# INEQUALITIES FOR NONCOMMUTATIVE SUBMARTINGALES AND THEIR STRONG DIFFERENTIAL SUBORDINATES

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Abstract. We introduce a notion of strong differential subordination of noncommutative semimartingales, extending Burkholder's definition from the classical case. Then we establish weak-type (1,1) and $L^1 \to L^p$ estimates ($0 < p < 1$) under the additional assumption that the dominating process is a submartingale. The proof rests on a significant extension of the maximal weak-type estimate of Cuculescu and a Gundy-type decomposition of an arbitrary noncommutative submartingale. We also show the corresponding $L^p$ estimates ($1 < p < \infty$) under the assumption that the dominating process is a nonnegative submartingale. This is accomplished by combining several techniques, including interpolation and noncommutative analogue of good-$\lambda$ inequalities.

## 1. Introduction

The motivation for the results obtained in this paper comes from the question about appropriate noncommutative extensions of several semimartingale inequalities studied by Burkholder [8] and Hammack [10] in the classical case. To briefly describe these estimates, assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, equipped with a filtration $(\mathcal{F}_n)_{n\geq 0}$, that is, a non-decreasing sequence of sub-$\sigma$-fields of $\mathcal{F}$. Suppose that $x = (x_n)_{n\geq 0}$ and $y = (y_n)_{n\geq 0}$ are adapted sequences of integrable random variables, with the corresponding differences $dx = (dx_n)_{n\geq 0}$, $dy = (dy_n)_{n\geq 0}$ given by $dx_0 = x_0$ and $dx_n = x_n - x_{n-1}$ for $n \geq 1$ (with an analogous formula for $dy$). Consider the following two conditions:

(DS) for any $n \geq 0$ we have $|dy_n| \leq |dx_n|$ almost surely;
(CDS) for any $n \geq 1$ we have $|\mathcal{E}_{n-1}(dy_n)| \leq |\mathcal{E}_{n-1}(dx_n)|$,

where for any nonnegative integer $n$, the symbol $\mathcal{E}_n$ stands for the conditional expectation with respect to the $\sigma$-field $\mathcal{F}_n$. If the requirement (DS) is satisfied, then $y$ is said to be differentially subordinate to $x$. If (CDS) holds true, then $y$ is conditionally differentially subordinate to $x$. Finally, if both (DS) and (CDS) are satisfied, then $y$ is *strongly differentially subordinate* to $x$.

The strong differential subordination implies many interesting estimates if we impose some additional structure on the dominating process. Suppose first that $x$ is a martingale. Then the condition (CDS) enforces $y$ to be a martingale as well and the strong differential subordination reduces to the requirement (DS). In such

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a case, Burkholder [7] proved the sharp weak-type (1,1) bound

$$\mathbb{P}\left(\sup_{n\geq 0}|y_n|\geq 1\right)\leq 2||x||_1$$

and the sharp strong-type estimate

$$||y||_p\leq (p^*-1)||x||_p, \qquad 1<p<\infty,$$

where $p^*=\max\{p,p/(p-1)\}$ and $\|x\|_p=\sup_{n\geq 0}\|x_n\|_p$. These celebrated estimates have been extended in numerous directions and found applications in harmonic analysis, functional analysis and the theory of quasiconformal mappings (see [1, 2, 3, 4, 5, 21, 22] and consult the references therein).

The strong differential subordination can also be exploited under slightly weaker assumptions on the dominated process. As Burkholder proved in [8], if $x$ is assumed to be a nonnegative submartingale, then the weak- and strong-type bounds also hold true. More precisely, we have

$$\mathbb{P}\left(\sup_{n\geq 0}|y_n|\geq 1\right)\leq 3||x||_1 \tag{1.1}$$

and

$$||y||_p\leq (p^{**}-1)||x||_p, \qquad 1<p<\infty, \tag{1.2}$$

where $p^{**}=\max\{2p,p/(p-1)\}$. Again, the constants in the estimates above are optimal. A few years later, Hammack [10] generalized the weak-type inequality (1.1) to the setting of arbitrary submartingales (i.e., with no assumption on the sign of the dominating process) and proved that then the optimal constant increases to 6. Furthermore, he showed that there is no version of (1.2) in this more general context, by constructing appropriate examples.

We will be interested in the analogs of the above results in the context of noncommutative (or quantum) probability. The theory of semimartingale inequalities in the non-commutative setting has gained a lot interest in the last twenty years. Starting with the seminal paper of Pisier and Xu [24], where the appropriate counterparts of Burkholder-Gundy inequalities were proposed, many important estimates have been successfully extended to the noncommutative realm. These include the analogue of Doob's maximal $L^p$ bound obtained by Junge [11] in the case $1<p<\infty$ and Randrianantoanina [29] for $0<p<1$, noncommutative Burkholder-Rosenthal inequalities investigated by Junge and Xu [12, 14], as well as appropriate weak-type versions due to Randrianantoanina [26, 27, 28].

Let us briefly describe the structure of the paper and say a few words about our approach. We should point out that the passage from noncommutative martingales to the context of noncommutative submartingales (and their strong differential subordinates) requires the development of new methods and techniques. Most of the arguments which are typically used in the area (e.g., standard interpolation, duality) cannot be successfully applied here, or their efficiency is limited. As we believe, the approach we present considerably extends the machinery which can be used in the theory of noncommutative semimartingales, and its appropriate modifications might play an important role in the further exploration of the subject.

The background on noncommutative semimartingale theory, which is necessary for the treatment of the above problems, is presented in Section 2. We also provide there the appropriate counterpart of the strong differential subordination and discuss some of its properties.

In Section 3 we establish the noncommutative analogue of Hammack's result (i.e., a maximal weak-type (1,1) estimate for arbitrary submartingales and their strong differential subordinates). This is accomplished by establishing a significant extension of Cuculescu's weak-type estimate [9], which is of independent interest, and an appropriate modification of noncommutative Gundy-type decomposition due to Parcet and Randrianantoanina [23]. We conclude by giving an important application, a localized $L^1 \to L^p$ estimate for $0 < p < 1$.

Section 4, the final part of the paper, is devoted to the nocommutative extension of the inequality (1.2). Quite interestingly, we will have to split the reasoning into two parts, corresponding to $1 < p \leq 2$ ("small $p$") and $p > 2$ ("big $p$"), in which our methods will be quite different. In the case $1 < p \leq 2$ we use a certain adaptation of Gundy-type decomposition and exploit arguments which can be interpreted as a version of real interpolation: in a sense, we will study a behavior of the $K$-functional associated with the subordinates. For $p > 2$, our approach depends heavily on the noncommutative analogue of good-lambda inequalities. In the classical case, this extrapolation technique was introduced by Burkholder in [6], and it has turned out to be very powerful in a number of problems arising in harmonic analysis and probability. The noncommutative counterpart of this method, recently obtained by the authors in [17], allows us to obtain a certain version of the noncommutative Doob's inequality (proved by Junge in [11]). This estimate, combined with a certain novel $L^p$ bound for submartingale differences, yields the moment inequality for the strong subordinates in the range $p > 2$.

## 2. Preliminaries

Throughout the paper, we use standard notation from the theory of operator algebras, we refer the reader to [19, 20, 30] for the detailed exposition. Let $H$ be a given Hilbert space and denote by $B(H)$ the algebra of all bounded operators acting on $H$. Let $\mathcal{M}$ be a von Neumann subalgebra of $B(H)$, equipped with a semifinite normal faithful trace $\tau$. A closed densely defined operator $a$ on $H$ is said to be affiliated with $\mathcal{M}$ if $u^*au = a$ for all unitary $u$ in the commutant $\mathcal{M}'$ of $\mathcal{M}$. Such an operator is said to be $\tau$-measurable if for any $\varepsilon > 0$ there exists a projection $e$ contained in its domain, satisfying $\tau(I - e) < \varepsilon$ (here and in what follows, the letter $I$ stands for the identity operator). The set of all $\tau$-measurable operators will be denoted by $L^0(\mathcal{M}, \tau)$. The trace $\tau$ can be extended to a positive tracial functional on the positive part $L^0_+(\mathcal{M}, \tau)$ of $L^0(\mathcal{M}, \tau)$ and this extension is still denoted by $\tau$. Suppose that $a$ is a self-adjoint $\tau$-measurable operator and let $a = \int_{-\infty}^{\infty} \lambda de_\lambda$ stand for its spectral decomposition. For any Borel subset $B$ of $\mathbb{R}$, the spectral projection of $a$ corresponding to the set $B$ is defined by $I_B(a) = \int_{-\infty}^{\infty} \chi_B(\lambda) de_\lambda$. Sometimes, with no risk of confusion, we will write $\tau(a \in B)$ instead of $\tau(I_B(a))$.

For $0 < p < \infty$, we recall that the noncommutative $L^p$-space associated with $(\mathcal{M}, \tau)$ is defined by $L^p(\mathcal{M}, \tau) = \{x \in L^0(\mathcal{M}, \tau) : \tau(|x|^p) < \infty\}$ equipped with the (quasi-)norm $\|x\|_p = (\tau(|x|^p))^{1/p}$, where $|x| = (x^*x)^{1/2}$ is the modulus of $x$. For $p = \infty$, the space $L^p(\mathcal{M}, \tau)$ coincides with $\mathcal{M}$ with its usual operator norm. We refer to the survey [25] and the references therein for more details.

The main subject of this paper is the theory of noncommutative semimartingales. Let us now present the general setup. Assume that $(\mathcal{M}_n)_{n\geq 0}$ is a filtration, that is, a nondecreasing sequence of von Neumann subalgebras of $\mathcal{M}$ whose union is weak$^*$-dense in $\mathcal{M}$. Then for any $n \geq 0$ there exists a normal conditional expectation $\mathcal{E}_n$ from $\mathcal{M}$ onto $\mathcal{M}_n$, which satisfies the following two conditions:

(i) $\mathcal{E}_n(axb) = a\mathcal{E}_n(x)b$ for all $a$, $b \in \mathcal{M}_n$ and $x \in \mathcal{M}$;
(ii) $\tau \circ \mathcal{E}_n = \tau$.

It can be verified readily that the conditional expectations enjoy the property $\mathcal{E}_m\mathcal{E}_n = \mathcal{E}_n\mathcal{E}_m = \mathcal{E}_{\min(m,n)}$ for all nonnegative integers $m$ and $n$. Furthermore, the operator $\mathcal{E}_n$ is trace preserving, and hence it can be extended to a contractive projection from $L^p(\mathcal{M},\tau)$ onto $L^p(\mathcal{M}_n,\tau_n)$ for all $1 \leq p \leq \infty$; here $\tau_n$ denotes the restriction of $\tau$ to $\mathcal{M}_n$.

A sequence $x = (x_n)_{n\geq 0}$ in $L^1(\mathcal{M})$ is called a noncommutative martingale (respectively, submartingale or supermartingale) adapted to $(\mathcal{M}_n)_{n\geq 0}$ if for any $n \geq 0$ we have

$$\mathcal{E}_n(x_{n+1}) = x_n$$

(respectively, $\mathcal{E}_n(x_{n+1}) \geq x_n$ or $\mathcal{E}_n(x_{n+1}) \leq x_n$). Note that the sub- and supermartingales need to consist of self-adjoint operators, so that the inequalities make sense. The associated difference sequence is defined as in the commutative case, with the use of the formulae $dx_0 = x_0$ and $dx_n = x_n - x_{n-1}$ for $n \geq 1$. Sometimes we will exploit the notation

$$\|x\|_p = \sup_{n\geq 0} \|x_n\|_p, \qquad 0 < p < \infty,$$

for the $p$-th norm of the sequence $x$. In the paper we will mostly deal with finite martingales $x = (x_n)_{n=0}^N$ (that is, consisting of a finite number of operators).

In what follows, we will need the so-called Burkholder-Rosenthal inequalities for finite, self-adjoint martingales (see [12], [28] for the general statement). Suppose that $x = (x_n)_{n=0}^N$ is such a sequence with terms belonging to $L^2(\mathcal{M})$. We define the associated conditional square function $s_N(x)$ by the formula

$$s_N(x) = \left(\sum_{n=0}^N \mathcal{E}_{n-1}(dx_n^2)\right)^{1/2}.$$

Then for any $p \geq 2$ there exists a constant $c_p$ depending only on $p$ such that

$$c_p^{-1}\|x_N\|_p \leq \|s_N(x)\|_p + \left(\sum_{n=0}^N \|dx_n\|_p^p\right)^{1/p} \leq c_p\|x_N\|_p.$$

Furthermore, $c_p$ can be taken to be of order $O(p)$ as $p \to \infty$ (see [28]).

We are ready to introduce the domination principle under which we will work in this paper.

**Definition 2.1.** Suppose that $x = (x_n)_{n\geq 0}$, $y = (y_n)_{n\geq 0}$ are self-adjoint adapted sequences in $L^1(\mathcal{M})$. We say that $y$ is strongly differentially subordinate to $x$ if the following two conditions are satisfied:

(DS) for any $n \geq 0$ and any projection $R \in \mathcal{M}_{n-1}$ we have

$$Rdy_nRdy_nR \leq Rdx_nRdx_nR;$$

(CDS) for any $n \geq 1$ we have

$$-|\mathcal{E}_{n-1}(dx_n)| \leq \mathcal{E}_{n-1}(dy_n) \leq |\mathcal{E}_{n-1}(dx_n)|.$$

Observe that in the commutative case this reduces to the usual definition of strong differential subordination formulated in the introductory section. Furthermore, note that the condition (CDS) is slightly weaker than the requirement

$$|\mathcal{E}_{n-1}(dy_n)| \leq |\mathcal{E}_{n-1}(dx_n)| \quad \text{for each } n \geq 1.$$

There is a weaker version of the differential subordination (i.e., the condition (DS)) which will also be of importance:

(WDS) for any $n \geq 0$ we have $dy_n^2 \leq dx_n^2$.

We refer the reader to [16, Lemma 3.3] for a detailed comparison on the conditions (DS) and (WDS).

We will show that some of the results (namely, the $L^p$ estimates in the range $p \geq 2$) holds true under the weaker assumption (WDS)+(CDS). On the other hand, as the authors exhibited in [16], this weaker set of conditions is not sufficient for the validity of $L^p$ estimates in the range $1 < p < 2$ even in the martingale setting. This justifies the use of the more complicated requirement (DS) in the case of "small exponent $p$".

## 3. A maximal weak-type estimate

We will now handle a maximal weak-type $(1,1)$ estimate for strong differential subordinates of arbitrary (no necessary to be nonnegative) noncommutative submartingales, which provides a noncommutative version of (1.1). Recall that (1.1) was firstly proved by Burkholder in [8] for nonnegative submartingales and then generalized to the general case by Hammack in [10]. The following is the precise statement.

**Theorem 3.1.** *Let $x = (x_n)_{n\geq 0}$ be an arbitrary submartingale and suppose that $y$ is strongly differentially subordinate to $x$. Then there exists a projection $q$ satisfying*

$$-q \leq qy_nq \leq q \qquad \text{for all } n \tag{3.1}$$

*and such that*

$$\tau(I-q) \leq 327||x||_1. \tag{3.2}$$

Two important observations are in order. In the commutative case, it is easy to see that the largest projection $q$ satisfying (3.1) is precisely the indicator function of the set $\{\sup_{n\geq 0}|y_n| \leq 1\}$ and then (3.2) becomes

$$\mathbb{P}\left(\sup_{n\geq 0}|y_n| > 1\right) \leq 327||x||_1.$$

This explains why we refer to (3.2) as to a maximal weak-type bound. The second comment is that the above result holds true in the particular case when $x$ is a martingale. Thus Theorem 3.1 generalizes the main result of [15], as it provides an estimate for a wider class of processes and under a weaker domination requirement.

We start with introducing certain families $(R_n)_{n\geq -1}$, $(D_n)_{n\geq 0}$ and $(U_n)_{n\geq 0}$ of projections which will play a key role in our considerations below. For an arbitrary submartingale $x = (x_n)_{n\geq 0}$, define $R_{-1} = I$ and for $n \geq 0$, inductively,

$$\begin{aligned} R_n &= R_{n-1} I_{(-1,1)}(R_{n-1}x_nR_{n-1}), \\ D_n &= I_{[1,\infty)}(R_{n-1}x_nR_{n-1}), \\ U_n &= I_{(-\infty,-1]}(R_{n-1}x_nR_{n-1}). \end{aligned}$$

Crucial properties of these objects, to be needed later, are gathered in the next lemma.

**Lemma 3.2.** *Let $x = (x_n)_{n\geq 0}$ be an $L^1$-bounded submartingale. Then the following statements hold true:*

(i) *for each $n \geq 0$ the projections $R_n$, $U_n$ and $D_n$ belong to $\mathcal{M}_n$ and $R_n + U_n + D_n = R_{n-1}$;*
(ii) *for each $n \geq 0$, the projections $R_n$, $U_n$ and $D_n$ commute with $R_{n-1}x_nR_{n-1}$;*
(iii) *for each $n \geq 0$ we have*

$$-R_n \leq R_nx_nR_n \leq R_n, \quad U_nx_nU_n \leq -U_n, \quad D_nx_nD_n \geq D_n;$$

(iv) *for any $N \geq 0$ we have*

$$\tau(I - R_N) \leq 2\tau(x_N^+) - \tau(x_0).$$

**Remark 3.3.** The expression on the right-hand side of (iv) can be bounded from above by a simple $3||x||_1$. However, we have decided to keep the above formulation. It should be stressed that both terms $\tau(x_0)$ and $\tau(x_N^+)$ (i.e., the measurements of the size of the starting and the terminating operator of $x$) are necessary, due to the submartingale structure of $x$ (cf. [10] for a similar phenomenon in the classical case). This should be contrasted with the martingale setting, where both terms could be replaced by the expression $\tau(|x_N|)$ involving just the terminating operator. A similar remark applies to three lemmas below.

*Proof of Lemma 3.2.* The first three properties are evident and the main difficulty lies in proving (iv). Note that for any $n \geq 1$ we have, by the submartingale property of $x$, the tracial property of $\tau$ and part (i) above,

$$\begin{aligned} \tau(R_{n-1}x_{n-1}R_{n-1}) &\leq \tau(R_{n-1}x_nR_{n-1}) \\ &= \tau(R_{n-1}x_n) \\ &= \tau(R_nx_n) + \tau(D_nx_n) + \tau(U_nx_n) \\ &= \tau(R_nx_nR_n) + \tau(D_nx_nD_n) + \tau(U_nx_nU_n). \end{aligned} \tag{3.3}$$

Now by Lemma 3.2 (iii), we have $\tau(U_nx_nU_n) \leq -\tau(U_n)$ and

$$\tau(D_nx_nD_n) \leq 2\tau(D_nx_nD_n) - \tau(D_n) \leq 2\tau(D_nx_ND_n) - \tau(D_n),$$

where in the last passage we have exploited the submartingale property. Putting all the above facts together, we see that we have proved that

$$\begin{aligned} \tau(R_{n-1}x_{n-1}R_{n-1}) - \tau(R_nx_nR_n) &\leq 2\tau(D_nx_ND_n) - \tau(D_n) - \tau(U_n) \\ &= 2\tau(D_nx_N) - \tau(R_{n-1} - R_n). \end{aligned}$$

Summing over all $1 \leq n \leq N$, we get

$$\tau(R_0x_0R_0) - \tau(R_Nx_NR_N) \leq 2\tau\left(\sum_{n=1}^{N} D_nx_N\right) - \tau(R_0 - R_N).$$

Adding to this estimate the trivial bounds $\tau(U_0x_0U_0) \leq -\tau(U_0)$ (which follows from part (iii)) and $\tau(D_0x_0D_0) \leq \tau(D_0x_ND_0) \leq 2\tau(D_0x_ND_0)$ (which is due to the submartingale property and the fact that $\tau(D_0x_0D_0) \geq 0$), we obtain that

$$\begin{aligned}\tau(x_0) &= \tau(R_0x_0R_0) + \tau(U_0x_0U_0) + \tau(D_0x_0D_0)\\ &\leq \tau\left(\left(R_N + 2\sum_{n=0}^N D_n\right)x_N\right) - \tau(R_0 - R_N) - \tau(U_0)\\ &\leq \tau\left(\left(R_N + 2\sum_{n=0}^N D_n\right)x_N^+\right) - \tau(I - R_N) \leq 2\tau(x_N^+) - \tau(I - R_N).\end{aligned}$$

This is precisely the claim. □

We will also need the following further properties of the projections $(R_n)_{n\geq -1}$.

**Lemma 3.4.** *Let $x = (x_n)_{n\geq 0}$ be an $L^1$-bounded submartingale. Then for any nonnegative integer $N$ we have*

$$\sum_{n=0}^N \tau(R_n dx_n R_{n-1} dx_n) \leq 4\tau(x_N^+) - 2\tau(x_0).$$

*Proof.* Let us first study a single summand of the above sum, corresponding to some $n \geq 1$. We have

$$\begin{aligned}\tau(R_n dx_n R_{n-1} dx_n) &= \tau(R_n(x_n - x_{n-1})R_{n-1}(x_n - x_{n-1}))\\ &= \tau(R_nx_nR_{n-1}x_n) + \tau(R_nx_{n-1}R_{n-1}x_{n-1})\\ &\quad - \tau(R_nx_nR_{n-1}x_{n-1}) - \tau(R_nx_{n-1}R_{n-1}x_n).\end{aligned}$$

The last two summands are equal, by the tracial property and the fact that $R_n$ commutes with $R_{n-1}x_nR_{n-1}$ (which implies $R_nx_nR_{n-1} = R_{n-1}x_nR_n$). This commuting property of $R_n$ implies also that $\tau(R_nx_nR_{n-1}x_n) = \tau(R_nx_nR_nx_n)$. Furthermore, because of the traciality of $\tau$ and the inequality $R_{n-1} \geq R_n$, we see that

$$\tau(R_nx_{n-1}R_{n-1}x_{n-1}) = \tau(R_{n-1}x_{n-1}R_nx_{n-1}R_{n-1}) \leq \tau(R_{n-1}x_{n-1}R_{n-1}x_{n-1}R_{n-1})$$

and hence we may write

$$\begin{aligned}\tau(R_ndx_nR_{n-1}dx_n) &\leq \tau(R_nx_nR_nx_n) - \tau(R_{n-1}x_{n-1}R_{n-1}x_{n-1})\\ &\quad + 2\tau\big(R_{n-1}x_{n-1}R_{n-1}(R_{n-1}x_{n-1}R_{n-1} - R_nx_nR_n)\big).\end{aligned} \tag{3.4}$$

Let us handle the latter expression. Since $R_nx_nR_n = R_{n-1}x_nR_n$, we have the splitting

$$\begin{aligned}&\tau\big(R_{n-1}x_{n-1}R_{n-1}(R_{n-1}x_{n-1}R_{n-1} - R_nx_nR_n)\big)\\ &= \tau\big(R_{n-1}x_{n-1}R_{n-1}(x_{n-1} - x_n)R_{n-1}\big)\\ &\quad + \tau\big(R_{n-1}x_{n-1}R_{n-1}x_n(R_{n-1} - R_n)\big).\end{aligned} \tag{3.5}$$

We have $x_{n-1} - x_n = -dx_n$, so by the properties of conditional expectation

$$\tau\big(R_{n-1}x_{n-1}R_{n-1}(x_{n-1} - x_n)R_{n-1}\big) = \tau\big(R_{n-1}x_{n-1}R_{n-1}(-\mathcal{E}_{n-1}(dx_n))\big).$$

Lemma 3.2 (iii) gives $R_{n-1}x_{n-1}R_{n-1} \geq -R_{n-1}$; furthermore, $x$ is a submartingale, so $\mathcal{E}_{n-1}(dx_n) \geq 0$. These two observations imply that the above expression does not exceed

$$\begin{aligned}\tau(R_{n-1}\mathcal{E}_{n-1}(dx_n)) &= \tau(R_{n-1}dx_nR_{n-1})\\ &= \tau(R_{n-1}x_nR_{n-1}) - \tau(R_{n-1}x_{n-1}R_{n-1})\\ &= \tau(R_nx_n) + \tau(U_nx_n) + \tau(D_nx_n) - \tau(R_{n-1}x_{n-1})\\ &\leq \tau(R_nx_n) + \tau(U_nx_n) + \tau(D_nx_N) - \tau(R_{n-1}x_{n-1}),\end{aligned} \tag{3.6}$$

where in the last line we have exploited the submartingale property of $x$. Let us now analyze the second term on the right-hand side of (3.5). We have $R_{n-1} - R_n = U_n + D_n$, so by the commuting properties of $U$ and $D$ described in Lemma 3.2, we obtain

$$\begin{aligned}&\tau\big(R_{n-1}x_{n-1}R_{n-1}x_n(R_{n-1} - R_n)\big)\\ &= \tau\big(R_{n-1}x_{n-1}R_{n-1}D_nx_nD_n)\big) + \tau\big(R_{n-1}x_{n-1}R_{n-1}U_nx_nU_n)\big).\end{aligned}$$

However, the operator $D_nx_nD_n$ is nonnegative, while $U_nx_nU_n$ is nonpositive; furthermore, we have $-R_{n-1} \leq R_{n-1}x_{n-1}R_{n-1} \leq R_{n-1}$. Consequently, the above expression does not exceed

$$\tau(D_nx_nD_n) - \tau(U_nx_nU_n) \leq \tau(D_nx_N) - \tau(U_nx_n),$$

where the last passage is due to the fact that $x$ is a submartingale. Plugging the above observations into (3.5), we get

$$\begin{aligned}&\tau\big(R_{n-1}x_{n-1}R_{n-1}(R_{n-1}x_{n-1}R_{n-1} - R_nx_nR_n)\big)\\ &\leq \tau(R_nx_n) - \tau(R_{n-1}x_{n-1}) + 2\tau(D_nx_N)\end{aligned}$$

and hence, returning to (3.4), we have shown that

$$\begin{aligned}\tau(R_ndx_nR_{n-1}dx_n) \leq \tau(R_nx_nR_nx_n) &- \tau(R_{n-1}x_{n-1}R_{n-1}x_{n-1})\\ &+ 2\big(\tau(R_nx_n) - \tau(R_{n-1}x_{n-1})\big) + 4\tau(D_nx_N).\end{aligned}$$

Consequently, using the equality $\tau(R_0dx_0R_{-1}dx_0) = \tau(R_0x_0R_0x_0)$, we get

$$\begin{aligned}&\sum_{n=0}^{N}\tau(R_ndx_nR_{n-1}dx_n)\\ &= \tau(R_0x_0R_{-1}x_0) + \sum_{n=1}^{N}\tau(R_ndx_nR_{n-1}dx_n)\\ &\leq \tau(R_Nx_NR_Nx_N) + 2\tau(R_Nx_N) - 2\tau(R_0x_0) + 4\tau\left(\sum_{n=1}^{N}D_nx_N\right).\end{aligned} \tag{3.7}$$

It remains to apply some final estimates. The operator $R_N(x_N + I)R_N$ is nonnegative and does not exceed $2R_N$ (see Lemma 3.2 (iii)), so

$$\begin{aligned}\tau(R_Nx_NR_Nx_N) + \tau(R_Nx_N) &= \tau(R_Nx_NR_N(x_N + I)R_N)\\ &\leq \tau(R_Nx_N^+R_N(x_N + I)R_N)\\ &\leq 2\tau(R_Nx_N^+R_N).\end{aligned} \tag{3.8}$$

Furthermore, $U_0x_0U_0$ is nonpositive, so using the submartingale property,

$$(3.9)\quad \begin{aligned} &\tau(R_Nx_N) - 2\tau(R_0x_0) + 4\tau\left(\sum_{n=1}^N D_nx_N\right) \\ &= -2\tau(x_0) + 2\tau(D_0x_0) + 2\tau(U_0x_0) + \tau\left(\left(R_N + 4\sum_{n=1}^N D_n\right)x_N\right) \\ &\leq -2\tau(x_0) + \tau\left(\left(R_N + 2D_0 + 4\sum_{n=1}^N D_n\right)x_N\right) \\ &\leq -2\tau(x_0) + \tau\left(\left(R_N + 4\sum_{n=0}^N D_n\right)x_N^+\right). \end{aligned}$$

Combining the estimates (3.7), (3.8) and (3.9), we obtain the desired result. □

We conclude the analysis of $(R_n)_{n\geq 0}$ by the following statement.

**Lemma 3.5.** *Let $x = (x_n)_{n\geq 0}$ be an $L^1$-bounded submartingale. Then for any nonnegative integer $N$ we have*

$$(3.10)\quad \left\|\sum_{n=0}^N \Big(|(R_{n-1} - R_n)dx_n(R_{n-1} - R_n)| + R_{n-1}\mathcal{E}_{n-1}(dx_n)R_{n-1}\Big)\right\|_1 \leq 4\tau(x_N^+) - 2\tau(x_0).$$

*(we interpret $\mathcal{E}_{-1}a = 0$ for each $a \in L^1$).*

*Proof.* Let us analyze a single summand of the above sum (note that each such summand is nonnegative). First we take a look at summands corresponding to $n \geq 1$. The trace of the term $R_{n-1}\mathcal{E}_{n-1}(dx_n)R_{n-1}$ can be handled as in (3.6). To deal with the reamining part, we apply the triangle inequality to get

$$\begin{aligned} &\tau(|(R_{n-1} - R_n)dx_n(R_{n-1} - R_n)|) \\ &\leq \tau\Big(|(R_{n-1} - R_n)x_n(R_{n-1} - R_n)| + |(R_{n-1} - R_n)x_{n-1}(R_{n-1} - R_n)|\Big). \end{aligned}$$

Now, by the commuting properties of $U$ and $D$, for any $n \geq 1$ we have

$$(R_{n-1} - R_n)x_n(R_{n-1} - R_n) = D_nx_nD_n + U_nx_nU_n.$$

The first term on the right is nonnegative, while the second is nonpositive, so

$$\tau(|(R_{n-1} - R_n)x_n(R_{n-1} - R_n)|) \leq \tau(D_nx_n) - \tau(U_nx_n) \leq \tau(D_nx_N) - \tau(U_nx_n),$$

where in the last passage we have exploited the submartingale property. Next, we have the estimate

$$|(R_{n-1} - R_n)x_{n-1}(R_{n-1} - R_n)| \leq R_{n-1} - R_n,$$

directly from Lemma 3.2 (iii). Thus, combining the above observations, we have shown that if $n \geq 1$, then

$$\begin{aligned} &\tau\Big(|(R_{n-1} - R_n)dx_n(R_{n-1} - R_n)| + R_{n-1}\mathcal{E}_{n-1}(dx_n)R_{n-1}\Big) \\ &\qquad \leq \tau(R_nx_n) - \tau(R_{n-1}x_{n-1}) + 2\tau(D_nx_N) + \tau(R_{n-1} - R_n). \end{aligned}$$

A similar reasoning to that above yields also the appropriate upper bound for the first term in (3.10):

$$\tau(|(I-R_0)dx_0(I-R_0)|)=\tau(|(I-R_0)x_0(I-R_0)|)\leq\tau(D_0x_N)-\tau(U_0x_0).$$

Summing over $n$, it follows from the fact $\tau(D_0x_0D_0)\leq\tau(D_0x_ND_0)$ (which is due to the submartingale property) that

$$\begin{aligned}
&\left\|\sum_{n=0}^{N}\Big(|(R_{n-1}-R_n)dx_n(R_{n-1}-R_n)|+R_{n-1}\mathcal{E}_{n-1}(dx_n)R_{n-1}\Big)\right\|_1\\
&=\sum_{n=0}^{N}\tau\Big(|(R_{n-1}-R_n)dx_n(R_{n-1}-R_n)|+R_{n-1}\mathcal{E}_{n-1}(dx_n)R_{n-1}\Big)\\
&=\tau(D_0x_N)-\tau(U_0x_0)\\
&\quad+\tau(R_Nx_N)-\tau(R_0x_0)+2\tau\left(\left(\sum_{n=1}^{N}D_n\right)x_N\right)+\tau(R_0-R_N)\\
&=\tau\left(\left(R_N+D_0+2\sum_{n=1}^{N}D_n\right)x_N\right)-\tau(x_0)+\tau(D_0x_0)+\tau(R_0-R_N)\\
&\leq\tau\left(\left(R_N+2\sum_{n=0}^{N}D_n\right)x_N\right)-\tau(x_0)+\tau(R_0-R_N)\\
&\leq 2\tau(x_N^+)-\tau(x_0)+\tau(I-R_N).
\end{aligned}$$

It remains to apply Lemma 3.2 (iv) to get the claim. □

The next step of our analysis is to introduce yet another families of projections $(S_n)_{n\geq-1}$, $(Q_n)_{n\geq0}$, $(T_n)_{n\geq0}$, this time associated with the dominated process $y$. Set $S_{-1}=I$ and for $n\geq0$, by induction,

$$\begin{aligned}
S_n&=S_{n-1}I_{(-1,1)}\left(S_{n-1}\left(\sum_{k=0}^{n}R_{k-1}dy_kR_{k-1}\right)S_{n-1}\right),\\
Q_n&=I_{[1,\infty)}\left(S_{n-1}\left(\sum_{k=0}^{n}R_{k-1}dy_kR_{k-1}\right)S_{n-1}\right),\\
T_n&=I_{(-\infty,-1]}\left(S_{n-1}\left(\sum_{k=0}^{n}R_{k-1}dy_kR_{k-1}\right)S_{n-1}\right).
\end{aligned}$$

The crucial property of $(S_n)_{n\geq-1}$ is described in the next lemma. Before we proceed, let us introduce a Gundy-type decomposition for $y$:

$$dy_n=d\alpha_n+d\beta_n+d\gamma_n+d\delta_n,$$

where

$$\begin{aligned}
d\alpha_n&=R_{n-1}dy_nR_n+R_ndy_nR_{n-1}-R_ndy_nR_n\\
&\quad-\mathcal{E}_{n-1}(R_{n-1}dy_nR_n+R_ndy_nR_{n-1}-R_ndy_nR_n),\\
d\beta_n&=\mathcal{E}_{n-1}(R_{n-1}dy_nR_n+R_ndy_nR_{n-1}-R_ndy_nR_n),\\
d\gamma_n&=R_ndy_n(I-R_{n-1}),\\
d\delta_n&=(I-R_n)dy_n-(R_{n-1}-R_n)dy_nR_n.
\end{aligned}$$

**Lemma 3.6.** *For any integer $N \geq -1$ we have*

$$\tau(I - S_N) \leq 216\tau(x_N^+) - 108\tau(x_0).$$

*Proof.* For the sake of clarity, it is convenient to split the quite lengthy reasoning into a few intermediate parts.

*Step 1 (Preliminary observations).* We start with the identity

$$\tau(I - S_N) = \sum_{n=0}^{N} \tau(S_{n-1} - S_n) = \sum_{n=0}^{N} \big(\tau(Q_n) + \tau(T_n)\big). \tag{3.11}$$

Now by the very definition of $Q_n$ we have

$$\begin{aligned} \tau(Q_n) &= \tau\left(Q_n\left(\sum_{k=0}^{n} R_{k-1}dy_kR_{k-1}\right)Q_n \geq 1\right) \\ &= \tau\left(Q_n\left(\sum_{k=0}^{n} R_{k-1}(d\alpha_k + d\beta_k + d\gamma_k + d\delta_k)R_{k-1}\right)Q_n \geq 1\right). \end{aligned} \tag{3.12}$$

Note that $R_{k-1}d\gamma_kR_{k-1} = 0$, $R_{k-1}d\alpha_kR_{k-1} = d\alpha_k$, $R_{k-1}d\beta_kR_{k-1} = d\beta_k$ and $R_{k-1}d\delta_kR_{k-1} = (R_{k-1} - R_k)dy_k(R_{k-1} - R_k)$. Consequently, the above expression is not bigger than

(3.13)
$$\begin{aligned} &\tau\left(Q_n\alpha_nQ_n \geq 1/3\right) + \tau\left(Q_n\beta_nQ_n \geq 1/3\right) \\ &\qquad + \tau\left(Q_n\left(\sum_{k=0}^{n}(R_{k-1} - R_k)dy_k(R_{k-1} - R_k)\right)Q_n \geq 1/3\right). \end{aligned}$$

We will treat each of these three summands separately.

*Step 2 (Bound for the summand involving $\alpha$).* By Chebyshev's inequality and the fact that $\alpha$ is an $L^2$-bounded martingale, we obtain

$$\tau(Q_n\alpha_nQ_n \geq 1/3) \leq 9\tau((Q_n\alpha_nQ_n)^2) \leq 9\tau(Q_n\alpha_n^2Q_n) \leq 9\tau(Q_n\alpha_N^2).$$

Hence, summing over $n$ and using the fact that the sum of $Q_n$'s is not bigger than $I$, we get

$$\sum_{n=0}^{N} \tau(Q_n\alpha_nQ_n \geq 1/3) \leq 9\sum_{n=0}^{N} \tau(Q_n\alpha_N^2) \leq 9\tau(\alpha_N^2) = 9\sum_{n=0}^{N} \tau(d\alpha_n^2). \tag{3.14}$$

Directly from the definition of $d\alpha_n$ we infer that

$$\begin{aligned} \tau(d\alpha_n^2) &\leq \tau((R_{n-1}dy_nR_n + R_ndy_nR_{n-1} - R_ndy_nR_n)^2) \\ &= 2\tau(R_ndy_nR_{n-1}dy_n) - \tau(R_ndy_nR_ndy_n) \leq 2\tau(R_{n-1}dy_nR_ndy_n). \end{aligned} \tag{3.15}$$

Applying the differential subordination of $y$ to $x$, we obtain

$$R_{n-1}dy_nR_{n-1}dy_nR_{n-1} \leq R_{n-1}dx_nR_{n-1}dx_nR_{n-1}$$

and hence also $R_ndy_nR_{n-1}dy_nR_n \leq R_ndx_nR_{n-1}dx_nR_n$, since $R_n \leq R_{n-1}$. Passing to the trace, this implies

$$\tau(R_{n-1}dy_nR_ndy_n) = \tau(R_ndy_nR_{n-1}dy_nR_n) \leq \tau(R_ndx_nR_{n-1}dx_nR_n).$$

Plugging this into (3.15) and then returning to (3.14), we obtain

$$\sum_{n=0}^{N} \tau(Q_n\alpha_n Q_n \geq 1/3) \leq 18\tau\left(\sum_{n=0}^{N} R_n dx_n R_{n-1} dx_n R_n\right) \leq 72\tau(x_N^+) - 36\tau(x_0), \tag{3.16}$$

where in the last line we exploited the estimate of Lemma 3.4.

*Step 3 (Bound for the term involving $\beta$).* Let us first find an appropriate upper bound for $d\beta_n$. We have

$$\begin{aligned} d\beta_n &= \mathcal{E}_{n-1}(-R_{n-1}dy_nR_{n-1} + R_{n-1}dy_nR_n + R_ndy_nR_{n-1} - R_ndy_nR_n) \\ &\quad + R_{n-1}\mathcal{E}_{n-1}(dy_n)R_{n-1} \\ &= -\mathcal{E}_{n-1}((R_{n-1}-R_n)dy_n(R_{n-1}-R_n)) + R_{n-1}\mathcal{E}_{n-1}(dy_n)R_{n-1} \\ &\leq -\mathcal{E}_{n-1}((R_{n-1}-R_n)dy_n(R_{n-1}-R_n)) + R_{n-1}\mathcal{E}_{n-1}(dx_n)R_{n-1}, \end{aligned}$$

where in the last line we have exploited the conditional differential subordination of $y$ to $x$. Next, by the differential subordination of $y$ to $x$, we have

$$R_{n-1}dy_nR_{n-1}dy_nR_{n-1} \leq R_{n-1}dx_nR_{n-1}dx_nR_{n-1},$$

so

$$\begin{aligned} &(R_{n-1}-R_n)dy_n(R_{n-1}-R_n)dy_n(R_{n-1}-R_n) \\ &\leq (R_{n-1}-R_n)dy_nR_{n-1}dy_n(R_{n-1}-R_n) \\ &\leq (R_{n-1}-R_n)dx_nR_{n-1}dx_n(R_{n-1}-R_n) \\ &= (R_{n-1}-R_n)(x_n-x_{n-1})R_{n-1}(x_n-x_{n-1})(R_{n-1}-R_n) \\ &= (R_{n-1}-R_n)x_nR_{n-1}x_n(R_{n-1}-R_n) + (R_{n-1}-R_n)x_{n-1}R_{n-1}x_{n-1}(R_{n-1}-R_n) \\ &\quad - (R_{n-1}-R_n)x_{n-1}R_{n-1}x_n(R_{n-1}-R_n) - (R_{n-1}-R_n)x_nR_{n-1}x_{n-1}(R_{n-1}-R_n). \end{aligned}$$

Since $R_n$ commutes with $R_{n-1}x_nR_{n-1}$, the above sum is equal to

$$\begin{aligned} &(R_{n-1}-R_n)x_n(R_{n-1}-R_n)x_n(R_{n-1}-R_n) \\ &+ (R_{n-1}-R_n)x_{n-1}R_{n-1}x_{n-1}(R_{n-1}-R_n) \\ &- (R_{n-1}-R_n)x_{n-1}(R_{n-1}-R_n)x_n(R_{n-1}-R_n) \\ &- (R_{n-1}-R_n)x_n(R_{n-1}-R_n)x_{n-1}(R_{n-1}-R_n) \end{aligned}$$

(note that the second summand has not changed), which can be further transformed into

$$\begin{aligned} &(R_{n-1}-R_n)dx_n(R_{n-1}-R_n)dx_n(R_{n-1}-R_n) \\ &+ (R_{n-1}-R_n)x_{n-1}R_nx_{n-1}(R_{n-1}-R_n). \end{aligned} \tag{3.17}$$

Let us handle the second term in the latter expression. Since $R_n \leq R_{n-1}$, we have

$$\begin{aligned} &(R_{n-1}-R_n)x_{n-1}R_nx_{n-1}(R_{n-1}-R_n) \\ &\leq (R_{n-1}-R_n)x_{n-1}R_{n-1}x_{n-1}(R_{n-1}-R_n). \end{aligned}$$

This is not bigger than $R_{n-1}-R_n$. Indeed, by Lemma 3.2 (iii), we have the estimate $R_{n-1}x_{n-1}R_{n-1} \leq R_{n-1}$, which yields $R_{n-1}x_{n-1}R_{n-1}x_{n-1}R_{n-1} \leq R_{n-1}$ and hence also the desired inequality. This enables us to bound the expression in (3.17) from above by a convenient square:

$$(|(R_{n-1}-R_n)dx_n(R_{n-1}-R_n)| + R_{n-1}-R_n)^2.$$

Putting all the above facts together, we conclude that

$$\begin{aligned}(R_{n-1}-R_n)dy_n(R_{n-1}-R_n)dy_n(R_{n-1}-R_n)\\ \leq (|(R_{n-1}-R_n)dx_n(R_{n-1}-R_n)|+R_{n-1}-R_n)^2,\end{aligned}$$

which implies

$$\begin{aligned}|(R_{n-1}-R_n)dy_n(R_{n-1}-R_n)|\\ \leq |(R_{n-1}-R_n)dx_n(R_{n-1}-R_n)|+R_{n-1}-R_n\end{aligned} \tag{3.18}$$

and hence

$$d\beta_n \leq \mathcal{E}_{n-1}\big(|(R_{n-1}-R_n)dx_n(R_{n-1}-R_n)|+R_{n-1}-R_n\big)+R_{n-1}\mathcal{E}_{n-1}(dx_n)R_{n-1}.$$

Denoting the right-hand side by $d\tilde{\beta}_n$, we see that $\beta_n \leq \tilde{\beta}_n$ and $\tilde{\beta}_n$ is nonnegative. Therefore, by Chebyshev's inequality, we get

$$\tau(Q_n\beta_nQ_n \geq 1/3) \leq \tau(Q_n\tilde{\beta}_nQ_n \geq 1/3) \leq 3\tau(Q_n\tilde{\beta}_nQ_n) \leq 3\tau(Q_n\tilde{\beta}_N)$$

and hence, summing over $n$, we arrive at

$$\begin{aligned}\sum_{n=0}^{N}\tau(Q_n\beta_nQ_n \geq 1/3) &\leq 3\tau(\tilde{\beta}_N) = 3\sum_{n=0}^{N}\tau(d\tilde{\beta}_n)\\ &\leq 12\tau(x_N^+)-6\tau(x_0)+3\tau(I-R_N)\\ &\leq 18\tau(x_N^+)-9\tau(x_0),\end{aligned}$$

where the last two estimates follow from Lemma 3.2 (iv) and Lemma 3.5.

*Step 4 (Bound for the term involving $\delta$).* All the crucial observations has been made in the previous step. First, by Chebyshev's inequality, we have

$$\begin{aligned}&\tau\left(Q_n\left(\sum_{k=0}^{n}(R_{k-1}-R_k)dy_k(R_{k-1}-R_k)\right)Q_n \geq 1/3\right)\\ &\leq \tau\left(Q_n\left(\sum_{k=0}^{n}|(R_{k-1}-R_k)dy_k(R_{k-1}-R_k)|\right)Q_n \geq 1/3\right)\\ &\leq 3\tau\left(Q_n\left(\sum_{k=0}^{n}|(R_{k-1}-R_k)dy_k(R_{k-1}-R_k)|\right)Q_n\right)\\ &\leq 3\tau\left(Q_n\left(\sum_{k=0}^{N}|(R_{k-1}-R_k)dy_k(R_{k-1}-R_k)|\right)Q_n\right).\end{aligned} \tag{3.19}$$

Summing over $n$, we obtain that

$$\begin{aligned}\sum_{n=0}^{N}\tau\left(Q_n\left(\sum_{k=0}^{N}(R_{k-1}-R_k)dy_k(R_{k-1}-R_k)\right)Q_n \geq 1/3\right)\\ \leq 3\tau\left(\sum_{k=0}^{N}|(R_{k-1}-R_k)dy_k(R_{k-1}-R_k)|\right).\end{aligned}$$

In the light of (3.18), this can be bounded from above by

$$3\tau(I-R_N)+3\tau\left(\sum_{k=0}^{N}|(R_{k-1}-R_k)dx_k(R_{k-1}-R_k)|\right).$$

The first trace can be handled with the use of Lemma 3.2 (iv). The second trace is not bigger than the left-hand side of (3.10), by the submartingale property of $x$. Combining these observations with (3.19), we finally obtain

$$\tau\left(Q_n\left(\sum_{k=0}^{n}(R_{k-1}-R_k)dy_k(R_{k-1}-R_k)\right)Q_n\geq 1/3\right)\leq 18\tau(x_N^+)-9\tau(x_0).$$

*Step 5 (Conclusion).* Having completed the analysis of the three terms in (3.13), we combine it with (3.12) and obtain

$$\sum_{n=0}^{N}\tau(Q_n)\leq 108\tau(x_N^+)-54\tau(x_0).$$

The same analysis (or simply the replacement of $y$ with $-y$ in all relevant places) gives the corresponding bound for the projections $T$:

$$\sum_{n=0}^{N}\tau(T_n)\leq 108\tau(x_N^+)-54\tau(x_0).$$

Summing the last two estimates yields the claim, by virtue of (3.11). □

We are ready for the proof of the weak-type estimate.

*Proof of Theorem 3.1.* The desired projection $q$ is given by the intersection

$$q=\bigwedge_{n=-1}^{\infty}S_n\wedge\bigwedge_{n=-1}^{\infty}R_n.$$

To show that $-q\leq qy_nq\leq q$ for each $n$, fix a vector $\xi\in q(H)$. Then $\xi\in S_n(H)$ and $\xi\in R_{k-1}(H)$ for each $k\leq n$, so

$$\begin{aligned}|\langle y_n\xi,\xi\rangle| &= \left|\sum_{k=0}^{n}\langle dy_kR_{k-1}S_n\xi,R_{k-1}S_n\xi\rangle\right|\\ &=\left|\left\langle S_n\left(\sum_{k=0}^{n}R_{k-1}dy_nR_{k-1}\right)S_n\xi,\xi\right\rangle\right|<||\xi||^2,\end{aligned}$$

where the last estimate follows from the very definition of $S_n$. It remains to show the upper bound for $\tau(I-q)$. This is an immediate consequence of Lemma 3.2 (iv) and Lemma 3.6: indeed, we have

$$\tau(I-S_N\wedge R_N)\leq\tau(I-S_N)+\tau(I-R_N)\leq 218\tau(x_N^+)-109\tau(x_0)\leq 327||x||_1.$$

Letting $N\to\infty$ completes the proof. □

We conclude this section by a simple, yet important application of the above weak-type bound, which serves as a motivation for our further considerations.

**Theorem 3.7.** *Suppose that $\tau(I)=1$ and $0<p<1$. If $x=(x_n)_{n\geq 0}$ is an arbitrary submartingale and $y$ is strongly differentially subordinate to $x$, then*

$$\|y\|_p\leq\frac{654}{(1-p)^{-1}}\|x\|_1.$$

*The order $(1-p)^{-1}$ is the best possible, as it is already optimal in the case when $x$ is assumed to be a classical martingale.*

*Proof.* We will first show that for any nonnegative integer $N$ and any positive number $t$ we have

$$t\tau(|y_N| > t) \leq 654\|x\|_1. \tag{3.20}$$

By homogeneity, we may assume that $t = 1$. By Theorem 3.1, there is a projection $q$ such that $-q \leq qy_Nq \leq q$ and $\tau(I-q) \leq 327\|x\|_1$. Observe that the projection $I_{(1,\infty)}(y_N)$ is equivalent to a subprojection of $I-q$. Indeed, if a vector $\xi$ belongs to $q(H)$, then $\langle y_N\xi, \xi\rangle = \langle qy_Nq\xi, \xi\rangle \leq \|\xi\|^2$ and hence this vector cannot lie in $I_{(1,\infty)}(y_N)(H)$, unless $\xi = 0$. This gives the equivalence and implies the inequality

$$\tau(y_N > 1) \leq \tau(I-q) \leq 327\|x\|_1.$$

We prove analogously the symmetric estimate $\tau(y_N < -1) \leq 327\|x\|_1$ and hence (3.20) follows. Consequently, if we set $a = 654\|x\|_1$, then

$$\begin{aligned}
\|y_N\|_p^p &= p\int_0^\infty t^{p-1}\tau(|y_N| > t)\mathrm{d}t \\
&= p\int_0^a t^{p-1}\tau(|y_N| > t)\mathrm{d}t + p\int_a^\infty t^{p-1}\tau(|y_N| > t)\mathrm{d}t \\
&\leq p\int_0^a t^{p-1}\mathrm{d}t + p\int_a^\infty t^{p-2}a\mathrm{d}t \\
&= \frac{a^p}{1-p}.
\end{aligned}$$

Since $N$ was arbitrary, the proof is complete. □

## 4. Moment estimates for $1 < p < \infty$

The primary goal of this section is to prove the strong type $(p,p)$ inequality $(1 < p < \infty)$ for noncommutative submartingales and their strong differential subordinates. As we mentioned in the introduction, this inequality fails in general even in the classical case, and to overcome this problem one imposes the additional sign assumption on the dominating process. We will proceed similarly and assume, throughout this section, that $x = (x_n)_{n\geq 0}$ is a nonnegative submartingale. We shall establish the following noncommutative version of (1.2).

**Theorem 4.1.** *For any $1 < p < \infty$ there is a finite constant $C_p$ depending only on $p$ such that the following holds: if $y$ is strongly differentially subordinate to $x$ (i.e., $y$ and $x$ satisfy the conditions (DS)+(CDS)), then*

$$\|y_N\|_p \leq C_p\|x_N\|_p, \qquad N = 0, 1, 2, \ldots. \tag{4.1}$$

*Furthermore, if $p \geq 2$, the same inequality holds true under the "weaker" strong differential subordination (WDS)+(CDS) of Section 2.*

**Remark 4.2.** Our proof will give $C_p$ of orders $O((p-1)^{-1})$ as $p \to 1+$ and $O(p^4)$ as $p \to \infty$. The first order is optimal, as it is already the best possible in the commutative setting. Unfortunately, the second order does not seem to be the optimal, and our guess is that the best order is $O(p^2)$, the same as in the noncommutative Doob's inequality. The reason for this conjecture is the following. Let $a = (a_n)_{n\geq 0}$ be a sequence of positive operators adapted to some filtration $(\mathcal{M}_n)_{n\geq 0}$. Consider the "extended" filtration $\widetilde{\mathcal{M}} = (\mathcal{M}_0, \mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_2, \ldots)$, in which each algebra $\mathcal{M}_k$ appears twice. Then the process $x = (x_n)_{n\geq 0}$ with the difference defined by $dx_{2n} = \mathcal{E}_{n-1}a_n$ and $dx_{2n+1} = a_n - \mathcal{E}_{n-1}(a_n)$, $n = 0, 1, 2, \ldots$, is a nonnegative submartingale with respect to $\widetilde{\mathcal{M}}$. Furthermore, the sequence $y = (y_n)_{n\geq 0}$ defined by $dy_{2n} = \mathcal{E}_{n-1}a_n$ and $dy_{2n+1} = 0$, is strongly differentially subordinate to $x$ (no matter which domination - (DS)+(CDS) or (WDS)+(CDS) - we consider). Therefore, (4.1) yields

$$\left\| \sum_{n=0}^{N} \mathcal{E}_{n-1}(a_n) \right\|_p \leq C_p \left\| \sum_{n=0}^{N} a_n \right\|_p ,$$

for $N = 0, 1, 2, \ldots$. If we drop the adaptedness assumption on the sequence $a = (a_n)_{n\geq 0}$, the above estimate is just the noncommutative Doob's $L^p$ inequality established by Junge in [11]. As Junge and Xu showed in [13], the optimal order of the constant in this inequality, as $p \to \infty$, is $O(p^2)$. Thus, we believe that the optimal order of the constant in (4.1) should also be quadratic.

The proof of the $L^p$ bound between $x$ and $y$ will involve the Doob-Meyer decompositions of these sequences, which we briefly recall. For any $n \geq 0$, we may write

$$dx_n = du_n + da_n, \qquad dy_n = dv_n + db_n,$$

where $du_n = dx_n - \mathcal{E}_{n-1}(dx_n)$, $da_n = \mathcal{E}_{n-1}(dx_n)$ and, similarly, $dv_n = dy_n - \mathcal{E}_{n-1}(dy_n)$, $db_n = \mathcal{E}_{n-1}(dy_n)$. Note that $du$ and $dv$ are martingale differences, while $da$, $db$ are predictable processes and $da$ consists of nonnegative operators. The $L^p$ bound for $y$ will be obtained by providing appropriate estimates for $\|v\|_p$ and $\|b\|_p$. As we have already mentioned in the introductory section, the analysis in the cases $1 < p \leq 2$ and $p > 2$ will be quite different; we have decided to split the remaining part of this section accordingly.

### 4.1. The case $1 < p \leq 2$.

We start with the estimate for the finite variation term $\|b_N\|_p$. It is quite interesting to note that the lemma below provides the *best* constants, even in the commutative case (see Wang [31]).

**Lemma 4.3.** *For $1 \leq p \leq 2$, we have $\|b_N\|_p \leq p\|x_N\|_p$.*

*Proof.* By the conditional differential subordination (i.e., the condition (CDS)), we have $-a_n \leq b_n \leq a_n$ and hence $||b_N||_p \leq \|a_N\|_p$ for each $N$. Therefore, it suffices to prove the bound $\|a_N\|_p \leq p\|x_N\|_p$. To this end, consider the sequence

$$w_n = a_n^p - pa_n^{p-1}x_n, \qquad n = 0, 1, 2, \ldots, N.$$

We will prove that $w_n$ is trace-decreasing. Recall that $(a_n)_{n\geq 0}$ is predictable, so for any $n \geq 0$,

$$\begin{aligned}\tau\left(a_{n+1}^p - pa_{n+1}^{p-1}x_{n+1}\right) &= \tau\left(\mathcal{E}_n(a_{n+1}^p - pa_{n+1}^{p-1}x_{n+1})\right) \\ &= \tau\left(a_{n+1}^p - pa_{n+1}^{p-1}\mathcal{E}_n(x_{n+1})\right) \\ &= \tau\left(a_{n+1}^p - pa_{n+1}^{p-1}(x_n + da_{n+1})\right).\end{aligned}$$

We have $a_n \leq a_{n+1}$ and $1 \leq p \leq 2$: therefore, $a_n^{p-1} \leq a_{n+1}^{p-1}$ and

$$\tau\left(a_{n+1}^{p-1}x_n\right) = \tau\left(x_n^{1/2}a_{n+1}^{p-1}x_n^{1/2}\right) \geq \tau\left(x_n^{1/2}a_n^{p-1}x_n^{1/2}\right) = \tau\left(a_n^{p-1}x_n\right).$$

Furthermore, by Young's inequality,

$$\tau\left(a_{n+1}^{p-1}a_n\right) \leq \frac{p-1}{p}\tau\left(a_{n+1}^p\right) + \frac{1}{p}\tau\left(a_n^p\right),$$

which is equivalent to

$$\tau\left(a_{n+1}^p\right) \leq \tau\left(a_n^p\right) + p\tau\left(a_{n+1}^{p-1}(a_{n+1}-a_n)\right) = \tau\left(a_n^p\right) + p\tau\left(a_{n+1}^{p-1}da_{n+1}\right).$$

Putting these observations above, we get

$$\tau(w_{n+1}) = \tau\left(a_{n+1}^p - pa_{n+1}^{p-1}x_{n+1}\right) \leq \tau\left(a_n^p - pa_n^{p-1}x_n\right) = \tau(w_n),$$

as we have claimed. Consequently, we have $\tau(w_N) \leq \tau(w_0) = 0$. Using Young's inequality, we get

$$p\tau\left(a_N^{p-1}x_N\right) = \tau\left(a_N^{p-1}(px_N)\right) \leq \frac{p-1}{p}\tau\left(a_N^p\right) + \frac{1}{p}\tau\big((px_N)^p\big),$$

or, equivalently,

$$\tau(w_N) \geq \frac{1}{p}\tau\left(a_N^p - p^p x_N^p\right).$$

Combining this with $\tau(w_N) \leq 0$, we obtain that $\|a_N\|_p \leq p\|x_N\|_p$. The proof is complete. □

*Proof of Theorem 4.1 for $p = 2$.* As we have just shown above, we have $\|b_N\|_2 \leq 2\|x_N\|_2$, so it remains to provide an $L^2$ bound for $v$. This sequence is a martingale, so, by properties of conditional expectations,

$$\begin{aligned}\|v_N\|_2^2 = \sum_{n=0}^N \tau(dv_n^2) &= \sum_{n=0}^N \tau\Big(\big(dy_n - \mathcal{E}_{n-1}(dy_n)\big)^2\Big)\\ &= \sum_{n=0}^N \tau\Big(dy_n^2 - \big(\mathcal{E}_{n-1}(dy_n)\big)^2\Big) \leq \sum_{n=0}^N \tau\left(dy_n^2\right).\end{aligned}$$

Furthermore, since $x$ is a nonnegative submartingale,

$$\tau\left(x_n^2\right) = \tau\left(dx_n^2 + 2x_{n-1}dx_n + x_{n-1}^2\right) \geq \tau\left(dx_n^2 + x_{n-1}^2\right).$$

Hence,

$$\sum_{n=0}^N \tau(dx_n^2) \leq \tau(x_N^2).$$

It remains to use the "weak" differential subordination of $y$ to $x$ (that is, the condition (WDS)) to obtain $\|v_N\|_2 \leq \|x_N\|_2$. This implies $\|y_N\|_2 \leq \|v_N\|_2 + \|b_N\|_2 \leq 3\|x_N\|_2$ and completes the proof. □

For $1 < p < 2$ the analysis of $\|v_N\|_p$ will be more elaborate and will exploit the projections $(R_n)_{n\geq 0}$ studied in the previous sections. Since $x$ is nonnegative, these objects are given by the recursive formula $R_{-1} = 0$ and $R_n = R_{n-1}I_{[0,1)}(R_{n-1}x_nR_{n-1})$ for $n \geq 0$. We will require appropriate versions of lemmas studied in Section 3, taking into account that $x$ is nonnegative. We start with the following version of property (iv) of Lemma 3.2.

**Lemma 4.4.** *For any $N \geq 0$ we have $\tau(I - R_N) \leq \tau\big((I - R_N)x_N\big)$.*

*Proof.* We repeat the argument of Cuculescu [9]: for any $n \geq 0$, by the very definition of $R_n$ and the submartingale property of $x$,

$$\begin{aligned}\tau(R_{n-1}-R_n) &\leq \tau\big((R_{n-1}-R_n)x_n(R_{n-1}-R_n)\big)\\ &\leq \tau\big((R_{n-1}-R_n)x_N(R_{n-1}-R_n)\big)=\tau\big((R_{n-1}-R_n)x_N\big).\end{aligned}$$

Summing the estimate over $n = 0, 1, 2, \ldots, N$ we get the claim. □

We also need the following version of Lemma 3.4.

**Lemma 4.5.** *For any nonnegative integer $N$ we have*

$$\sum_{n=0}^{N}\tau(R_n dx_n R_{n-1}dx_n)\leq \tau(R_N x_N R_N x_N)+2\tau\big((I-R_N)x_N\big).$$

*Proof.* Arguing as the proof of Lemma 3.4, we show that

$$\begin{aligned}\tau(R_n dx_n R_{n-1}dx_n) &\leq \tau(R_n x_n R_n x_n)-\tau(R_{n-1}x_{n-1}R_{n-1}x_{n-1})\\ &\quad+2\tau\big(R_{n-1}x_{n-1}R_{n-1}(R_{n-1}x_{n-1}R_{n-1}-R_n x_n R_n)\big)\end{aligned}\tag{4.2}$$

(this is (3.4)). We will now analyze the last trace. Since $x$ is a nonnegative submartingale, we have

$$\begin{aligned}&\tau\big(R_{n-1}x_{n-1}R_{n-1}x_{n-1}R_{n-1}\big)\\ &\qquad=\tau\big((R_{n-1}x_{n-1}R_{n-1})^{1/2}x_{n-1}(R_{n-1}x_{n-1}R_{n-1})^{1/2}\big)\\ &\qquad\leq \tau\big((R_{n-1}x_{n-1}R_{n-1})^{1/2}\mathcal{E}_{n-1}(x_n)(R_{n-1}x_{n-1}R_{n-1})^{1/2}\big)\\ &\qquad=\tau\big(R_{n-1}x_{n-1}R_{n-1}x_nR_{n-1}\big).\end{aligned}$$

Consequently, using the equality $R_nx_nR_n = R_nx_nR_{n-1}$, we may write

$$\begin{aligned}&2\tau\big(R_{n-1}x_{n-1}R_{n-1}(R_{n-1}x_{n-1}R_{n-1}-R_nx_nR_n)\big)\\ &\leq 2\tau\big(R_{n-1}x_{n-1}R_{n-1}(R_{n-1}-R_n)x_nR_{n-1}\big)\\ &=2\tau\big(R_{n-1}x_{n-1}R_{n-1}(R_{n-1}-R_n)x_n(R_{n-1}-R_n)\big)\end{aligned}$$

(the last equality follows from the fact that $R_n$, and hence also $R_{n-1}-R_n$, commutes with $R_{n-1}x_nR_{n-1}$). But $R_{n-1}x_{n-1}R_{n-1} \leq I$, so we obtain

$$\begin{aligned}2\tau\big(R_{n-1}x_{n-1}R_{n-1}(R_{n-1}x_{n-1}R_{n-1}-R_nx_nR_n)\big) &\leq 2\tau\big((R_{n-1}-R_n)x_n\big)\\ &\leq 2\tau\big((R_{n-1}-R_n)x_N\big)\end{aligned}$$

Plugging this into (4.2) and summing over $n = 0, 1, 2, \ldots, N$, we get

$$\sum_{n=0}^{N}\tau(R_n dx_n R_{n-1}dx_n)\leq \tau(R_N x_N R_N x_N)+2\tau\big((I-R_N)x_N\big).$$

□

Finally, we will need the following analogue of Lemma 3.5.

**Lemma 4.6.** *For any nonnegative integer $N$ we have*

$$\left\|\sum_{n=0}^{N}\big|(R_{n-1}-R_n)dx_n(R_{n-1}-R_n)\big|\right\|_1\leq 2\tau\big((I-R_N)x_N\big).$$

*Proof.* By the triangle inequality and the fact that $x$ is a nonnegative submartingale satisfying $R_n x_n R_n \leq I$ for each $n$, we see that

$$\begin{aligned}
&\tau\Big(\big|(R_{n-1}-R_n)dx_n(R_{n-1}-R_n)\big|\Big)\\
&\leq \tau\Big((R_{n-1}-R_n)x_n(R_{n-1}-R_n)\Big)+\tau\Big((R_{n-1}-R_n)x_{n-1}(R_{n-1}-R_n)\Big)\\
&\leq \tau\Big((R_{n-1}-R_n)x_N(R_{n-1}-R_n)\Big)+\tau\big(R_{n-1}-R_n\big).
\end{aligned}$$

It remains to sum over $n = 0, 1, 2, \ldots, N$ and use Lemma 4.4. □

The proof of the $L^p$ bound for $v$ will rest on the following intermediate weak-type inequality.

**Theorem 4.7.** *Suppose that $x = (x_n)_{n\geq 0}$ is a nonnegative submartingale and $y = (y_n)_{n\geq 0}$ is strongly differentially subordinate to $x$. Then for any nonnegative integer $N$ we have*

$$\tau\big(|v_N| \geq 4\big) \leq 2\tau(R_N x_N R_N x_N) + 9\tau\big((I-R_N)x_N\big). \tag{4.3}$$

**Remark 4.8.** The inequality (4.3) can be interpreted in the language of real interpolation theory. The left-hand side is a tail of $v_N$, which after homogenization and integration leads to the $p$-th norm of $v_N$ up to some multiplicative constant. The right-hand side can be viewed as a $K$-functional of the operator $x_N$, corresponding to the interpolating spaces $L^1$ and $L^2$. Indeed, let us look at this expression in the commutative context. As we have already discussed earlier, the projection $R_N$ corresponds to the indicator function of the set $\{\max_{0\leq n\leq N} x_n \leq 1\}$ and hence, roughly speaking, the operator

$$2R_N x_N R_N x_N R_N + 9(I-R_N)x_N$$

is equal to the quadratic term $x_N^2$ when $x_N$ is small and to the linear term $9x_N$ when $x_N$ is large. This is precisely the intuition behind the $K$-functional.

*Proof of Theorem 4.7.* We decompose $dv_n$ using a splitting similar to that introduced in Section 3:

$$dv_n = d\alpha_n + d\beta_n + d\gamma_n + d\delta_n,$$

where this time,

$$\begin{aligned}
d\alpha_n &= R_{n-1}dy_nR_n + R_ndy_nR_{n-1} - R_ndy_nR_n\\
&\qquad - \mathcal{E}_{n-1}\big(R_{n-1}dy_nR_n + R_ndy_nR_{n-1} - R_ndy_nR_n\big),\\
d\beta_n &= -\mathcal{E}_{n-1}\big((R_{n-1}-R_n)dy_n(R_{n-1}-R_n)\big),\\
d\gamma_n &= \big(R_ndy_n - \mathcal{E}_{n-1}(dy_n)\big)(I-R_{n-1}),\\
d\delta_n &= (I-R_n)dy_n - (R_{n-1}-R_n)dy_nR_n - (I-R_{n-1})\mathcal{E}_{n-1}(dy_n)R_{n-1}.
\end{aligned}$$

We write

$$\tau\big(|v_N| \geq 4\big) \leq \tau\big(|\alpha_N| \geq 1\big) + \tau\big(|\beta_N| \geq 1\big) + \tau\big(|\gamma_N| \geq 1\big) + \tau\big(|\delta_N| \geq 1\big)$$

and analyze each term on the right separately. Arguing as in the proof of Lemma 3.6, we obtain

$$\begin{aligned}\tau(|\alpha_N|\geq 1)\leq \tau(\alpha_N^2)&=\sum_{n=0}^N \tau(d\alpha_n^2)\\ &\leq 2\sum_{n=0}^N \tau(R_n dy_n R_{n-1} dy_n)\\ &\leq 2\sum_{n=0}^N \tau(R_n dx_n R_{n-1} dx_n)\\ &\leq 2\tau\big(R_N x_N R_N x_N\big)+4\tau\big((I-R_N)x_N\big),\end{aligned}$$

where in the last line we exploited Lemma 4.5. Furthermore, again by the reasoning presented in the proof of Lemma 3.6 (see (3.18)), we get

$$\begin{aligned}\tau\big(|\beta_N|\geq 1\big)\leq \tau\big(|\beta_N|\big)&\leq \left\|\sum_{n=0}^N |d\beta_n|\right\|_1\\ &\leq \left\|\sum_{n=0}^N \Big(|(R_{n-1}-R_n)dx_n(R_{n-1}-R_n)|+(R_{n-1}-R_n)\Big)\right\|_1\\ &\leq 3\tau\big((I-R_N)x_N\big),\end{aligned}$$

where the latter passage is due to Lemmas 4.4 and 4.6. To handle the terms $\tau(|\gamma_N|\geq 1)$ and $\tau(|\delta_N|\geq 1)$, note that the right support of $d\gamma_n$ satisfies $r(d\gamma_n)\leq I-R_{n-1}\leq I-R_N$, so

$$\bigvee_{n=0}^N r(d\gamma_n)\leq I-R_N.$$

Therefore, by Lemma 4.4,

$$\tau\big(|\gamma_N|\geq 1\big)\leq \tau\left(\bigvee_{n=0}^N r(d\gamma_n)\right)\leq \tau\big((I-R_N)x_N\big).$$

A similar analysis of the left support of $d\delta_n$ gives

$$\tau\left(|\delta_N|\geq 1\right)\leq \tau\left(\bigvee_{n=0}^N \ell(d\delta_n)\right)\leq \tau\big((I-R_N)x_N\big).$$

Putting all the above facts together we get the claim. □

Arguing as in Jiao et al. [16, Theorem 5.1 (i)], the weak-type bound (4.3) yields the following $L^p$ estimate.

**Theorem 4.9.** *For any nonnegative integer $N$ and any $B>1$ we have*

$$\|v_N\|_p\leq \frac{4B^{p-1}}{B^{p-1}-1}\left(9B^p-3+\frac{4B^p(B^p-1)}{1-B^{p-2}}\right)^{1/p}\|x_N\|_p.$$

*Proof of Theorem 4.1 for $1<p<2$.* Combining Theorem 4.9 with Lemma 4.3, we get the desired $L^p$ estimate between $x$ and $y$:

$$\|y_N\|_p\leq \left(p+\frac{4B^{p-1}}{B^{p-1}-1}\left(9B^p-3+\frac{4B^p(B^p-1)}{1-B^{p-2}}\right)^{1/p}\right)\|x_N\|_p.$$

Note that the constant is indeed of order $O\left((p-1)^{-1}\right)$ as $p \to 1+$. □

4.2. **The case** $p > 2$**.** In [12], Junge and Xu proved that if $z = (z_n)_{n=0}^N$ is a martingale, then we have

$$\left(\sum_{n=0}^{N} ||dz_n||_p^p\right)^{1/p} \leq 2^{1-2/p}||z_N||_p. \tag{4.4}$$

We will prove that the same inequality is true if $z$ is assumed to be a nonnegative submartingale. To this end, we first strengthen (4.4) slightly.

**Lemma 4.10.** *Let $z = (z_n)_{n=0}^N$ be a martingale. Then we have the estimate*

$$\left(2^{p-2}||dz_0||_p^p + \sum_{n=1}^{N} ||dz_n||_p^p\right)^{1/p} \leq 2^{1-2/p}||z_N||_p. \tag{4.5}$$

*Proof.* We argue as in [12]. With no loss of generality we may assume that $||z_N||_p = 1$. We have

$$\left(2^{p-2}\|dz_0\|_p^p + \sum_{n=1}^{N} \|dz_n\|_p^p\right)^{1/p} = \tau(b_0 \cdot 2^{1-2/p}dz_0) + \sum_{1\leq k\leq N} \tau(b_k dz_k), \tag{4.6}$$

where $\sum_{0\leq k\leq N} \|b_k\|_{p'}^{p'} \leq 1$. By approximation and the interpolation results of Kosaki [18], there exist continuous functions $Z$, $B_k : \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\} \to \mathcal{M}$ analytic in the interior of the strip such that $z_N = Z(2/p)$, $b_k = B_k(2/p)$ and

$$\begin{gathered}\sup_{t\in\mathbb{R}} \max\left\{\|Z(it)\|_\infty, \|Z(1+it)\|_2\right\} \leq 1,\\ \sup_{t\in\mathbb{R}} \max\left\{\sum_{0\leq k\leq N} \|B_k(it)\|_1, \left(\sum_{0\leq k\leq N} \|B_k(1+it)\|_2^2\right)^{1/2}\right\} \leq 1.\end{gathered} \tag{4.7}$$

Consider the analytic function

$$F(z) = \tau\left(2^{1-z}B_0(z)\mathcal{E}_0 Z(z) + \sum_{0<k\leq N} B_k(z)(\mathcal{E}_k - \mathcal{E}_{k-1})Z(z)\right).$$

By Hölder's inequality,

$$\begin{aligned}|F(it)| \leq \tau&\left(\sum_{0\leq k\leq N} |B_k(it)|\right)\\ &\times \max\left\{||2\mathcal{E}_0 Z(it)||_\infty, ||(\mathcal{E}_1 - \mathcal{E}_0)Z(it)||_\infty, \ldots, ||(\mathcal{E}_N - \mathcal{E}_{N-1})Z(it)||_\infty\right\},\end{aligned}$$

which is not bigger than 2, by the assumptions in (4.7). Similarly, we have

$$\begin{aligned}|F(1+it)| \leq \tau&\left(\sum_{0\leq k\leq N} |B_k(1+it)|^2\right)^{1/2}\\ &\times \tau\left(|\mathcal{E}_0 Z(1+it)|^2 + \sum_{0<k\leq N} |(\mathcal{E}_k - \mathcal{E}_{k-1})Z(1+it)|^2\right)^{1/2} \leq 1.\end{aligned}$$

Hence, by the three lines lemma, we get $F(2/p) \leq 2^{1-2/p}$, which is the desired claim (see (4.6)). $\square$

The estimate (4.5) allows to obtain the following trace inequality, which is of independent interest.

**Lemma 4.11.** *For any $a$, $b \in \mathcal{M}$ we have*

$$\tau\big(|a+b|^p\big) \geq \tau\big(|a|^p\big) + p\tau\big(|a|^{p-2}ab\big) + 2^{2-p}\tau\big(|b|^p\big).$$

*Proof.* Let $s$ be a positive number and introduce the centered random variable $\xi$ with the distribution

$$\mathbb{P}(\xi = -s) = \frac{1}{s+1} = 1 - \mathbb{P}(\xi = 1).$$

Consider the martingale given by $z_0 = 1 \otimes a$, $z_1 = 1 \otimes a + \xi \otimes b$ (on the von Neumann algebra $L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \otimes \mathcal{M}$ with the natural filtration). By (4.5), we get

$$\begin{aligned} 2^{p-2}\tau(|a|^p) + \frac{s}{s+1}\tau(|b|^p) + \frac{s^p}{s+1}\tau(|b|^p) \\ \leq 2^{p-2}\left(\frac{s}{s+1}\tau(|a+b|^p) + \frac{1}{s+1}\tau(|a-sb|^p)\right), \end{aligned}$$

or, equivalently,

$$2^{p-2}\left(\frac{\tau(|a+b|^p) - \tau(|a|^p)}{s+1} + \frac{\tau(|a-sb|^p) - \tau(|a|^p)}{s(s+1)}\right) \geq \frac{\tau(|b|^p)}{s+1} + \frac{s^{p-1}\tau(|b|^p)}{s+1}.$$

It remains to let $s \to 0$ to obtain the claim. $\square$

Now we prove the following "submartingale" version of (4.4).

**Theorem 4.12.** *Suppose that $x = (x_n)_{n=0}^N$ is a nonnegative submartingale. Then*

$$\left(\sum_{n=0}^N \|dx_n\|_p^p\right)^{1/p} \leq 2^{1-2/p}\|x_N\|_p. \tag{4.8}$$

*Proof.* By the previous lemma, applied to $a = x_{n-1}$ and $b = dx_n$, we have

$$\|x_n\|_p^p - \|x_{n-1}\|_p^p \geq p\tau(x_{n-1}^{p-1}dx_n) + 2^{2-p}\|dx_n\|_p^p \geq 2^{2-p}\|dx_n\|_p^p.$$

Summing over $n$ completes the proof. $\square$

We turn our attention to the upper bound for $\|b_N\|_p$. We will exploit the noncommutative good-$\lambda$ inequalities developed by Jiao et al. in [16, 17]. Let us briefly recall the framework. Let $(\mathcal{M}, \tau)$ be a von Neumann algebra equipped with some filtration $(\mathcal{M}_n)_{n\geq 0}$. Suppose that $q = (q_n)_{n=0}^N$ is an adapted finite self-adjoint martingale, and $r$, $s$ are self-adjoint operators. For any $\lambda > 0$, consider the projections $S_{-1}^\lambda$, $S_0^\lambda$, $S_1^\lambda$, $\ldots$, $S_N^\lambda$ given by $S_{-1}^\lambda = I$ and $S_n^\lambda = S_{n-1}^\lambda I_{(-\lambda,\lambda)}(S_{n-1}^\lambda q_n S_{n-1}^\lambda)$ for $n = 1, 2, \ldots, N$.

**Definition 4.13.** The triple $\big(q, r, s\big)$ is said to satisfy the *good-$\lambda$ testing condition* if the following two requirements are fulfilled.

(i) For all $\lambda > 0$, we have

$$\sum_{n=0}^N \sum_{k=n+1}^N \tau\left(\left(S_{n-1}^\lambda - S_n^\lambda\right) dq_k S_{n-1}^\lambda dq_k \left(S_{n-1}^\lambda - S_n^\lambda\right)\right) \leq \tau\left(\left(I - S_N^\lambda\right) r^2\right).$$

(ii) For each $0 \leq n \leq N$ and any projection $P \in \mathcal{M}_n$,

$$\tau\left(Pdq_n^2P\right) \leq \tau\left(Ps^2P\right).$$

Let us stress here that we do not assume that $r$, $s$ are $\mathcal{M}_N$-measurable. One of the main results of [17] is the following.

**Theorem 4.14.** *If $(q, r, s)$ satisfies the good-$\lambda$ testing condition, then for any $p > 2$,*

$$||q_N||_p \leq \frac{12p}{\left(1 - \left(1 + \frac{1}{p}\right)^{2-p}\right)^{1/2}} \left(||r||_p^2 + ||s||_p^2\right)^{1/2}. \tag{4.9}$$

Equipped with the above statement, we will prove the following statement.

**Theorem 4.15.** *Fix $p > 2$. Then for any finite nonnegative submartingale $x = (x_n)_{n\geq 0}$ we have*

$$\|b_N\|_p \leq \|a_N\|_p \leq C_p \|x_N\|_p, \tag{4.10}$$

*where $C_p$ is of order $O(p^2)$ as $p \to \infty$.*

*Proof.* Fix $p > 2$. It suffices to show the second estimate in (4.10). We start with an appropriate complication of the underlying von Neumann algebra which will enable us to fit the assertion into the framework of good-$\lambda$ inequalities. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a classical probability space and let $\varepsilon_0$, $\varepsilon_1$, $\varepsilon_2$, $\ldots$, $\varepsilon_N$ be a sequence of independent Rademacher variables. Consider the algebra $\mathcal{N} = \mathbb{M}_{N+2} \otimes L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \otimes \mathcal{M}$, where $\mathbb{M}_{N+2}$ denotes the algebra of $(N+2) \times (N+2)$ matrices with the standard trace. So, we may interpret $\mathcal{N}$ as the algebra of $(N+2) \times (N+2)$ matrices, whose entries are random elements of $\mathcal{M}$. We equip $\mathcal{N}$ with the usual tensor trace $\nu$ and the filtration $(\mathcal{N}_n)_{n=0}^N = (\mathbb{M}_{N+2} \otimes L^\infty(\Omega, \mathcal{F}_n, \mathbb{P}) \otimes \mathcal{M}_{n-1})_{n=0}^N$, where $\mathcal{F}_n$ stands for the $\sigma$-field generated by the variables $\varepsilon_0$, $\varepsilon_1$, $\varepsilon_2$, $\ldots$, $\varepsilon_n$. Notice that we have used the algebra $\mathcal{M}_{n-1}$ on the third factor (with the convention that $\mathcal{M}_{-1} = \mathcal{M}_0$). Consider the operator

$$\begin{aligned} r = s &= e_{1,1} \otimes 1 \otimes x_N^{1/2} + \sum_{n=0}^N e_{n+2,n+2} \otimes 1 \otimes |dx_n|^{1/2} \\ &= \begin{bmatrix} x_N^{1/2} & 0 & 0 & \ldots & 0 \\ 0 & |dx_0|^{1/2} & 0 & \ldots & 0 \\ 0 & 0 & |dx_1|^{1/2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & |dx_N|^{1/2} \end{bmatrix} \end{aligned}$$

and the sequence $q = (q_n)_{n=0}^N$ given by

$$q_n = \sum_{k=0}^{n} (e_{1,k+2} + e_{k+2,1}) \otimes \varepsilon_k \otimes da_k^{1/2}$$
$$= \begin{bmatrix} 0 & \varepsilon_0 da_0^{1/2} & \varepsilon_1 da_1^{1/2} & \dots & \varepsilon_n da_n^{1/2} & 0 & \dots & 0 \\ \varepsilon_0 da_0^{1/2} & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \varepsilon_1 da_1^{1/2} & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \varepsilon_n da_n^{1/2} & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Of course, $q$ is a martingale adapted to $(\mathcal{N}_n)_{n=0}^N$: this is due to the fact that the sequence $da$ is predictable. Let us verify that the triple $(q, r, s)$ satisfies the good-$\lambda$ testing condition. We have

$$dq_k^2 = ((e_{1,k+2} + e_{k+2,1}) \otimes \varepsilon_k \otimes da_k^{1/2})^2 = (e_{1,1} + e_{k+2,k+2}) \otimes 1 \otimes da_k,$$

so

$$\sum_{n=0}^{N} \sum_{k=n+1}^{N} \nu\big((S_{n-1}^\lambda - S_n^\lambda) dq_k S_{n-1}^\lambda dq_k (S_{n-1}^\lambda - S_n^\lambda)\big)$$
$$\leq \nu\left(\sum_{n=0}^{N} \sum_{k=n+1}^{N} (S_{n-1}^\lambda - S_n^\lambda) dq_k^2 (S_{n-1}^\lambda - S_n^\lambda)\right)$$
$$= \nu\left(\sum_{n=0}^{N} \sum_{k=n+1}^{N} (S_{n-1}^\lambda - S_n^\lambda) \left((e_{1,1} + e_{k+2,k+2}) \otimes 1 \otimes da_k\right)\right)$$
$$= \nu\left(\sum_{n=0}^{N} \sum_{k=n+1}^{N} (S_{n-1}^\lambda - S_n^\lambda) \left((e_{1,1} + e_{k+2,k+2}) \otimes 1 \otimes dx_k\right)\right).$$

We split the latter expression into two parts:

$$\nu\left(\sum_{n=0}^{N} \sum_{k=n+1}^{N} \left(S_{n-1}^\lambda - S_n^\lambda\right) (e_{1,1} \otimes 1 \otimes dx_k)\right)$$
$$+ \nu\left(\sum_{n=0}^{N} \sum_{k=n+1}^{N} \left(S_{n-1}^\lambda - S_n^\lambda\right) (e_{k+2,k+2} \otimes 1 \otimes dx_k)\right)$$
$$\leq \sum_{n=0}^{N} \nu\left(\left(S_{n-1}^\lambda - S_n^\lambda\right) (e_{1,1} \otimes 1 \otimes (x_N - x_n))\right)$$
$$+ \nu\left(\sum_{k=0}^{N} \sum_{n=0}^{k-1} \left(S_{n-1}^\lambda - S_n^\lambda\right) (e_{k+2,k+2} \otimes 1 \otimes |dx_k|)\right)$$
$$\leq \nu\left(\left(I - S_N^\lambda\right) \left(e_{1,1} \otimes 1 \otimes x_N + \sum_{k=0}^{N} e_{k+2,k+2} \otimes 1 \otimes |dx_k|\right)\right)$$
$$= \nu\left(\left(I - S_N^\lambda\right) r^2\right),$$

which is the condition (i). Concerning the assumption (ii), we check that for any projection $P \in \mathcal{N}_n$,

$$\nu\big(Pdq_n^2P\big) = \tau\Big(P((e_{1,1} + e_{n+2,n+2}) \otimes 1 \otimes da_n)P\Big)$$
$$= \tau\Big(P((e_{1,1} + e_{n+2,n+2}) \otimes 1 \otimes dx_n)P\Big) \leq \tau\big(Ps^2P\big),$$

where in the second passage we have used the equality $da_n = \mathcal{E}_{n-1}(dx_n)$ (and the fact that the third factor in $\mathcal{N}_n = \mathbb{M}_{N+2} \otimes L^\infty(\Omega, \mathcal{F}_n, \mathbb{P}) \otimes \mathcal{M}_{n-1}$ is the algebra $\mathcal{M}_{n-1}$). So, the good-$\lambda$ testing condition holds true and therefore the inequality (4.9) gives

$$||q_N||_{2p} \leq \frac{24p}{\left(1 - \left(1 + \frac{1}{2p}\right)^{2-2p}\right)^{1/2}} \cdot 2^{1/2}||r||_{2p}. \tag{4.11}$$

However, we have $q_N^2 \geq e_{1,1} \otimes 1 \otimes a_N$, which implies $||q_N||_{2p}^2 \geq \|a_N\|_p$. Furthermore, we have

$$||r||_{2p}^2 = \left(\|x_N\|_p^p + \sum_{n=0}^N \|dx_n\|_p^p\right)^{1/p}.$$

By Theorem 4.12, the expression on the right does not exceed $(1 + 2^{p-2})^{1/p}||x_N||_p$. Putting all the above observations together, we get the desired estimate (4.10). $\square$

We turn our attention to the estimate for the martingale part $\|v\|_p$. This is the only missing part of the proof of Theorem 4.1.

**Theorem 4.16.** *We have $\|v_N\|_p \leq \kappa_p\|x_N\|_p$, where $\kappa_p$ is of order $O(p^4)$ as $p \to \infty$.*

*Proof.* First apply Burkholder-Rosenthal inequality (see [12]) to obtain

$$\|v_N\|_p \leq c_p \left[\|s_N(v)\|_p + \left(\sum_{n=0}^N \|dv_n\|_p^p\right)^{1/p}\right],$$

where $c_p$ is of order $O(p)$ as $p \to \infty$. By the assumption (WDS), we have

$$\mathcal{E}_{n-1}(dv_n^2) = \mathcal{E}_{n-1}((dy_n - \mathcal{E}_{n-1}(dy_n))^2) \leq \mathcal{E}_{n-1}(dy_n^2)$$
$$\leq \mathcal{E}_{n-1}(dx_n^2) = \mathcal{E}_{n-1}(du_n^2) + da_n^2,$$

so $s_N(v)^2 \leq s_N(u)^2 + s_N(a)^2$ and $\|s_N(v)\|_p \leq \|s_N(u)\|_p + \|s_N(a)\|_p$. In addition, the triangle inequality and (WDS)+(CDS) assumptions give that

$$\left(\sum_{n=0}^N \|dv_n\|_p^p\right)^{1/p} \leq \left(\sum_{n=0}^N \|du_n\|_p^p\right)^{1/p} + \left(\sum_{n=0}^N \|da_n\|_p^p\right)^{1/p}.$$

Combining the above observations, we obtain

$$\begin{aligned}
&\|s_N(v)\|_p + \left(\sum_{n=0}^N \|dv_n\|_p^p\right)^{1/p} \\
&\leq \|s_N(u)\|_p + \left(\sum_{n=0}^N \|du_n\|_p^p\right)^{1/p} + \|s_N(a)\|_p + \left(\sum_{n=0}^N \|da_n\|_p^p\right)^{1/p} \\
&\leq \tilde{c}_p||u_N||_p + (1+2^{1-2/p})||a_N||_p \\
&\leq \tilde{c}_p||x_N||_p + (\tilde{c}_p + 1 + 2^{1-2/p})||a_N||_p,
\end{aligned}$$

with $\tilde{c}_p$ of order $O(p)$. Here in the second passage we have used the reverse Burkholder-Rosenthal inequality (which gave the estimate for the terms involving $u$), the inequality

$$||s_N(a)||_p = \left\| \left(\sum_{n=0}^N da_n^2\right)^{1/2} \right\|_p \leq \|a_N\|_p$$

and Theorem 4.12 applied to the submartingale $(a_n)_{n=0}^N$. Putting all the above facts together and combining them with the estimate (4.10), we obtain the claim. □

*Proof of Theorem 4.1 for* $2 < p < \infty$. Combining Theorem 4.15 with Theorem 4.16, we get the desired $L^p$ estimate between $x$ and $y$:

$$\|y_N\|_p \leq c_p \|x_N\|_p,$$

where $c_p$ is of order $O(p^4)$ as $p \to \infty$. □

We conclude with the following interesting question.

**Remark 4.17.** Let us write the estimates of Lemma 4.3 and Theorem 4.15 in the language of noncommutative Doob's inequality. We have shown that if $x = (x_n)_{n\geq 0}$ is a nonnegative submartingale, then for $1 \leq p < \infty$ we have

$$\left\| \sum_{n=0}^N \mathcal{E}_{n-1}(dx_n) \right\|_p \leq C_p \left\| \sum_{n=0}^N dx_n \right\|_p,$$

with $C_p = O(p^2)$ as $p \to \infty$. Is this order optimal? This is not clear, since the sequence $dx$ above needs to be adapted and hence the estimate does not generalize Doob's inequality.

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