mathrm{d}s + K\int_0^1 \exp\left(K^{-1}\int_s^1 \frac{w^*(t)}{t}\mathrm{d}t - 1\right)\mathrm{d}s,$$

which is the desired estimate. To see that this bound is sharp, fix $K > 0$ and an arbitrary weight w satisfying

$$K\int_0^1 \exp\left(K^{-1}\int_s^1 \frac{w^*(t)}{t}\mathrm{d}t - 1\right)\mathrm{d}s < \infty. \tag{2.3}$$

Consider the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) = ((0,1], \mathcal{B}((0,1]), |\cdot|)$ and equip it with the filtration $(\tilde{\mathcal{F}}_t)_{t\geq 0}$, given as follows: if $t \in [0,1)$, then $\tilde{\mathcal{F}}_t$ is a σ-algebra generated by $(0, 1-t)$ and all Borel subsets of $[1-t, 1]$; if $t \geq 1$, then $\mathcal{F}_t = \mathcal{B}((0,1])$. Then $\tilde{w} = w^*$ can be regarded as a weight on this new probability space. Furthermore, the random variable

$$\tilde{X}_\infty(s) = \exp\left(\frac{1}{K}\int_s^1 \frac{w^*(r)}{r}\mathrm{d}r - 1\right), \qquad s \in (0,1],$$

is integrable, in the light of (2.3), and hence $(\tilde{X}_t)_{t\geq 0} = \left(\tilde{\mathbb{E}}\left[\tilde{X}_\infty|\tilde{\mathcal{F}}_t\right]\right)_{t\geq 0}$ defines a uniformly integrable martingale. Directly from the definition of $\tilde{\mathcal{F}}_t$, we infer that

$\tilde{X}_t = \tilde{X}_\infty$ for $t \geq 1$, while for $t \in [0,1)$,

$$\tilde{X}_t(\omega) = \begin{cases} \frac{1}{1-t}\int_0^{1-t} \exp\left(\frac{1}{K}\int_s^1 \frac{w^*(r)}{r}\mathrm{d}r - 1\right)\mathrm{d}s & \text{if } \omega \in [0, 1-t), \\ \exp\left(\frac{1}{K}\int_\omega^1 \frac{w^*(r)}{r}\mathrm{d}r - 1\right) & \text{if } \omega \in [1-t, 1]. \end{cases}$$

Consequently,

$$\mathcal{M}X(\omega) \geq \lim_{t\uparrow 1-\omega} X_t(\omega) = \frac{1}{\omega}\int_0^\omega \exp\left(\frac{1}{K}\int_s^1 \frac{w^*(r)}{r}\mathrm{d}r - 1\right)\mathrm{d}s$$

and hence

$$\begin{aligned} \tilde{\mathbb{E}}(\mathcal{M}X)\,\tilde{w} &= \int_0^1 \mathcal{M}X(s)\tilde{w}(s)\mathrm{d}s \\ &\geq \int_0^1 \frac{1}{t}\int_0^t \exp\left(\frac{1}{K}\int_s^1 \frac{w^*(r)}{r}\mathrm{d}r - 1\right)\mathrm{d}s\, w^*(t)\mathrm{d}t \\ &= \int_0^1 \int_s^1 \frac{w^*(r)}{r}\mathrm{d}r \exp\left(\frac{1}{K}\int_s^1 \frac{w^*(r)}{r}\mathrm{d}r - 1\right)\mathrm{d}s \\ &= K\tilde{\mathbb{E}}|X_\infty|\log|X_\infty| + K\int_0^1 \exp\left(K^{-1}\int_s^1 \frac{w^*(t)}{t}\mathrm{d}t - 1\right)\mathrm{d}s. \end{aligned}$$

This establishes the sharpness and completes the proof of the theorem. □

3. Proof of Theorem 1.1

3.1. **On the value of $L^S(K,n)$.** We start with the analysis of the operator S, for which the reasoning is simpler. First we will consider the case $K > 1$. Fix $R > 0$ and a positive integer n. It is enough to study the inequality (1.6) for nonnegative f only, since the passage $f \to |f|$ does not affect the right-hand side and does not decrease the left-hand side. Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the open ball $B(0,R)$ in $\mathbb{R}^n$, $\mathcal{F}$ is the σ-algebra of all Borel subsets of $B(0,R)$ and $\mathbb{P}$ is the normalized Lebesgue measure. Introduce the filtration $(\mathcal{F}_t)_{t\geq 0}$, where $\mathcal{F}_t = \mathcal{F}$ for $t \geq R$, and for $t \in [0,R)$, $\mathcal{F}_t$ is the σ-algebra generated by the open ball $B(0,R-t)$ and all Borel subsets of $B(0,R)\setminus B(0,R-t)$. Now, given an integrable function f on $B(0,R)$, we treat it as a terminal variable of the martingale

$$X_t = \mathbb{E}\left[f|\mathcal{F}_t\right], \qquad t \geq 0.$$

By the very definition of $\mathcal{F}_t$, we see that $X_t = f$ for $t \geq R$; in addition, if $t \in [0,R)$, we have

$$X_t(\omega) = \begin{cases} \frac{1}{|B(0,R-t)|}\int_{B(0,R-t)} f(x)\mathrm{d}x & \text{if } |\omega| \in [0, R-t) \\ f(\omega) & \text{if } |\omega| \in [R-t, R]. \end{cases}$$

This implies

$$\mathcal{M}X(\omega) \geq \lim_{t\uparrow R-\omega} X_t(\omega) = Sf(\omega).$$

Now we apply the inequality (2.2) with the weight $w \equiv 1$. As the result, we obtain

$$\fint_{B(0,R)} Sf(x)\mathrm{d}x \leq \mathbb{E}\mathcal{M}X \leq K\mathbb{E}X_\infty \log X_\infty + K\int_0^1 \exp\left(K^{-1}\int_s^1 \frac{\mathrm{d}t}{t} - 1\right)\mathrm{d}s$$
$$= \fint_{B(0,R)} f(x)\log f(x)\mathrm{d}x + \frac{K^2e^{-1}}{K-1},$$

which is (1.6). To see that this bound is sharp, consider the function f on $B(0,R)$ given by $f(x) = e^{-1}(R/|x|)^{n/K}$. We have $B(0,|x|) = n^{-1}\omega_{n-1}|x|^n$ (recall that ω_{n-1} denotes the area of unit sphere S^{n-1} in $\mathbb{R}^n$), so passing to polar coordinates gives

$$Sf(x) = \fint_{B(0,|x|)} f(x)\mathrm{d}x = \frac{n}{\omega_{n-1}|x|^n}\int_0^{|x|} e^{-1}\left(\frac{R}{r}\right)^{n/K}\omega_{n-1}r^{n-1}\mathrm{d}r$$
$$= \frac{Ke^{-1}R^{n/K}}{K-1}|x|^{-n/K}.$$

Repeating this computation, we obtain

$$\fint_{B(0,|x|)} Sf(x)\mathrm{d}x = \frac{K^2e^{-1}}{(K-1)^2}.$$

On the other hand, a similar calculation gives

$$\fint_{B(0,|x|)} f(x)\log f(x)\mathrm{d}x = \frac{Ke^{-1}}{(K-1)^2} - \frac{Ke^{-1}}{K-1},$$

and hence we obtain equality in (1.6). It remains to consider the case $K \leq 1$. Suppose that $K' > 1$ is an arbitrary number. Since $s\log s \geq -e^{-1}$ for all $s \geq 0$, we have

$$\begin{aligned}
\fint_{B(0,R)} Sf(x)\mathrm{d}x &\leq K\fint_{B(0,R)} |f(x)|\log|f(x)|\mathrm{d}x + L^S(K,n)\\
&\leq K\left(\fint_{B(0,R)} |f(x)|\log|f(x)|\mathrm{d}x + e^{-1}\right) - Ke^{-1} + L^S(K,n)\\
&\leq K'\left(\fint_{B(0,R)} |f(x)|\log|f(x)|\mathrm{d}x + e^{-1}\right) - Ke^{-1} + L^S(K,n)\\
&= K'\fint_{B(0,R)} |f(x)|\log|f(x)|\mathrm{d}x + (K'-K)e^{-1} + L^S(K,n).
\end{aligned}$$

This implies $L^S(K',n) \leq (K'-K)e^{-1} + L^S(K,n)$. However, as we have proved above, $L^S(K',n)$ explodes as $K' \downarrow 1$, and hence $L^S(K,n) = \infty$ for $K \leq 1$. This completes the analysis of the constant L^S.

3.2. **On the value of $L^T(K,n)$.** The reasoning for the operator T will be more involved. As previously, it is enough to study (1.7) for nonnegative functions. Pick such an f, assume that $K > 2^{n-1}$ and note that

$$\fint_{B(0,R/2)} Tf(x)\mathrm{d}x = \frac{n^2 2^n}{\omega_{n-1}^2 R^n}\int_{B(0,R/2)} \frac{1}{|x|^n}\int_{B(0,R)} f(y)\chi_{\{|x-y|\leq|x|\}}\mathrm{d}y\mathrm{d}x$$
$$= \frac{n^2 2^n}{\omega_{n-1}^2 R^n}\int_{B(0,R)} f(y)\int_{B(0,R/2)} \frac{\chi_{\{|x-y|\leq|x|\}}}{|x|^n}\mathrm{d}x\mathrm{d}y.$$

By the triangle inequality, $|x-y| \leq |x|$ implies $|x| \geq |y|/2$. Furthermore, passing to the polar coordinates $x = r\theta$, $y = |y|\phi$, we see that $|x-y| \leq |x|$ if and only if $\theta \cdot \phi \geq |y|/(2r)$. Hence we get that

$$\begin{aligned}\int_{B(0,R/2)} \frac{\chi_{\{|x-y|\leq|x|\}}}{|x|^n}\mathrm{d}x &= \int_{|y|/2}^{R/2} |\{\theta \in S^{n-1} : \theta\cdot\phi \geq |y|/(2r)\}| \frac{\mathrm{d}r}{r} \\ &= \int_{|y|}^{R} |\{\theta \in S^{n-1} : \theta\cdot\phi \geq s/R\}| \frac{\mathrm{d}s}{s},\end{aligned}$$

where in the last equality we have used the substitution $r = |y|R/(2s)$. Clearly, the latter integrand does not depend on ϕ. Thus if we define $w : B(0,R) \to \mathbb{R}_+$ by

$$w(u) = |\{\theta \in S^{n-1} : \theta\cdot\phi \geq |u|/R\}|,$$

then w does not depend on y. We continue by

$$\begin{aligned}&\fint_{B(0,R/2)} Tf(x)\mathrm{d}x \\ &= \frac{n^2 2^n}{\omega_{n-1}^2 R^n}\int_{B(0,R)} f(y)\int_{|y|}^{R} |\{\theta \in S^{n-1} : \theta\cdot\phi \geq s/R\}|\frac{\mathrm{d}s}{s}\mathrm{d}y \\ &= \frac{n^2 2^n}{\omega_{n-1}^2 R^n}\int_{B(0,R)} \frac{f(y)}{\omega_{n-1}}\int_{B(0,R)} \frac{w(x)\chi_{\{|x|\geq|y|\}}}{|x|^n}\mathrm{d}x\mathrm{d}y \qquad (3.1)\\ &= \frac{n^2 2^n}{\omega_{n-1}^3 R^n}\int_{B(0,R)} \frac{1}{|x|^n}\int_{B(0,R)} f(y)\mathrm{d}y w(x)\mathrm{d}x \\ &= \fint_{B(0,R)}\fint_{B(0,R)} f(y)\mathrm{d}y\,\frac{2^n w(x)}{\omega_{n-1}}\mathrm{d}x.\end{aligned}$$

Now we use the same probability space and the same martingale as in the analysis of the constant $L^S(K,n)$. An application of (2.2) yields

$$\begin{aligned}\fint_{B(0,R/2)} Tf\mathrm{d}x &\leq \mathbb{E}(\mathcal{M}X)\frac{2^n w}{\omega_{n-1}} \\ &\leq K\int_{B(0,R)} f\log f\mathrm{d}x + K\int_0^1 \exp\left(K^{-1}\int_s^1 \frac{2^n w^*(t)}{\omega_{n-1}t}\mathrm{d}t - 1\right)\mathrm{d}s.\end{aligned}$$

But, directly from the definition, we see that $w(x)$ increases as $|x|$ decreases. Consequently, we have $w^*(t) = w(x)$, where $t = |B(0,|x|)|/|B(0,R)| = (|x|/R)^n$; that is,

$$\begin{aligned}w^*(t) = |\{\theta \in S^{n-1} : \theta\cdot\phi \geq t^{1/n}\}| &= |\{\theta \in S^{n-1} : \theta_1 \geq t^{1/n}\}| \\ &= \omega_{n-2}\int_{t^{1/n}}^1 (1-r^2)^{(n-3)/2}\mathrm{d}r.\end{aligned}$$

Therefore, by Fubini's theorem, we see that for any $s \in [0,1]$,

$$\int_s^1 \frac{w^*(t)}{t}\mathrm{d}t = \omega_{n-2}\int_{s^{1/n}}^1 (1-r^2)^{(n-3)/2}\ln\frac{r^n}{s}\mathrm{d}r$$

and hence the above inequality implies

$$L^T(K,n) \leq K\int_0^1 \exp\left(\frac{2^n\omega_{n-2}}{K\omega_{n-1}}\int_{s^{1/n}}^1 (1-r^2)^{(n-3)/2}\ln\frac{r^n}{s}\mathrm{d}r - 1\right)\mathrm{d}s. \qquad (3.2)$$

Note that the constant on the right is finite if and only if $K > 2^{n-1}$. Indeed,

$$\int_{s^{1/n}}^1 (1-r^2)^{(n-3)/2} \ln \frac{r^n}{s} \mathrm{d}r = \int_{s^{1/n}}^1 (1-r^2)^{(n-3)/2} \ln r^n \mathrm{d}r - \int_{s^{1/n}}^1 (1-r^2)^{(n-3)/2} \mathrm{d}r \, \ln s$$

and the first integral converges to a constant as $s \to 0$. Furthermore,

$$\exp\left(-\frac{2^n \omega_{n-2}}{K\omega_{n-1}} \int_{s^{1/n}}^1 (1-r^2)^{(n-3)/2} \mathrm{d}r \, \ln s\right) \simeq \exp\left(-\frac{2^n \omega_{n-2}}{K\omega_{n-1}} \int_0^1 (1-r^2)^{(n-3)/2} \mathrm{d}r \, \ln s\right) = s^{-2^{n-1}/K},$$

where we have used the convention $A \simeq B$ if $\lim_{s\to 0} A/B = 1$. Thus, the right-hand side of (3.2) explodes as $K \downarrow 2^{n-1}$.

To see that we have equality in (3.2), let $f : B(0,R) \to \mathbb{R}$ be given by

$$f(y) = \exp\left[\frac{n2^n}{K\omega_{n-1}} \int_{|y|}^R |\{\theta \in S^{n-1} : \theta_1 \geq s/R\}| \frac{\mathrm{d}s}{s} - 1\right] = \exp\left[\frac{n2^n}{K\omega_{n-1}^2} \int_{B(0,R)} \frac{w(x)\chi_{\{|x|\geq|y|\}}}{|x|^n} \mathrm{d}x - 1\right].$$

This function is integrable and using the calculations from (3.1), we get

$$⨍_{B(0,R/2)} Tf \mathrm{d}x = \frac{n^2 2^n}{\omega_{n-1}^2 R^n} \int_{B(0,R)} f(y) \, \frac{K\omega_{n-1}}{n2^n} (\log f(y) + 1) \mathrm{d}y = K ⨍_{B(0,R)} f(y) \log f(y) \mathrm{d}y + K ⨍_{B(0,R)} f(y) \mathrm{d}y,$$

and the second term in the latter expression is precisely the right-hand side of (3.2).

It remains to show that $L^T(K,n) = \infty$ for $K \leq 2^{n-1}$. This is done exactly in the same manner as in the analysis of S, and follows directly from the fact that the constant $L^T(K,n)$ explodes as $K \downarrow 2^{n-1}$. We leave the easy details to the reader.

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References

[1] M. Christ and L. Grafakos, *Best constants for two nonconvolution inequalities*, Proc. Amer. Math. Soc. **123** (1995), 1687–1693.
[2] J. L. Doob, *Stochastic processes*, Wiley, New York, 1953.
[3] D. Gilat, *The best bound in the LlogL inequality of Hardy and Littlewood and its martingale counterpart*, Proc. Amer. Math. Soc. **97** (1986), 429–436.
[4] S. E. Graversen and G. Peskir, *Optimal stopping in the LlogL-inequality of Hardy and Littlewood*, Bull. London Math. Soc. **30** (1998), 171–181.
[5] G. Hardy, J. Littlewood, and G. Plya, *Inequalities*, The University Press, Cambridge, 1959.

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