SHARP ESTIMATES FOR LIPSCHITZ CLASS

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Abstract. Let I be an interval contained in $\mathbb{R}$. For a given function $f : I \to \mathbb{R}$, $u \in I$ and any $0 < \alpha \leq 1$, set

$$f^\sharp_\alpha(u) = \sup \frac{1}{|J|^\alpha} \left(\frac{1}{|J|} \int_J \left| f(x) - \frac{1}{|J|} \int_J f(y)\mathrm{d}y \right|^2 \mathrm{d}x \right)^{1/2},$$

where the supremum is taken over all subintervals $J \subseteq I$ which contain u. The paper contains the proofs of the estimates

$$\ell(\alpha)||f^\sharp_\alpha||_{L^\infty(I)} \leq ||f||_{\mathrm{Lip}_\alpha(I)} \leq L(\alpha)||f^\sharp_\alpha||_{L^\infty(I)},$$

where

$$\ell(\alpha) = 2\sqrt{2\alpha+1}, \quad L(\alpha) = \frac{(4\alpha+4)^{(\alpha+1)/(2\alpha+1)}\sqrt{2\alpha+1}}{2\alpha}$$

are the best possible. The proof rests on the evaluation of Bellman functions associated with the above estimates.

1. Introduction

A real-valued locally integrable function f defined on $\mathbb{R}^n$ is said to be in BMO, the space of functions of bounded mean oscillation, if

$$||f||_{BMO^1} = \sup_Q \left\langle |f - \langle f\rangle_Q| \right\rangle_Q < \infty. \tag{1.1}$$

Here the supremum is taken over all cubes Q in $\mathbb{R}^n$ with edges parallel to the coordinate axes, and

$$\langle f\rangle_Q = \frac{1}{|Q|} \int_Q f(x)\mathrm{d}x$$

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denotes the average of f over Q. One can consider a slightly less restrictive setting in which only the cubes Q within a given Q^0 are considered; to stress the dependence on Q^0, one uses the notation $BMO(Q^0)$. The BMO class, introduced by John and Nirenberg in [5], plays an important role in analysis and probability, since many classical operators (maximal, singular integral, etc.) map L^∞ into BMO. Another remarkable result, due to Fefferman [3], asserts that BMO is a dual to the Hardy space H^1. It is well-known that the functions of bounded mean oscillation have very strong integrability properties (see e.g. [5]). In particular, for each $1 < p < \infty$, the p-oscillation

$$||f||_{BMO^p} := \sup_Q \left\langle |f - \langle f\rangle_Q|^p \right\rangle_Q^{1/p}$$

is finite for any $f \in BMO$ and forms an equivalent seminorm on $BMO(\mathbb{R}^n)$. It is often more convenient to work with $||\cdot||_{BMO^2}$ than with $||\cdot||_{BMO^1}$; the L^2-based seminorm admits the identity

$$||f||_{BMO^2} = \sup_Q \left\{ \langle f^2\rangle_Q - \langle f\rangle_Q^2 \right\}^{1/2},$$

which allows certain algebraic manipulations.

In recent years, there has been a considerable interest in obtaining various sharp estimates for the BMO class in the one-dimensional setting. Probably the first result in this direction is that of Slavin and Vasyunin [16], which identifies the optimal constants in the so-called integral form of John-Nirenberg inequality. Specifically, if I is a subinterval of $\mathbb{R}$ and $\varphi : I \to \mathbb{R}$ satisfies $||\varphi||_{BMO^2(I)} < 1$, then

$$\langle e^\varphi\rangle_I \leq \frac{\exp(-||\varphi||_{BMO^2(I)})}{1 - ||\varphi||_{BMO^2(I)}}\, e^{\langle\varphi\rangle_I}.$$

Furthermore, this bound is sharp in the sense that for each $\varepsilon < 1$ there is a function φ satisfying $||\varphi||_{BMO^2(I)} = \varepsilon$ and $\langle e^\varphi\rangle_I = e^{-\varepsilon} e^{\langle\varphi\rangle_I}/(1-\varepsilon)$. In particular, this shows that there is no exponential estimate of the above type when $||\varphi||_{BMO^2(I)} \geq 1$. For related results in this direction, consult the papers [10] by the author, Ivanishvili et. al. [4], Slavin and Vasyunin [17], Vasyunin [18], and Vasyunin and Volberg [21].

In this paper we will be interested in a slightly different problem. Given $\alpha \in [0, n]$, $1 \le p < \infty$ and an integrable function f defined on some cube $Q^0 \subset \mathbb{R}^n$, define the associated α-sharp function

$$f^\sharp_{\alpha,p}(u) = \sup |Q|^{-\alpha/n} \big\langle |f - \langle f\rangle_Q|^p \big\rangle_Q^{1/p},$$

where the supremum is taken over all cubes $Q \subseteq Q^0$ containing u. We see that $f \in BMO(Q^0)$ if and only if $||f^\sharp_{0,p}||_{L^\infty(Q^0)} < \infty$ for some (equivalently, all) $1 \le p < \infty$. Hence, by the above discussion, those f for which the corresponding functions $f^\sharp_{0,p}$ are bounded, have very strong integrability properties. This gives rise to the very natural question about the analogue of this statement for positive α. It is not difficult to see that for such α's, the boundedness of $f^\sharp_{\alpha,1}$ implies the α-Lipschitz property of f (up to a set of measure 0). Indeed, for any cubes Q and $Q' \subset Q$ satisfying $|Q'|/|Q| = 2^{-n}$, we have

$$|\langle f\rangle_{Q'} - \langle f\rangle_Q| \le \frac{1}{|Q'|}\int_{Q'} |f - \langle f\rangle_Q| \le \frac{2^n}{|Q|}\int_Q |f - \langle f\rangle_Q| \le 2^n |Q|^{\alpha/n} ||f^\sharp_\alpha||_{L^\infty(Q^0)},$$

which implies, by Lebesgue's differentiation theorem,

$$|f(x) - \langle f\rangle_Q| \le \frac{2^n|Q|^{\alpha/n}}{1 - 2^{-n\alpha}} ||f^\sharp_\alpha||_{L^\infty(Q^0)}$$

for almost all $x \in Q$. Thus, modifying f on a set of measure 0, we obtain an α-Lipschitz function and

$$||f||_{\mathrm{Lip}_\alpha(Q^0)} = \sup_{a,b\in Q^0} \frac{|f(a) - f(b)|}{|a-b|^\alpha} \le C_{n,\alpha} ||f^\sharp_\alpha||_{L^\infty(Q^0)},$$

where $C_{n,\alpha}$ depends only on the parameters indicated. In particular, this also means that all functions $f^\sharp_{\alpha,p}$, corresponding to different values of p, are equivalent.

The primary goal of this paper is to study sharp bounds for the Lipschitz constants in terms of $||f^\sharp_{\alpha,2}||_{L^\infty}$ in the one-dimensional setting. We have chosen to work with $f^\sharp_{\alpha,2}$ because of the convenient identity

$$f^\sharp_{\alpha,2}(u) = \sup\left\{ |J|^{-\alpha}\Big(\langle f^2\rangle_J - \langle f\rangle_J^2\Big)^{1/2} \,:\, J \ni u\right\}. \tag{1.2}$$

In what follows, we will skip the lower index 2 and write $f^\sharp_\alpha$ instead of $f^\sharp_{\alpha,2}$.

Here is the formulation of our main result.

Theorem 1.1. *Let I be an interval contained in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be a continuous function. Then*

$$\ell(\alpha)||f^\sharp_\alpha||_{L^\infty(I)} \leq ||f||_{\mathrm{Lip}_\alpha(I)} \leq L(\alpha)||f^\sharp_\alpha||_{L^\infty(I)}, \tag{1.3}$$

where

$$\ell(\alpha) = 2\sqrt{2\alpha+1} \qquad \text{and} \qquad L(\alpha) = \frac{(4\alpha+4)^{(\alpha+1)/(2\alpha+1)}\sqrt{2\alpha+1}}{2\alpha}$$

are the best possible.

Note that for $\alpha = 1$, we have $\ell(\alpha) = L(\alpha) = 2\sqrt{3}$ and hence

$$||f||_{\mathrm{Lip}_1(I)} = 2\sqrt{3}||f^\sharp_1||_{L^\infty(I)},$$

which is not difficult to prove directly.

Our approach will rest on a modification of the Bellman function method, a powerful and general tool used widely in probability and harmonic analysis. This approach has its origins in the theory of stochastic optimal control, and its fruitful connection with other areas of mathematics was firstly observed by Burkholder, during the study of certain sharp inequalities for martingale transforms [2] and the unconditional constant of the Haar system [1]; for the continuation of this probabilistic path, consult the monograph [9]. Another line of research, which pushed the method towards applications in harmonic analysis, was initiated by the seminal paper [7] by Nazarov and Treil (inspired by the preprint version of [8]). Since then, the technique has been exploited and extended in numerous settings; see e.g. the works of Kovač [6], Pereyra [11], Petermichl [12], Petermichl and Wittwer [13], Rey and Reznikov [14], Vasyunin and Volberg [19, 20], Wittwer [22], as well as the papers on BMO cited above.

Let us describe the organization of the paper. Theorem 1.1 will be proved in the three sections below. In the next section, we provide a sharp upper bound for $\sup_I f$ in terms of $\langle f\rangle_I$, $\langle f^2\rangle_I$ and $||f^\sharp_\alpha||_{L^\infty(I)}$; precisely, we will identify the associated Bellman function

$$B_\alpha(x,y,t)=\sup\left\{\sup_I f\,:\,\langle f\rangle_I=x,\,\langle f^2\rangle_I=y,||f^\sharp_\alpha||^{1/\alpha}_{L^\infty(I)}|I|=t\right\}.$$

Section 3 is devoted to a dual version of this result. Namely, we will find there the explicit expression for

$$C_\alpha(x,y,t)=\sup\left\{\inf_I f\,:\,\langle f\rangle_I=x,\,\langle f^2\rangle_I=y,||f^\sharp_\alpha||^{1/\alpha}_{L^\infty(I)}|I|=t\right\}.$$

In the final part of the paper, we combine these two objects and provide the proof of Theorem 1.1.

2. Upper bound for $\sup_I f$

We start this section with introducing certain special objects which will be used throughout the paper. Let α be a fixed number belonging to the interval $(0,1]$. The symbol $\mathcal{D}_\alpha$ will stand for the parabolic-type region $\{(x,y,t)\in\mathbb{R}\times[0,\infty)\times[0,\infty):0\leq y-x^2\leq t^{2\alpha}\}$. A crucial property of this set, which will be freely used below, is the following: if $f:I\to\mathbb{R}$ is a function satisfying $||f^\sharp_\alpha||_{L^\infty(I)}\leq 1$, then $(\langle f\rangle_J,\langle f^2\rangle_J,|J|)$ lies in $\mathcal{D}_\alpha$ for any subinterval J of I. This fact is an immediate consequence of (1.2).

Next, we will need a certain special function s on $\mathcal{D}_\alpha$. Let us establish the following technical fact.

Lemma 2.1. *For any $(x,y,t)\in\mathcal{D}_\alpha$ with $y>x^2$, there is a unique number $s=s(x,y,t)\geq 0$ satisfying*

$$\frac{s+1+(2\alpha+1)s}{(s+1)^{2\alpha+2}}=\frac{y-x^2}{t^{2\alpha}}. \tag{2.1}$$

Furthermore, the obtained function $s:\mathcal{D}_\alpha\setminus\{(x,y,t):y>x^2\}\to[0,\infty)$ is continuous and of class C^1 in the interior of $\mathcal{D}_\alpha$.

Proof. Denote the left-hand side of (2.1) by $F(s)$. The function F is strictly decreasing on $[0,\infty)$: indeed, we have

$$F'(s) = -\frac{(2\alpha+2)(2\alpha+1)s}{(s+1)^{2\alpha+3}} < 0$$

when $s > 0$. Thus, the existence and uniqueness of $s(x,y,t)$ follows from the observation that $F(0) = 1 \geq (y-x^2)/t^{2\alpha} > 0 = \lim_{s\to\infty} F(s)$. The second part of the lemma, concerning the regularity of s, is a consequence of standard theorems on implicit functions. □

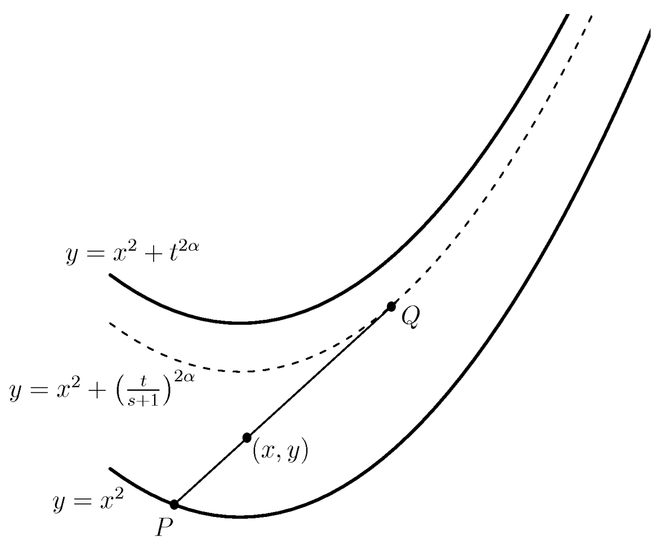

FIGURE 1. The graphical interpretation of the parameter $s = s(x,y,t)$. The point (x,y) splits the tangent line segment PQ in the ratio $1 : s$.

In what follows, we will often write s instead of $s(x,y,t)$; this should not lead to any confusion. Before we proceed, let us describe some geometrical meaning behind the definition of the parameter s. This will be quite helpful in our further considerations, as the Bellman functions B_α, C_α will depend somewhat on this graphical interpretation. Namely, $s = s(x,y,t)$ is the unique number with the following property. There is a point $P = P(x,y,t)$ lying on the parabola $y = x^2$ and a point $Q = Q(x,y,t)$ lying on the parabola $y = x^2 + \left(\frac{t}{s+1}\right)^{2\alpha}$ such that the line segment PQ is tangent to the latter parabola, $(x,y)\in PQ$ and

$$\frac{\mathrm{dist}(P,(x,y))}{\mathrm{dist}(Q,(x,y))}=\frac{1}{s}.$$

See Figure 1. One easily finds the coordinates of the points P and Q: we have

$$P=\left(x-\frac{\sqrt{2\alpha+1}\,t^\alpha}{(s+1)^{\alpha+1}},\left(x-\frac{\sqrt{2\alpha+1}\,t^\alpha}{(s+1)^{\alpha+1}}\right)^2\right) \tag{2.2}$$

and

$$Q=\left(x+\frac{\sqrt{2\alpha+1}st^\alpha}{(s+1)^{\alpha+1}},\left(x+\frac{\sqrt{2\alpha+1}st^\alpha}{(s+1)^{\alpha+1}}\right)^2+\left(\frac{t}{s+1}\right)^{2\alpha}\right). \tag{2.3}$$

A direct differentiation of (2.1) with respect to variables x, y and t yields

$$\begin{aligned}
s_x&=\frac{-2x(s+1)^{2\alpha+2}}{(2\alpha+2)\big(t^{2\alpha}-(y-x^2)(s+1)^{2\alpha+1}\big)}=\frac{2x(s+1)^{2\alpha+3}}{(2\alpha+2)(2\alpha+1)st^{2\alpha}},\\
s_y&=\frac{(s+1)^{2\alpha+2}}{(2\alpha+2)\big(t^{2\alpha}-(y-x^2)(s+1)^{2\alpha+1}\big)}=-\frac{(s+1)^{2\alpha+3}}{(2\alpha+2)(2\alpha+1)st^{2\alpha}},\\
s_t&=\frac{2\alpha\big((2\alpha+2)s+1\big)(s+1)}{(2\alpha+2)(2\alpha+1)st}.
\end{aligned} \tag{2.4}$$

We are ready to introduce the Bellman function, which will be used in the proof of the upper bound for $\sup_I f$. Define $B_\alpha:\mathcal{D}_\alpha\to\mathbb{R}$ by the formula

$$B_\alpha(x,y,t)=\begin{cases} x & \text{if } y=x^2,\\ x+\sqrt{2\alpha+1}\left(\dfrac{t}{s+1}\right)^\alpha\left[\dfrac{\alpha+1}{\alpha}-\dfrac{1}{s+1}\right] & \text{if } y>x^2.\end{cases}$$

By (2.4), we easily compute the formulae for partial derivatives of B_α:

$$\begin{aligned}
B_{\alpha x}(x,y,t)&=1-\frac{\sqrt{2\alpha+1}(\alpha+1)st^\alpha}{(s+1)^{\alpha+2}}s_x=1-\frac{x(s+1)^{\alpha+1}}{\sqrt{2\alpha+1}\,t^\alpha},\\
B_{\alpha y}(x,y,t)&=-\frac{\sqrt{2\alpha+1}(\alpha+1)st^\alpha}{(s+1)^{\alpha+2}}s_y=\frac{(s+1)^{\alpha+1}}{2\sqrt{2\alpha+1}\,t^\alpha},\\
B_{\alpha t}(x,y,t)&=\frac{(\alpha+1)t^{\alpha-1}}{\sqrt{2\alpha+1}(s+1)^\alpha}.
\end{aligned} \tag{2.5}$$

The key property of B_α is described in the statement below. In a sense, it can be regarded as a concavity-type condition (the "main inequality", in the terminology introduced in [7]).

Theorem 2.2. *Suppose that* (x,y,t), $(x_\pm,y_\pm,t_\pm)\in\mathcal{D}_\alpha$ *satisfy*

$$x=ax_++(1-a)x_-,\quad y=ay_++(1-a)y_-,\quad t_+=at,\quad t_-=(1-a)t \tag{2.6}$$

for some $a\in(0,1)$. *Then*

$$B_\alpha(x,y,t)\geq\max\{B_\alpha(x_-,y_-,t_-),B_\alpha(x_+,y_+,t_+)\}. \tag{2.7}$$

Proof. Let us first exclude the trivial case $(x_+,y_+)=(x,y)$. If this equation holds true, then we also have $(x_-,y_-)=(x,y)$ and the assertion follows at once from the estimates $t_\pm\leq t$ and $B_{\alpha t}\geq 0$ (see (2.5)).

So, we may assume that (x_-,y_-) and (x_+,y_+) are different from (x,y). By symmetry, it is enough to establish the bound $B_\alpha(x,y,t)\geq B_\alpha(x_+,y_+,t_+)$. We start with a certain geometric, optimization-type argument. Namely, fix t and the positions of (x,y) and (x_+,y_+); next, start moving the point (x_-,y_-) along the line passing through (x,y) and (x_+,y_+). Clearly, any alteration of the position of (x_-,y_-) changes the values of a, t_+ and t_- (which are still assumed to satisfy (2.6)). Of course, if (x_-,y_-) moves away from (x,y), then the number a increases and hence so do t_+ and $B_\alpha(x_+,y_+,t_+)$ (here we again use the bound $B_{\alpha t}\geq 0$). Consequently, it suffices to show the inequality $B_\alpha(x,y,t)\geq B_\alpha(x_+,y_+,t_+)$ under the assumption that (x_-,y_-) lies in the farthest position from (x,y). On the other hand, from the assumption $(x_-,y_-,t_-)\in\mathcal{D}_\alpha$, we infer that $y_-\geq x_-^2$. Thus, the farthest position of (x_-,y_-) is precisely the intersection point of the halfline starting from (x_+,y_+) and passing through (x,y), with the parabola $y=x^2$. Summarizing, we will be done if we show the inequality $B_\alpha(x,y,t)\geq B_\alpha(x_+,y_+,t_+)$ under the assumption $x_-=u$, $y_-=u^2$ for some $u\in\mathbb{R}$.

To achieve this goal, it is natural to join the points (x,y,t) and (x_+,y_+,t_+) with some curve and then show the monotonicity of B_α along this set. Precisely, consider the function

$$G(\delta) = G_{x,y,t,u}(\delta)$$
$$= B_\alpha\left(x+\delta(x-u), \min\left\{y+\delta(y-u^2), (x+\delta(x-u))^2+\left(\frac{t}{1+\delta}\right)^{2\alpha}\right\}, \frac{t}{1+\delta}\right).$$

Observe that $G(0) = B_\alpha(x,y,t)$ and $G(1/a-1) = B_\alpha(x_+,y_+,t_+)$. The reason why we use the above complicated formula instead of much simpler (and more natural) function

$$\delta \mapsto B_\alpha\left(x+\delta(x-u), y+\delta(y-u^2), \frac{t}{1+\delta}\right)$$

is that the point $\left(x+\delta(x-u), y+\delta(y-u^2), \frac{t}{1+\delta}\right)$ need not lie in $\mathcal{D}_\alpha$; the above correction assures this inclusion.

So, we will be done if we show that G is nonincreasing on $[0, 1/a-1]$ or, equivalently, $G'(\delta+) \leq 0$ for $\delta \in [0, 1/a-1)$. Observe that we have the translation-type property

$$G_{x,y,t,u}(\delta+\eta) = G_{x+\delta(x-u), y+\delta(y-u^2), t/(1+\delta), u}\left(\frac{\eta}{1+\delta}\right)$$

for any δ, $\eta \geq 0$ with $\delta+\eta \leq 1/a-1$. Hence it is enough to prove that $G'(0+) \leq 0$ for all "base" parameters x, y, t and u. Assume first that $y - x^2 < t^{2\alpha}$. Then for small values of δ we have $G(\delta) = B_\alpha\left(x+\delta(x-u), y+\delta(y-u^2), \frac{t}{1+\delta}\right)$ and hence

$$G'(0+) = (x-u)B_{\alpha x}(x,y,t) + (y-u^2)B_{\alpha y}(x,y,t) - tB_{\alpha t}(x,y,t).$$

By (2.5), we have $B_{\alpha y}(x,y,t) > 0$. Consequently, the right-hand side above, considered as a function of u, attains its global maximum at

$$u_{max} = -\frac{B_{\alpha x}(x,y,t)}{2B_{\alpha y}(x,y,t)} = x - \frac{\sqrt{2\alpha+1}\,t^\alpha}{(s+1)^{\alpha+1}}.$$

It is easy to verify that for this particular choice of u, we have $G'(0+) = 0$, as desired. Next, suppose that $y - x^2 = t^{2\alpha}$. We consider two cases. If $y - u^2 \leq 2x(x-u) - 2\alpha t^{2\alpha}$, then a direct differentiation gives $y+\delta(y-u^2) \leq (x+\delta(x-u))^2 + \left(\frac{t}{1+\delta}\right)^{2\alpha}$ for δ close to 0. Thus $G(\delta) = B_\alpha\left(x+\delta(x-u), y+\delta(y-u^2), \frac{t}{1+\delta}\right)$ for such δ and the whole reasoning from the preceding case is still valid. Hence it remains to analyze the case $y - x^2 = t^{2\alpha}$ and $y - u^2 > 2x(x-u) - 2\alpha t^{2\alpha}$. Subtracting these two algebraic facts, we get $(x-u)^2 < (2\alpha+1)t^{2\alpha}$, or

$$|x - u| < \sqrt{2\alpha + 1}\, t^{\alpha}. \tag{2.8}$$

Next, for such x, y, t and u, we have

$$G(\delta) = B_\alpha \left(x + \delta(x-u), (x + \delta(x-u))^2 + \left(\frac{t}{1+\delta}\right)^{2\alpha}, \frac{t}{1+\delta} \right)$$

when $\delta > 0$ is sufficiently small. Hence

$$\begin{aligned} G'(0+) &= (x-u)B_{\alpha x}(x,y,t) + \big(2x(x-u) - 2\alpha t^{2\alpha}\big)B_{\alpha y}(x,y,t) - tB_{\alpha t}(x,y,t) \\ &= x - u - \sqrt{2\alpha+1}\, t^{\alpha} < 0, \end{aligned}$$

by virtue of (2.8). The proof is complete. □

We are ready to establish the main result of this section.

Theorem 2.3. *For any α and an arbitrary continuous function $f : I \to \mathbb{R}$ with $||f_\alpha^\sharp||_{L^\infty(I)} < \infty$ we have the estimate*

$$\sup_I f \leq B_\alpha \left(\langle f \rangle_I, \langle f^2 \rangle_I, ||f_\alpha^\sharp||_{L^\infty(I)}^{1/\alpha} |I| \right). \tag{2.9}$$

This inequality is sharp: for any $\alpha \in (0,1]$ and any x, y with $x^2 < y \leq x^2 + |I|^{2\alpha}$, there is a function $f : I \to \mathbb{R}$ satisfying $\langle f \rangle_I = x$, $\langle f^2 \rangle_I = y$ and $||f_\alpha^\sharp||_{L^\infty(I)} = 1$, for which both sides above are equal.

Before we turn to the proof, let us make some important observations. First, by passing from f to $-f$, we get the following.

Corollary 2.4. *For any α and an arbitrary continuous function $f : I \to \mathbb{R}$ with $||f_\alpha^\sharp||_{L^\infty(I)} < \infty$ we have the estimate*

$$\inf_I f \geq -B_\alpha \left(-\langle f \rangle_I, \langle f^2 \rangle_I, ||f_\alpha^\sharp||_{L^\infty(I)}^{1/\alpha} |I| \right). \tag{2.10}$$

This inequality is sharp: for any $\alpha \in (0,1]$ *and any* x, y *with* $x^2 < y \le x^2 + |I|^{2\alpha}$, *there is a function* $f : I \to \mathbb{R}$ *satisfying* $\langle f\rangle_I = x$, $\langle f^2\rangle_I = y$ *and* $||f^\sharp_\alpha||_{L^\infty(I)} = 1$, *for which both sides above are equal.*

By (2.5), we have $B_{\alpha y} \ge 0$; furthermore, we have $\langle f^2\rangle_I \le \langle f\rangle_I^2 + ||f^\sharp_\alpha||^2_{L^\infty(I)}|I|^{2\alpha}$. Combining these two observations gives

$$B_\alpha(\langle f\rangle_I, \langle f^2\rangle_I, ||f^\sharp_\alpha||^{1/\alpha}_{L^\infty(I)}|I|) \le B_\alpha(\langle f\rangle_I, \langle f\rangle_I^2 + ||f^\sharp_\alpha||^2_{L^\infty(I)}|I|^{2\alpha}, ||f^\sharp_\alpha||^{1/\alpha}_{L^\infty(I)}|I|)$$
$$= \langle f\rangle_I + \frac{\sqrt{2\alpha+1}}{\alpha}||f^\sharp_\alpha||_{L^\infty(I)}|I|^\alpha.$$

Combining this with the above theorem and its corollary, we get the following result.

Corollary 2.5. *For any* α *and an arbitrary function* $f : I \to \mathbb{R}$ *with* $||f^\sharp_\alpha||_{L^\infty(I)} < \infty$ *we have the sharp estimates*

$$\sup_I f \le \langle f\rangle_I + \frac{\sqrt{2\alpha+1}}{\alpha}||f^\sharp_\alpha||_{L^\infty(I)}|I|^\alpha, \quad \inf_I f \ge \langle f\rangle_I - \frac{\sqrt{2\alpha+1}}{\alpha}||f^\sharp_\alpha||_{L^\infty(I)}|I|^\alpha.$$

Proof of (2.9). By homogeneity, we may and do assume that $||f^\sharp_\alpha||_{L^\infty(I)} \le 1$. For any nonnegative integer n, let $\mathcal{I}^n$ denote the n-th dyadic generation of I. That is, put $\mathcal{I}^0 = \{I\}$, $\mathcal{I}^1 = \{I^\ell, I^r\}$ (here and below, I^ℓ and I^r denote the left and the right half of an interval I), $\mathcal{I}^2 = \{I^{\ell\ell}, I^{\ell r}, I^{r\ell}, I^{rr}\}$, and so on. Let f_n be the conditional expectation of f with respect to $\mathcal{I}^n$: f_n is constant on each element of $\mathcal{I}^n$ and $f_n|_J \equiv \langle f\rangle_J$ for each $J \in \mathcal{I}^n$. Let g_n denote the conditional expectation of f^2 with respect to $\mathcal{I}_n$.

The key point in the proof is to show that the sequence $\big(\sup_I B_\alpha(f_n, g_n, |I|/2^n)\big)_n$ is nonincreasing. To see this, pick an arbitrary n and an interval $J \in \mathcal{I}^n$. We have

$$f_n|_J = \frac{1}{|J|}\int_J f = \frac{1}{|J|}\left(\int_{J^\ell} f + \int_{J^r} f\right) = \frac{1}{2}\left(\frac{1}{|J^\ell|}\int_{J^\ell} f + \frac{1}{|J^r|}\int_{J^r} f\right)$$
$$= \frac{1}{2}f_{n+1}|_{J^\ell} + \frac{1}{2}f_{n+1}|_{J^r}$$

and, similarly,

$$g_n|_J = \frac{1}{2}g_{n+1}|_{J^\ell} + \frac{1}{2}g_{n+1}|_{J^r}.$$

Furthermore, since $f^\sharp_\alpha \leq 1$, we see that

$$g_n|_J - (f_n|_J)^2 = \frac{1}{|J|}\int_J (f - \langle f\rangle_J)^2 \leq |J|^{2\alpha} = (|I|/2^n)^{2\alpha},$$

and similarly for $g_n|_{J^\ell} - (f_n|_{J^\ell})^2$, $g_n|_{J^r} - (f_n|_{J^r})^2$. Consequently, by Theorem 2.2, we get

$$\begin{aligned} &B_\alpha(f_n|_J, g_n|_J, |I|/2^n) \\ &\quad \geq \max\Big\{B_\alpha(f_{n+1}|_{J^\ell}, g_{n+1}|_{J^\ell}, |I|/2^{n+1}), B_\alpha(f_{n+1}|_{J^r}, g_{n+1}|_{J^r}, |I|/2^{n+1})\Big\}. \end{aligned}$$

Taking the supremum over all J, we obtain

$$\sup_I B_\alpha(f_n, g_n, |I|/2^n) \geq \sup_I B_\alpha(f_{n+1}, g_{n+1}, |I|/2^{n+1})$$

and hence for each n and each $x \in I$,

$$B_\alpha(f_n(x), g_n(x), |I|/2^n) \leq B_\alpha(f_0, g_0, |I|) = B_\alpha(\langle f\rangle_I, \langle f^2\rangle_I, |I|).$$

However, we have $B_\alpha(x, y, t) \geq x$, which is evident from the very definition of B_α. Plugging this estimate above gives $f_n(x) \leq B_\alpha(\langle f\rangle_I, \langle f^2\rangle_I, |I|)$. It remains to use the pointwise convergence $f_n \to f$ as $n \to \infty$ (which follows by the continuity of f) to obtain (2.9). □

In the proof of the sharpness of (2.9), we will require the following auxiliary fact.

Lemma 2.6. *For any $\alpha \in (0, 1]$ and any $1 \leq c \leq r$, consider the function $\varphi^{(c)}$: $[0, r] \to \mathbb{R}$, given by*

$$\varphi^{(c)}(p) = \left(1 + \frac{1}{\alpha}\right)\sqrt{2\alpha + 1}\,\min\{p, c\}^\alpha.$$

Then $||\varphi^\sharp_\alpha||_{L^\infty([0,r])} = 1$.

Proof. Let $[a, b]$ be an arbitrary subinterval of $[0, r]$. First we will prove that

$$\langle(\varphi^{(c)})^2\rangle_{[a,b]} - \langle\varphi^{(c)}\rangle^2_{[a,b]}$$

does not decrease as c increases from 1 to r. This monotonicity is clear if $b \le 1$, since then the above expression does not depend on c. This monotonicity is also clear when c runs over the interval $[b, r]$. Thus we need to examine the local behavior of the above difference when $c \in [a, b]$. For such c, we have

$$\begin{aligned}
&\langle (\varphi^{(c)})^2 \rangle_{[a,b]} - \langle \varphi^{(c)} \rangle_{[a,b]}^2 \\
&= \frac{c^{2\alpha+1} - a^{2\alpha+1}}{(2\alpha+1)(b-a)} + \frac{c^{2\alpha}(b-c)}{b-a} - \left(\frac{c^{\alpha+1} - a^{\alpha+1}}{(\alpha+1)(b-a)} + \frac{c^{\alpha}(b-c)}{b-a} \right)^2 .
\end{aligned}$$

Denote the right-hand side by $F(c)$. A little calculation yields

$$F'(c) = \frac{2\alpha c^{\alpha-1}(b-c)}{(b-a)^2(\alpha+1)} \left[(\alpha+1)c^{\alpha}(c-a) - (c^{\alpha+1} - a^{\alpha+1}) \right] \ge 0,$$

where the latter bound follows from the mean-value property. This gives the aforementioned monotonicity and implies that

$$\langle (\varphi^{(c)})^2 \rangle_{[a,b]} - \langle \varphi^{(c)} \rangle_{[a,b]}^2 \le \langle (\varphi^{(r)})^2 \rangle_{[a,b]} - \langle \varphi^{(r)} \rangle_{[a,b]}^2 .$$

Now we will prove that the right hand side does not exceed $(b-a)^{2\alpha}$. We compute that

$$\langle \varphi^{(r)} \rangle_{[a,b]} = \left(1 + \frac{1}{\alpha}\right) \sqrt{2\alpha+1} \frac{b^{\alpha+1} - a^{\alpha+1}}{(\alpha+1)(b-a)}$$

and

$$\langle (\varphi^{(r)})^2 \rangle_{[a,b]} = \left(1 + \frac{1}{\alpha}\right)^2 (2\alpha+1) \frac{b^{2\alpha+1} - a^{2\alpha+1}}{(2\alpha+1)(b-a)},$$

so we may write

$$\begin{aligned}
&\frac{1}{(b-a)^{2\alpha}} \left(\langle (\varphi^{(r)})^2 \rangle_{[a,b]} - \langle \varphi^{(r)} \rangle_{[a,b]}^2 \right) \\
&= \left(1 + \frac{1}{\alpha}\right)^2 (2\alpha+1) \left[\frac{b^{2\alpha+1} - a^{2\alpha+1}}{(b-a)^{2\alpha+1}(2\alpha+1)} - \left(\frac{b^{\alpha+1} - a^{\alpha+1}}{(b-a)^{\alpha+1}(\alpha+1)} \right)^2 \right] \\
&= \left(1 + \frac{1}{\alpha}\right)^2 (2\alpha+1) \left[\frac{w^{2\alpha+1} - (w-1)^{2\alpha+1}}{2\alpha+1} - \left(\frac{w^{\alpha+1} - (w-1)^{\alpha+1}}{\alpha+1} \right)^2 \right],
\end{aligned}$$

where $w = b/(b-a) \ge 1$. If $a = 0$, then $w = 1$ and the above expression is equal to 1; in general, this is the largest possible value which can be attained. To see this, denote the expression in the square brackets by $f(w)$ and note that

$$\begin{aligned} f'(w) &= w^{2\alpha} - (w-1)^{2\alpha} - 2\left(\frac{w^{\alpha+1}-(w-1)^{\alpha+1}}{\alpha+1}\right)(w^\alpha - (w-1)^\alpha) \\ &= 2(w^\alpha - (w-1)^\alpha)\left[\frac{w^\alpha + (w-1)^\alpha}{2} - \int_{w-1}^{w} v^\alpha \mathrm{d}v\right] \leq 0, \end{aligned}$$

since $v \mapsto v^\alpha$ is concave. This completes the proof of the bound $||(\varphi^{(c)})^\sharp_\alpha||_{L^\infty([0,r])} \leq 1$. Actually, we have equality here: it suffices to compute the averages over the interval $[0,1]$. $\square$

Sharpness of (2.9). Fix x, y with $y > x^2$ and put $t = |I|$. Let $s = s(x,y,t)$ be the number given in (2.1). First we will construct a slightly different function than it was mentioned in the statement of Theorem 2.3. Namely, we will find $f : [0, 1+s] \to \mathbb{R}$ satisfying

$$\langle f\rangle_{[0,1+s]} = x,\ \langle f^2\rangle_{[0,1+s]} = y,\ ||f^\sharp_\alpha||_{L^\infty([0,1+s])} = (t/(s+1))^\alpha \tag{2.11}$$

and such that

$$\sup_I f = B_\alpha\left(\langle f\rangle_I, \langle f^2\rangle_I, ||f^\sharp_\alpha||^{1/\alpha}_{L^\infty([0,1+s])}\big|[0,1+s]\big|\right) = B_\alpha(x,y,t). \tag{2.12}$$

To get the function of Theorem 2.3, let T be the affine mapping which sends I onto $[0,1+s]$; then the function $p \mapsto f(T(p))$ is the desired object. This follows from the observation that composing f with an affine mapping preserves the averages and multiplies the α-sharp function by the constant $(T')^\alpha$.

The function satisfying (2.11) and (2.12) is given by

$$f(p) = x + \frac{\sqrt{2\alpha+1}\,st^\alpha}{(s+1)^{\alpha+1}} + \frac{\sqrt{2\alpha+1}}{\alpha}\left(\frac{t}{s+1}\right)^\alpha - \left(1+\frac{1}{\alpha}\right)\sqrt{2\alpha+1}\left(\frac{t}{s+1}\right)^\alpha p^\alpha$$

if $p \in [0,1]$, and $f(p) = x - \sqrt{2\alpha+1}\, t^\alpha (s+1)^{-\alpha-1}$ when $p \in (1, 1+s]$. Let us first verify that the appropriate averages are correct. We have

$$\begin{aligned}\langle f\rangle_{[0,1+s]} = x + \frac{1}{s+1}\Bigg[&\frac{\sqrt{2\alpha+1}\, st^\alpha}{(s+1)^{\alpha+1}} + \frac{\sqrt{2\alpha+1}}{\alpha}\left(\frac{t}{s+1}\right)^\alpha \\ &- \left(1+\frac{1}{\alpha}\right)\sqrt{2\alpha+1}\left(\frac{t}{s+1}\right)^\alpha \cdot \frac{1}{\alpha+1} - \frac{\sqrt{2\alpha+1}\, t^\alpha}{(s+1)^{\alpha+1}}\cdot s\Bigg] \\ = x\end{aligned}$$

and

$$\begin{aligned}\langle f^2\rangle_{[0,1+s]} = \frac{1}{s+1}\Bigg[&\left(x + \frac{\sqrt{2\alpha+1}\, st^\alpha}{(s+1)^{\alpha+1}} + \frac{\sqrt{2\alpha+1}}{\alpha}\left(\frac{t}{s+1}\right)^\alpha\right)^2 \\ &- 2\left(x + \frac{\sqrt{2\alpha+1}\, st^\alpha}{(s+1)^{\alpha+1}} + \frac{\sqrt{2\alpha+1}}{\alpha}\left(\frac{t}{s+1}\right)^\alpha\right) \times \\ &\times \left(1+\frac{1}{\alpha}\right)\sqrt{2\alpha+1}\left(\frac{t}{s+1}\right)^\alpha \cdot \frac{1}{\alpha+1} \\ &+ \left(1+\frac{1}{\alpha}\right)^2 (2\alpha+1)\left(\frac{t}{s+1}\right)^{2\alpha} \cdot \frac{1}{2\alpha+1} \\ &+ \left(x - \frac{\sqrt{2\alpha+1}\, t^\alpha}{(s+1)^{\alpha+1}}\right)^2 s\Bigg] \\ = \quad x^2 + &\frac{t^{2\alpha}}{(s+1)^{2\alpha+2}}\big[s+1+(2\alpha+1)s\big] = y,\end{aligned}$$

where the latter equality follows from the definition (2.1) of the parameter s. Next, observe that $\sup_{p\in[0,1+s]} f(p) = f(0) = B_\alpha(x,y,t)$, so it remains to show that $||f^\sharp_\alpha||_{L^\infty([0,1+s])} \leq (t/(s+1))^\alpha$. This is equivalent to saying that for any $[a,b] \subseteq [0,1+s]$,

$$\langle f^2\rangle_{[a,b]} - \langle f\rangle^2_{[a,b]} \leq \left(\frac{t}{s+1}\right)^{2\alpha} (b-a)^{2\alpha}. \tag{2.13}$$

The left-hand side does not change if we alter the sign of f and/or add a constant to f. Thus, if instead of working with f, we consider the function

$$\begin{aligned} p \mapsto -f(p) + x + \frac{\sqrt{2\alpha+1}\, st^\alpha}{(s+1)^{\alpha+1}} + \frac{\sqrt{2\alpha+1}}{\alpha}\left(\frac{t}{s+1}\right)^\alpha \\ = \left(1+\frac{1}{\alpha}\right)\sqrt{2\alpha+1}\left(\frac{t}{s+1}\right)^\alpha \min\{p,1\}^\alpha = \left(\frac{t}{s+1}\right)^\alpha \varphi^{(1)}(p), \end{aligned}$$

then the left-hand side of (2.13) remains the same. But, as we have verified in the previous lemma, we have $||(\varphi^{(c)})^\sharp_\alpha||_{L^\infty([0,1+s])} = 1$; this implies $||f^\sharp_\alpha||_{L^\infty([0,1+s])} \leq (t/(s+1))^\alpha$ and the proof is complete. □

3. Lower bound for $\sup_I f$

Now we will study a dual statement to Theorem 2.3. The Bellman function $C_\alpha : \mathcal{D}_\alpha \to \mathbb{R}$, corresponding to this new problem, is defined by the formula

$$C_\alpha(x,y,t) = \begin{cases} x & \text{if } y = x^2, \\ x - \dfrac{\sqrt{2\alpha+1}\, t^\alpha}{(s+1)^{\alpha+1}} & \text{if } y > x^2. \end{cases}$$

Here $s = s(x,y,t)$ is the special parameter defined in (2.1). The function C_α has a very nice geometrical interpretation (see Figure 1): $C_\alpha(x,y,t)$ is just the x-coordinate of the point $P = P(x,y,t)$.

Directly from (2.4) and the above definition, we compute that

$$\begin{aligned} C_{\alpha x}(x,y,t) &= 1 + \frac{x(s+1)^{\alpha+2}}{\sqrt{2\alpha+1}\, st^\alpha}, \\ C_{\alpha y}(x,y,t) &= -\frac{\sqrt{2\alpha+1}\,(s+1)^{\alpha+1}}{2(2\alpha+1)st^\alpha} < 0, \\ C_{\alpha t}(x,y,t) &= \frac{\alpha t^{\alpha-1}}{\sqrt{2\alpha+1}\, s(s+1)^\alpha} > 0. \end{aligned} \tag{3.1}$$

We will require the following auxiliary technical fact.

Lemma 3.1. *Let $c < 1$ be fixed. Then for any triple $(x,y,t) \in \mathbb{R}\times[0,\infty)\times[0,\infty)$ such that $x^2 \leq y \leq x^2 + c^2t^{2\alpha}$, we have $s(x,y,t) \geq s_0$, where $s_0 > 0$ depends only on c.*

Proof. This follows at once from the proof of Lemma 2.1. By the above assumptions on (x, y, t), we see that s must satisfy

$$\frac{s+1+(2\alpha+1)s}{(s+1)^{2\alpha+2}} \leq \frac{1}{c^2}.$$

This is equivalent to $s \geq s_0$, where s_0 is the unique strictly positive solution to

$$\frac{s_0+1+(2\alpha+1)s_0}{(s_0+1)^{2\alpha+2}} = \frac{1}{c^2}.$$

This is precisely the claim. □

We turn our attention to the analogue of Theorem 2.2.

Theorem 3.2. *For any fixed $c < 1$, there is a constant $\kappa \in (0, 1/2)$, depending only on c, such that the following holds. If $f : I \to \mathbb{R}$ satisfies $||f^\sharp_\alpha||_{L^\infty(I)} \leq c$, then there is a splitting $I = I_- \cup I_+$ for which $\kappa < |I_+|/|I| < 1 - \kappa$ and*

$$C_\alpha(\langle f\rangle_I, \langle f^2\rangle_I, |I|) \geq C_\alpha(\langle f\rangle_{I_+}, \langle f^2\rangle_{I_+}, |I_+|).$$

Proof. Let $s_0 = s_0(c)$ be the number guaranteed by the preceding lemma. We will show that $\kappa = s_0/(s_0+1)$ works fine. Pick an arbitrary $f : I \to \mathbb{R}$ and let $x = \langle f\rangle_I$, $y = \langle f^2\rangle_I$ and $t = |I|$. If $y = x^2$, then f is constant on I and hence one can take the splitting of I into halves. So, from now on, we assume that $y > x^2$. Suppose that $I = [a, b]$; for any $r \in (a, b)$, let $I_-(r) = [a, r]$ and $I_+(r) = (r, b]$. We will also use the notation $x_\pm(r) = \langle f\rangle_{I_\pm(r)}$, $y_\pm(r) = \langle f^2\rangle_{I_\pm(r)}$ and $t_\pm(r) = |I_\pm(r)|$. Clearly, $x_\pm$, $y_\pm$ and $t_\pm$ are continuous functions of the splitting parameter r. Recall the points $P = P(x, y, t)$, $Q = Q(x, y, t)$ given by (2.2) and (2.3) (see also Figure 1). We know that (x, y) belongs to the line segment PQ. Next, it is not difficult to check (and follows from the geoemtrical interpretation of the parameter s) that if $R = (R_x, R_y)$ is another point from the segment PQ, then

$$P\left(R_x, R_y, t\frac{x-P_x}{R_x-P_x}\right) = P, \qquad Q\left(R_x, R_y, t\frac{x-P_x}{R_x-P_x}\right) = Q.$$

In particular, this implies

$$C_\alpha\left(R_x, R_y, t\frac{x-P_x}{R_x-P_x}\right) = P_x = C_\alpha(x,y,t). \tag{3.2}$$

Now, we consider two cases. Let us first suppose that there is r satisfying $\delta < |I_+(r)|/|I| < 1-\delta$, such that the point (x_+, y_+) lies on the line PQ. Then (x_-, y_-) also belongs to this line, since (x, y) and $(x_\pm, y_\pm)$ are colinear. Furthermore, (x, y) is the average of $(x_\pm, y_\pm)$, so one of the points (x_-, y_-), (x_+, y_+) lies between P and (x, y). By symmetry, we may assume that (x_+, y_+) has this property. Then $x_+ \leq x$ and $t_+ \leq t \leq t(x-P_x)/(x_+-P_x)$, so (3.2) and the inequality $C_{\alpha t} \geq 0$ give

$$C_\alpha(x_+, y_+, t_+) \leq C_\alpha\left(x_+, y_+, t\frac{x-P_x}{x_+-P_x}\right) = C_\alpha(x,y,t)$$

and hence the splitting $I = I_- \cup I_+$ has the desired property. Thus, it remains to consider the case in which for any r such that $\kappa < |I_+(r)|/|I| < 1-\kappa$, the points $(x_-(r), y_-(r))$ lie on the same side of the line of PQ (and hence all $(x_+(r), y_+(r))$ lie on the opposite side). By symmetry, we may assume that all $(x_+(r), y_+(r))$ lie above the line PQ; then all $(x_-(r), y_-(r))$ lie below and hence in particular $x_-(r) \geq C_\alpha(x, y, t)$. Pick r such that $|I_+(r)|/|I| = 1-\kappa$: this gives the required splitting, as we will show now. Since $C_{\alpha y} \leq 0$, we have

$$C_\alpha(x_+, y_+, t_+) \leq C_\alpha(x_+, y', t_+),$$

where y' is the unique positive number such that (x_+, y') lies on the line PQ. Observe that

$$x_+ = \frac{x-\kappa x_-}{1-\kappa} \leq \frac{x-\kappa C_\alpha(x,y,t)}{1-\kappa} = x + \frac{\kappa}{1-\kappa}\frac{\sqrt{2\alpha+1}\,t^\alpha}{(s+1)^{\alpha+1}} \leq x + \frac{\sqrt{2\alpha+1}\,st^\alpha}{(s+1)^{\alpha+1}} = Q_x,$$

since $s \geq \kappa/(1-\kappa)$ (here we use the previous lemma). Consequently, (x_+, y') lies between P and Q; if it lies between (x, y) and P, we get $C_\alpha(x_+, y', t_+) \leq C_\alpha(x, y, t)$ by the same argument as above. If it lies between (x, y) and Q, then

$$C_\alpha(x_+, y', t_+) = C_\alpha\left(x_+, y', t\frac{x-x_-}{x_+-x_-}\right) \leq C_\alpha\left(x_+, y', t\frac{x-P_x}{x_+-P_x}\right) = C_\alpha(x,y,t),$$

again by the inequality $C_{\alpha t} \geq 0$ and (3.2). This proves the claim. $\square$

Equipped with the above theorem, we are ready to establish the following fact, which is the main result of this section.

Theorem 3.3. *For any continuous function $f : I \to \mathbb{R}$ satisfying $||f_\alpha^\sharp||_{L^\infty(I)} < \infty$, we have the inequality*

$$\inf_I f \leq C_\alpha(\langle f\rangle_I, \langle f^2\rangle_I, ||f_\alpha^\sharp||_{L^\infty(I)}^{1/\alpha}|I|). \tag{3.3}$$

This bound is sharp: for any x, y with $x^2 < y \leq x^2 + |I|^{2\alpha}$, there is $f : I \to \mathbb{R}$ satisfying $\langle f\rangle_I = x$, $\langle f^2\rangle_I = y$ and $||f_\alpha^\sharp||_{L^\infty(I)} = 1$, for which both sides above are equal.

Remark 3.4. As in the previous section, the passage from f to $-f$ yields the corresponding lower bound for $\sup_I f$.

Proof of (3.3). By homogeneity, we may assume that $||f_\alpha^\sharp||_{L^\infty(I)} \leq 1$. Fix $c < 1$ and let $\tilde{f} = cf$. By an inductive use of the preceding theorem, we construct an increasing family $(\mathcal{I}^n)_{n\geq 0}$ of partitions of I with respect to $\tilde{f}$. Namely, put $\mathcal{I}^0 = \{I\}$; next, having successfully defined $\mathcal{I}^n$, we pick an arbitrary $J \in \mathcal{I}^n$ and apply the above theorem to the function $\tilde{f}|_J$; this gives us the splitting $J = J_- \cup J_+$, and we set $\mathcal{I}^{n+1} = \{J_\pm : J \in \mathcal{I}^n\}$. Note that the diameter of $\mathcal{I}^{n+1}$ converges to 0: we have $\sup_n\{|J| : J \in \mathcal{I}^n\} \leq (1-\kappa)^n|I|$, where $\kappa = \kappa(c)$ is the number coming from the preceding theorem. Now, define f_n, g_n to be the conditional expectations of $\tilde{f}$ and $\tilde{f}^2$ with respect to $\mathcal{I}^n$; furthermore, for any $x \in I$ and any integer n, let $I^n(x)$ be the unique element of $\mathcal{I}^n$ which contains x. Then, essentially in the same manner as in the proof of Theorem 2.3, we show that the sequence $(\inf\{C_\alpha(f_n(x), g_n(x), |I^n(x)|) : x \in I\})_{n\geq 0}$, is nonincreasing. Consequently, we get

$$\inf_{x\in I} C_\alpha(f_n(x), g_n(x), |I^n(x)|) \leq C_\alpha(\langle \tilde{f}\rangle_I, \langle \tilde{f}^2\rangle_I, |I|).$$

But $C_\alpha(x,y,t) \geq x - \sqrt{2\alpha+1}\,t^\alpha$ for all (x,y,t) and, as we have just noted, $\operatorname{diam} \mathcal{I}^n \leq (1-\kappa)^n$. Combining these observations with the above bound gives

$$\inf_I \tilde{f} - \sqrt{2\alpha+1}(1-\kappa)^{n\alpha} \leq \inf_I f_n - \sqrt{2\alpha+1}(1-\kappa)^{n\alpha} \leq C_\alpha(\langle \tilde{f}\rangle_I, \langle \tilde{f}^2\rangle_I, |I|).$$

It remains to let $n \to \infty$ to get $\inf_I \tilde{f} \leq C_\alpha(\langle \tilde{f}\rangle_I, \langle \tilde{f}^2\rangle_I, |I|)$. Finally, letting $c \uparrow 1$ yields the claim. □

Sharpness. Fix x, y with $y > x^2$ and put $t = |I|$. Let $s = s(x,y,t)$ be the number given in (2.1). As previously, we will construct a slightly different function than it was announced in Theorem 3.3. Namely, it is more convenient to find $f : [0, 1+s] \to \mathbb{R}$ such that

$$\langle f\rangle_{[0,1+s]} = x,\ \langle f^2\rangle_{[0,1+s]} = y,\ ||f_\alpha^\sharp||_{L^\infty(I)} \leq (t/(s+1))^\alpha$$

and such that

$$\inf_I f \leq C_\alpha\left(\langle f\rangle_I, \langle f^2\rangle_I, ||f_\alpha^\sharp||_{L^\infty(I)}^{1/\alpha}\big|[0,1+s]\big|\right) = C_\alpha(x,y,t).$$

To get the function of Theorem 3.3, we compose f with an appropriate affine mapping. Actually, we exploit the same functions as in Theorem 2.3: we immediately check that the function f constructed there satisfies

$$\inf_I f = f(1) = C_\alpha(x,y,t). \quad \square$$

4. Two-sided bounds

In this part of the paper we combine the results obtained in the preceding two sections to establish the sharp comparison of Lipschitz constants and the size of the function $f_\alpha^\sharp$. We start with the right inequality of (1.3).

Theorem 4.1. *Let $f : I \to \mathbb{R}$ be a continuous function satisfying $||f_\alpha^\sharp||_{L^\infty(I)} < \infty$. Then*

$$(4.1) \qquad ||f||_{\mathrm{Lip}_\alpha(I)} \leq \frac{(4\alpha+4)^{(\alpha+1)/(2\alpha+1)}\sqrt{2\alpha+1}}{2\alpha}\,||f_\alpha^\sharp||_{L^\infty(I)}$$

and the constant on the right is the best possible.

At the first glance, the above bound seems to be just a trivial combination of the estimates of Corollary 2.5. Indeed, for any a, $b \in I$, $a < b$, these estimates imply

$$\begin{aligned} |f(a) - f(b)| \leq \sup_{[a,b]} f - \inf_{[a,b]} f &\leq \frac{2\sqrt{2\alpha+1}}{\alpha} ||f^\sharp_\alpha||_{L^\infty([a,b])} |a-b|^\alpha \\ &\leq \frac{2\sqrt{2\alpha+1}}{\alpha} ||f^\sharp_\alpha||_{L^\infty(I)} |a-b|^\alpha. \end{aligned}$$

This gives $||f||_{\mathrm{Lip}_\alpha(I)} \leq \frac{2\sqrt{2\alpha+1}}{\alpha} ||f^\sharp_\alpha||_{L^\infty(I)}$, which is slightly worse than (4.1). To obtain the best constant, we will have to proceed a bit more carefully.

Proof of (4.1). By homogeneity, we may assume that $||f^\sharp_\alpha||_{L^\infty(I)} \leq 1$. We need to show that for all a, $b \in I$ with $a < b$, we have

$$|f(a) - f(b)| \leq \frac{(2\alpha+2)^{(\alpha+1)/(2\alpha+1)}\sqrt{2\alpha+1}}{\alpha 2^\alpha} |a-b|^\alpha.$$

Actually, by restricting to $[a, b]$ if necessary, we may assume from the very beginning that a and b are the endpoints of I. It is convenient to split the reasoning into two steps.

Step 1. Some reductions. It suffices to show the assertion for f satisfying the antisymmetric condition

$$f(x) = -f(a+b-x) \qquad \text{for all } x \in I. \tag{4.2}$$

To see this, consider the function $\tilde{f}$ on I, given by $\tilde{f}(x) = (f(x) - f(a+b-x))/2$. This function satisfies (4.2), we have $|\tilde{f}(a) - \tilde{f}(b)| = |f(a) - f(b)|$ and

$$\begin{aligned} \tilde{f}^\sharp_\alpha(x) &= \sup_{J \ni x} \frac{1}{|J|^\alpha} \left\langle |\tilde{f} - \langle \tilde{f} \rangle_J|^2 \right\rangle_J^{1/2} \\ &\leq \sup_{J \ni x} \frac{1}{|J|^\alpha} \left(\left\langle |f - \langle f \rangle_J|^2 \right\rangle_J^{1/2} + \left\langle |f(a+b-\cdot) - \langle f(a+b-\cdot) \rangle_J|^2 \right\rangle_J^{1/2} \right) \\ &= \sup_{J \ni x} \frac{1}{|J|^\alpha} \left(\left\langle |f - \langle f \rangle_J|^2 \right\rangle_J^{1/2} + \left\langle |f - \langle f \rangle_{a+b-J}|^2 \right\rangle_{a+b-J}^{1/2} \right) \\ &\leq ||f^\sharp_\alpha||_{L^\infty(I)}, \end{aligned}$$

where we have used the standard notation $a+b-J=\{a+b-j : j\in J\}$. Hence $||\tilde{f}^\sharp_\alpha||_{L^\infty(I)} \leq ||f^\sharp_\alpha||_{L^\infty(I)}$, and having established (4.1) for functions f satisfying (4.2), we deduce this result in the general setting as well. The next reduction is that we may assume that

$$f \text{ is nonpositive on } I^\ell \text{ and nonnegative on } I^r, \tag{4.3}$$

where, as previously, I^ℓ, I^r denote the left and the right half of I. To see this, take an arbitrary f satisfying (4.2) and put

$$\tilde{f}(x)=\begin{cases} -|f(x)| & \text{if } x\in I^\ell,\\ |f(x)| & \text{if } x\in I^r.\end{cases}$$

Then $\tilde{f}$ has the appropriate sign on I^ℓ and I^r; furthermore, we have $|\tilde{f}(a)-\tilde{f}(b)| = |f(a)|+|f(b)|\geq |f(a)-f(b)|$. Finally, note that for any $x\in I$,

$$\tilde{f}^\sharp_\alpha(x)=\sup_{J\ni x}\frac{1}{|J|^\alpha}\left(\langle \tilde{f}^2\rangle_J-\langle \tilde{f}\rangle_J^2\right)^{1/2}\leq \sup_{J\ni x}\frac{1}{|J|^\alpha}\left(\langle f^2\rangle_J-\langle f\rangle_J^2\right)^{1/2}.$$

To justify the last passage, we use the identity $\tilde{f}^2=f^2$ and the fact that f and $\tilde{f}$ satisfy the symmetry condition (4.2): setting $\tilde{J}=J\setminus(a+b-J)$, we may write

$$|\langle f\rangle_J|=\frac{1}{|J|}\left|\int_J f\right|=\frac{1}{|J|}\left|\int_{\tilde{J}} f\right|\leq \frac{1}{|J|}\left|\int_J \tilde{f}\right|=\frac{1}{|J|}\left|\int_{\tilde{J}} \tilde{f}\right|=|\langle \tilde{f}\rangle_J|.$$

So $||\tilde{f}^\sharp_\alpha||_{L^\infty(I)}\leq ||f^\sharp_\alpha||_{L^\infty(I)}$ and it is enough to show (4.1) under (4.2) and (4.3).

Step 2. Calculations. Pick an arbitrary function satisfying $||f^\sharp_\alpha||_{L^\infty(I)}\leq 1$ and the conditions (4.2), (4.3). Consider the splitting $I=I^\ell\cup I^r$ and set $x=\langle f\rangle_{I^r}$, $y=\langle f^2\rangle_{I^r}$, $t=|I^r|$. Then Theorem 2.3 implies $\sup_{I^r} f\leq B_\alpha(x,y,t)$. Furthermore, by (4.2), we have $\langle f\rangle_{I^\ell}=-x$, $\langle f^2\rangle_{I^\ell}=y$ and hence, again by Theorem 2.3, $\inf_{I^\ell} f\geq -B_\alpha(x,y,t)$. These arguments show the estimate

$$|f(a)-f(b)|\leq \sup_{I^r} f-\inf_{I^\ell} f\leq 2B_\alpha(x,y,t).$$

Next, we have that f is nonnegative on I^r and hence Theorem 3.3 yields $0 \leq \inf_{I^r} f \leq C_\alpha(x,y,t)$. Consequently,

$$|f(a) - f(b)| \leq 2 \sup B_\alpha(x,y,t),$$

where the supremum is taken over all x, y such that $C_\alpha(x,y,t) \geq 0$. To handle this supremum on the right, fix y and take a look at the set $\{x \geq 0 : C_\alpha(x,y,t) \geq 0\}$. We have $C_\alpha(0,y,t) \leq 0$, $C_\alpha(\sqrt{y},y,t) = \sqrt{y} \geq 0$ and $C_{\alpha x}(x,y,t) \geq 0$ for $x \geq 0$ (see (3.1)). Thus the set $\{x \geq 0 : C_\alpha(x,y,t) \geq 0\}$ is a certain interval of the form $[x_0(y), \sqrt{y}]$, where $C_\alpha(x_0(y),y,t) = 0$. In addition, observe that the inequality $C_\alpha \geq 0$ is equivalent to $B_{\alpha x} \leq 0$. This shows that

$$\sup\{2B_\alpha(x,y,t) : C_\alpha(x,y,t) \geq 0\} = \sup\{2B_\alpha(x,y,t) : C_\alpha(x,y,t) = 0\}.$$

However, $C_\alpha(x,y,t) = 0$ is equivalent to

$$x = \sqrt{2\alpha+1}\, t^\alpha/(s+1)^{\alpha+1}, \tag{4.4}$$

and then, by (2.1), $y = (2\alpha+2)t^{2\alpha}(s+1)^{-2\alpha-1}$. However,

$$y = \langle f^2 \rangle_I = \langle f^2 \rangle_I - \langle f \rangle_I^2 \leq |I|^{2\alpha} = (2t)^{2\alpha},$$

which yields $s+1 \geq \left(\frac{2\alpha+2}{2^{2\alpha}}\right)^{1/(2\alpha+1)} = \frac{1}{2}(4\alpha+4)^{1/(2\alpha+1)}$. On the other hand, under (4.4) we have

$$B_\alpha(x,y,t) = \frac{(\alpha+1)\sqrt{2\alpha+1}}{\alpha} \left(\frac{t}{s+1}\right)^\alpha$$

and hence the lower bound for $s+1$ just obtained above gives

$$\begin{aligned} 2\sup B_\alpha(x,y,t) &\leq \frac{2(\alpha+1)\sqrt{2\alpha+1}}{\alpha} \frac{(2t)^\alpha}{(2s+2)^\alpha} \\ &= \frac{(4\alpha+4)^{(\alpha+1)/(2\alpha+1)}\sqrt{2\alpha+1}}{2\alpha}|b-a|^\alpha, \end{aligned}$$

which is the desired estimate. □

Sharpness. The reasoning consists of several separate parts. We have decided to split the proof accordingly.

Step 1. The extremal function and some initial calculations. Let $s = \frac{1}{2}(4\alpha + 4)^{1/(2\alpha+1)} - 1$ and $t = s+1$. Furthermore, let x be given by (4.4) and put $y = (2t)^{2\alpha}$. Then (2.1) is satisfied. Introduce the function $f : [0, 1+s] \to \mathbb{R}$ by

$$\begin{aligned} f(p) &= \left(1 + \frac{1}{\alpha}\right)\sqrt{2\alpha+1}\left(\frac{t}{s+1}\right)^{\alpha} \max\{1 - p^{\alpha}, 0\} \\ &= \left(1 + \frac{1}{\alpha}\right)\sqrt{2\alpha+1} \max\{1 - p^{\alpha}, 0\}. \end{aligned}$$

We have already considered this function in the proof of the sharpness of (2.9). In particular, we have checked that $||f_\alpha^\sharp||_{L^\infty([0,1+s])} = (t/(s+1))^\alpha = 1$. Let us extend this function to the interval $[0, 2+2s]$ by setting $f(p) = -f(2 + 2s - p)$. Then

$$||f||_{\mathrm{Lip}([0,2+2s])} \geq \frac{|f(0) - f(2s+2)|}{(2s+2)^\alpha} = \frac{2f(0)}{(2s+2)^\alpha} = L(\alpha).$$

Thus, we will be done if we show that $||f_\alpha^\sharp||_{L^\infty([0,2s+2])} \leq 1$.

Step 2. We must show that for any $0 \leq a < b \leq 2s+2$, we have $\langle f^2\rangle_{[a,b]} - \langle f\rangle^2_{[a,b]} \leq (b-a)^{2\alpha}$. This is equivalent to saying that

$$F(a,b) = \int_a^b f^2 - \frac{1}{b-a}\left(\int_a^b f\right)^2 - (b-a)^{2\alpha+1} \leq 0.$$

If $[a,b] \subseteq [0, 1+s]$ or $[a,b] \subseteq [1+s, 2+2s]$, then this bound follows from the fact that $||f_\alpha^\sharp||_{L^\infty([0,1+s])} = 1$, which we established in the proof of Theorem 2.3. Thus,

$$F(a, 1+s) \leq 0 \qquad \text{for all } a < 1+s, \tag{4.5}$$

and we need to check what happens for $a \leq 1+s \leq b$. Note that

$$\begin{aligned} \frac{\partial F}{\partial b}(a,b) &= f^2(b) + \frac{1}{(b-a)^2}\left(\int_a^b f\right)^2 - \frac{2f(b)\int_a^b f}{b-a} - (2\alpha+1)(b-a)^{2\alpha} \\ &= \left[\frac{\int_a^b f}{b-a} - f(b)\right]^2 - \left[\sqrt{2\alpha+1}(b-a)^\alpha\right]^2. \end{aligned}$$

The function f is nonincreasing, so the expression in the first square bracket is nonnegative and hence $\partial F/\partial b$ has the same sign as

$$G(a,b) := \frac{\int_a^b f}{b-a} - f(b) - \sqrt{2\alpha+1}(b-a)^\alpha.$$

Step 3. We will now show that $G(a, 1+s) \le 0$. We have

$$G(a,1+s) = \frac{\int_a^{1+s} f}{1+s-a} - \sqrt{2\alpha+1}(1+s-a)^\alpha = \frac{\int_a^1 f}{1+s-a} - \sqrt{2\alpha+1}(1+s-a)^\alpha.$$

The bound $G(a, 1+s) \le 0$ is evident if $a \in [1, 1+s]$. To show it for $a \le 1$, we denote $H(a) = (1+s-a)G(a, 1+s) = \int_a^1 f - \sqrt{2\alpha+1}(1+s-a)^{\alpha+1}$ and compute

$$\begin{aligned} H'(a) &= -f(a) + \sqrt{2\alpha+1}(\alpha+1)(1+s-a)^\alpha, \\ H''(a) &= \sqrt{2\alpha+1}(\alpha+1)(a^{\alpha-1} - \alpha(1+s-a)^{\alpha-1}), \\ H'''(a) &= \sqrt{2\alpha+1}(\alpha+1)(\alpha-1)(a^{\alpha-2} + \alpha(1+s-a)^{\alpha-2}). \end{aligned}$$

So, $H''' < 0$; furthermore, H'' is positive as a approaches 0. Consequently, H is either convex on $[0,1]$, or convex on $[0, a_0]$ and concave on $[a_0, 1]$ for some $a_0 \in (0,1)$; since $H'(1) > 0$, $H(1) < 0$ and $H(0) = \int_0^1 f - \sqrt{2\alpha+1}(1+s)^{\alpha+1} = \sqrt{2\alpha+1}(1 - (1+2s)^{\alpha+1}) < 0$, we obtain the desired bound $G(a, 1+s) \le 0$.

Step 4. We go back to the analysis of the sign of G. Clearly, $G(a, \cdot)$ is decreasing on $b \in [1+s, 1+2s]$ (the integral $\int_a^b f$ and $f(b)$ are constant on this interval) and hence it is negative there. For $b \ge 1+2s$, let

$$K(b) = \frac{G(a,b)}{b-a} = \int_a^b f - f(b)(b-a) - \sqrt{2\alpha+1}(b-a)^{\alpha+1}$$

and note that

$$\begin{aligned} K'(b) &= -f'(b)(b-a) - \sqrt{2\alpha+1}(\alpha+1)(b-a)^\alpha \\ &= \sqrt{2\alpha+1}(\alpha+1)(b-a)\big((2s+2-b)^{\alpha-1} - (b-a)^{\alpha-1}\big). \end{aligned}$$

Thus, we have two options. If $a/2 \leq s$, then $b \geq s+1+a/2$, or $2s+2-b \leq b-a$, for all b; this implies $K'(b) > 0$. If $a/2 > s$, then $K'(b) \leq 0$ for $b < s+1+a/2$ and $K'(b) > 0$ for remaining b. As we have shown above, $K(b) \leq 0$ for $b \in [1+s, 1+2s]$. Consequently, we have two possibilities: either K is nonpositive for all $b \in [1+s, 2+2s]$, or there is $b_0(a) \in [1+s, 2+2s]$ such that $K(b) \leq 0$ for $b \leq b_0(a)$ and $K(b) \geq b_0(a)$. Since $\partial F/\partial b$ shares the same sign with G and K, we see that the function $F(a, \cdot)$ is either decreasing on $[1+s, 2+2s]$, or is nonincreasing on $[1+s, b_0(a)]$ and nondecreasing on $[b_0(a), 2+2s]$. Combining this behavior with (4.5), we conclude that if $F(a,b) > 0$ for some $a < 1+s < b$, then also $F(a, 2+2s) > 0$. However, we have $F(a, 2+2s) = F(0, 2+2s-a)$, since $f(p) = -f(2+2s-p)$. So, repeating the above argumentation, we see that $F(0, 2+2s-a) > 0$ implies $F(0, 2+2s) > 0$. But this is a contradiction: we easily check that

$$F(0, 2+2s) = \int_0^{2s+2} f^2 - (2s+2)^{2\alpha+1} = 2\int_0^1 f^2 - (2s+2)^{2\alpha+1} = 0.$$

This proves that $||f_\alpha^\sharp||_{L^\infty([0,2s+2])} \leq 1$ and completes the proof. □

Finally, let us focus on the left inequality of (1.3). In comparison to the preceding estimate, here the reasoning will be much less technical.

Theorem 4.2. *Let $f : I \to \mathbb{R}$ be a continuous function satisfying $||f_\alpha^\sharp||_{L^\infty(I)} < \infty$. Then*

$$||f||_{\mathrm{Lip}_\alpha(I)} \geq 2\sqrt{2\alpha+1}\,||f_\alpha^\sharp||_{L^\infty(I)} \tag{4.6}$$

and the constant $2\sqrt{2\alpha+1}$ is the best possible.

Proof. By homogeneity, we may assume that $||f_\alpha^\sharp||_{L^\infty(I)} = 1$. For any $\varepsilon > 0$, there is a subinterval $[a,b] \subset I$ such that

$$\langle f^2\rangle_{[a,b]} - \langle f\rangle_{[a,b]}^2 > \big(1-\varepsilon\big)|b-a|^{2\alpha}.$$

By Theorem 3.3 and the remark following it, we have

$$\inf_{[a,b]} f \le C_\alpha(\langle f\rangle_{[a,b]}, \langle f^2\rangle_{[a,b]}, b-a)$$

and

$$\sup_{[a,b]} f \ge -C_\alpha(-\langle f\rangle_{[a,b]}, \langle f^2\rangle_{[a,b]}, b-a).$$

These two estimates imply

$$\begin{aligned} ||f||_{\mathrm{Lip}_\alpha(I)} &\ge \frac{\sup_{[a,b]} f - \inf_{[a,b]} f}{|b-a|^\alpha} \\ &\ge \frac{-C_\alpha(-\langle f\rangle_{[a,b]}, \langle f^2\rangle_{[a,b]}, b-a) - C_\alpha(\langle f\rangle_{[a,b]}, \langle f^2\rangle_{[a,b]}, b-a)}{|b-a|^\alpha}. \end{aligned}$$

Setting $s_\pm = s(\pm\langle f\rangle_{[a,b]}, \langle f^2\rangle_{[a,b]}, b-a)$, we see that the latter expression equals

$$\frac{\sqrt{2\alpha+1}}{(s_-+1)^{\alpha+1}} + \frac{\sqrt{2\alpha+1}}{(s_++1)^{\alpha+1}},$$

which can be made as close to $2\sqrt{2\alpha+1}$ as we wish, by taking ε sufficiently small: here we use the fact that $s = s(x,y,t)$ depends on x, y and t only through $(y-x^2)/t^{2\alpha}$; hence $s_\pm$ is actually a function of ε only.

To see that the constant $2\sqrt{2\alpha+1}$ is optimal, take $f : [-1,1] \to \mathbb{R}$ with $f(p) = 2^\alpha\sqrt{2\alpha+1}|p|^\alpha \operatorname{sgn} p$. Then $f \in \mathrm{Lip}_\alpha(I)$ and hence $||f^\sharp_\alpha||_{L^\infty(I)} < \infty$. Actually, since $\langle f\rangle_{[-1,1]} = 0$ and $\langle f^2\rangle_{[-1,1]} = 2^{2\alpha}$, we have the lower bound $||f^\sharp_\alpha||_{L^\infty(I)} \ge 1$. On the other hand, we see that

$$||f||_{\mathrm{Lip}_\alpha([-1,1])} = 2^\alpha\sqrt{2\alpha+1} \sup_{p,q\in[-1,1]} \frac{||p|^\alpha \operatorname{sgn} p - |q|^\alpha \operatorname{sgn} q|}{|p-q|^\alpha}.$$

A straightforward analysis shows that the above supremum is attained for $p = -q$, and equals $2^{1-\alpha}$. Consequently, we see that $||f||_{\mathrm{Lip}_\alpha([-1,1])} = 2\sqrt{2\alpha+1}$ and hence we must have equality in (4.6) (as a by-product, we also obtain that $||f^\sharp_\alpha||_{L^\infty([-1,1])} = 1$). Thus $\ell(\alpha)$ is indeed the best possible. □

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