SHARP $L^p \to L^{q,\infty}$ ESTIMATES FOR THE HILBERT TRANSFORM

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Abstract. For any $1 < q < p < \infty$, we identify the best constant $K_{p,q}$ with the following property. If $\mathcal{H}$ is the Hilbert transform on the unit circle $\mathbb{T}$ and $A \subset \mathbb{T}$ is an arbitrary measurable set, then

$$\int_A |\mathcal{H}f| \mathrm{d}m \leq K_{p,q} \|f\|_{L^p(\mathbb{T},m)} m(A)^{1-1/q}.$$

The proof rests on the construction of certain special superharmonic functions on the plane, which are of independent interest.

1. Introduction

Our motivation comes from a very basic question about the Hilbert transform $\mathcal{H}$ on the unit circle $\mathbb{T} \simeq (-\pi, \pi]$. Recall that this operator is given by the singular integral

$$\mathcal{H}f(e^{it}) = \text{p.v.} \int_{-\pi}^{\pi} f(s) \cot \frac{t-s}{2} \mathrm{d}m_{\mathbb{T}}(s) \qquad \text{for } f \in L^1(\mathbb{T}).$$

Here and below, the integrals and norms on $\mathbb{T}$ are taken with respect to $m_{\mathbb{T}}$, the normalized Haar measure on the circle. A classical result of M. Riesz [11] asserts that $\mathcal{H}$ is bounded as an operator on $L^p(\mathbb{T})$ when $1 < p < \infty$. For $p = 1$ the boundedness fails, but we have the substitute

$$m_{\mathbb{T}}(\{\theta \in \mathbb{T} : |\mathcal{H}f(\theta)| \geq 1\}) \leq C\|f\|_{L^1(\mathbb{T})},$$

proved by Kolmogorov in [7]: that is, $\mathcal{H}$ satisfies the weak-type $(1, 1)$ estimate. The precise values of the above L^p norms were evaluated by Pichorides [10] and Cole (unpublished; see [4]). Specifically, we have

$$\|\mathcal{H}\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})} = \cot \frac{\pi}{2p^*}, \qquad 1 < p < \infty,$$

where $p^* = \max\{p, p'\}$ and $p' = p/(p-1)$ denotes the conjugate exponent to p. Furthermore, as shown by Davis [3], the weak (1,1) norm is

$$\|\mathcal{H}\|_{L^1(\mathbb{T})\to L^{1,\infty}(\mathbb{T})} = \frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots}{1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots} = 1.347\ldots. \tag{1.1}$$

This gives rise to a very natural and very interesting question about the norms of $\mathcal{H}$ on other function spaces. Janakiraman [6] (see also [8]) extended (1.1) and proved that for $1 \leq p \leq 2$ we have the weak-type bound

$$\|\mathcal{H}\|_{L^p(\mathbb{T})\to L^{p,\infty}(\mathbb{T})} = \left(\frac{1}{\pi} \int_{\mathbb{R}} \frac{\left|\frac{2}{\pi} \log|t|\right|^p}{t^2+1} \mathrm{d}t \right)^{-1/p},$$

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where

$$\|\varphi\|_{L^{p,\infty}(\mathbb{T})} = \sup_{\lambda>0} \lambda m_{\mathbb{T}}(\{\theta \in \mathbb{T} : |\varphi(\theta)| \geq \lambda\})^{1/p}. \tag{1.2}$$

For $p > 2$, the problem of the explicit identification of $\|\mathcal{H}\|_{L^p(\mathbb{T})\to L^{p,\infty}(\mathbb{T})}$ is open, to the best of the authors' knowledge. In [9], a variant of this problem was studied, under the following equivalent norming of the Lorentz space $L^{p,\infty}$, $1 < p < \infty$:

$$\|\varphi\|_{L^{p,\infty}(\mathbb{T})} = \sup\left\{\frac{1}{m(E)^{1-1/p}}\int_E |\varphi|\mathrm{d}m_{\mathbb{T}} \, : \, E \subseteq \mathbb{T},\, m_{\mathbb{T}}(E) > 0\right\}. \tag{1.3}$$

Theorem 1.1. *Under the above norming, for $1 < p < \infty$ we have*

$$\|\mathcal{H}\|_{L^p(\mathbb{T})\to L^{p,\infty}(\mathbb{T})} = \begin{cases} \left[\dfrac{2^{p'+2}\Gamma(p'+1)}{\pi^{p'+1}}\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{p'+1}}\right]^{1/p'} & \text{if } 1 < p \leq 2, \\ \left[\dfrac{2^{p'+2}\Gamma(p'+1)}{\pi^{p'}}\displaystyle\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{p'}}\right]^{1/p'} & \text{if } 2 < p < \infty. \end{cases}$$

All the estimates mentioned above carry over, with unchanged constants, to the nonperiodic case (i.e., to the context of Hilbert transform on the real line).

The main objective of this paper is to study the weak-type estimates in the case of different exponents, when the Hilbert transform is considered as an operator from L^p to $L^{q,\infty}$, $p \neq q$. Obviously, this problem makes sense only if $q < p$ (for $q > p$ the norm is infinite). To formulate our main result, we need to introduce an auxiliary object. Given $0 < p < \infty$, let $\omega_p : [0,1] \to [0,\infty)$ be the L^p-modulus of continuity of the function $u \mapsto \frac{1}{\pi}\ln|\tan\frac{\pi u}{4}|$, $u \in (-2,2)$:

$$\omega_p(t) = \left(\frac{1}{4}\int_{-2}^{2}\left|\frac{1}{\pi}\ln\left|\tan\frac{\pi(s+t)}{4}\right| - \frac{1}{\pi}\ln\left|\tan\frac{\pi(s-t)}{4}\right|\right|^p \mathrm{d}s\right)^{1/p}. \tag{1.4}$$

We will prove the following fact.

Theorem 1.2. *For any $1 < q < p < \infty$ we have the sharp estimate*

$$\|\mathcal{H}f\|_{L^{q,\infty}(\mathbb{T})} \leq K_{p,q}\|f\|_{L^p(\mathbb{T})}, \tag{1.5}$$

where

$$K_{p,q} = \begin{cases} \left[\dfrac{2^{p'+2}\Gamma(p'+1)}{\pi^{p'+1}}\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{p'+1}}\right]^{1/p'} & \text{if } 1 < p \leq 2, \\ \displaystyle\sup_{0<t\leq 1}\left(t^{-1/q'}\omega_{p'}(t)\right) & \text{if } p > 2. \end{cases}$$

So, we see that if $1 < p \leq 2$, the weak norm does not change if we vary q: we have the identity $K_{p,q} = \|\mathcal{H}\|_{L^p(\mathbb{T})\to L^{p,\infty}(\mathbb{T})}$. Roughly speaking, this is a consequence of two facts: (i) for $1 < p \leq 2$, the extremal functions for (1.5) are the same; (ii) the "maximal sets" E for $\|\mathcal{H}f\|_{L^{q,\infty}(\mathbb{T})}$ (i.e., those for which the suprema defining the norms are attained, see (1.3)), coincide with $\mathbb{T}$. On the contrary, for $p > 2$ there is a nontrivial dependence on q. There is a natural question whether, for $p > 2$, the constant can be expressed in a more explicit form, but we believe that this is not possible. Another natural question concerns the possibility of extending the result to $\mathbb{R}$ or $\mathbb{Z}$ (in the case $q < p$). The answer is negative, since both spaces have infinite measure. Here is a quick example for $\mathbb{R}$ (a similar calculation for $\mathbb{Z}$ is left to the reader). We pick a positive number a and consider the characteristic function $f = \chi_{[-a,a]}$. Then $\mathcal{H}^{\mathbb{R}} f$, the nonperiodic Hilbert transform of f, equals

$$\mathcal{H}^{\mathbb{R}} f(x) = \frac{1}{\pi} \log \frac{|x+a|}{|x-a|}$$

(see [5], p. 251) and hence

$$\|\mathcal{H}^{\mathbb{R}} f\|_{L^{q,\infty}(\mathbb{R})} \geq \frac{1}{|[0,a]|^{1-1/q}} \int_0^a |\mathcal{H}^{\mathbb{R}} f| \mathrm{d}x = \frac{2\ln 2}{\pi} \cdot a^{1/q}.$$

Since $\|f\|_{L^p(\mathbb{R})} = (2a)^{1/p}$, the ratio $\|\mathcal{H}^{\mathbb{R}} f\|_{L^{q,\infty}(\mathbb{R})} / \|f\|_{L^p(\mathbb{R})}$ tends to infinity as $a \to \infty$; thus the weak-type bound fails to hold.

Our approach will rest on the construction of a certain special superharmonic function on the strip $\{z \in \mathbb{C} : |\operatorname{Re} z| \leq 1\}$, which will satisfy an appropriate majorization condition: see Section 2. This function will directly lead us to the desired estimate, and its sharpness will be obtained by the construction of the extremal examples.

2. A special superharmonic function

In our argumentation below, we will often use the identification $\mathbb{C} \simeq \mathbb{R}^2$ and switch from $z = x + iy$ to (x, y) and back; this should not lead to any confusion. Throughout, $a > 0$ and $1 < p \leq 2$ are fixed parameters. Consider the planar domain $D = D_a = ([-1,1] \times \mathbb{R}) \setminus \{(0,y) : |y| \geq a\}$ and let $H = H_a$ be the map given by

$$H(z) = i \left(\frac{e^{\pi a - i\pi z} - 1}{e^{\pi a} - e^{-i\pi z}} \right)^{1/2}, \qquad z \in \mathbb{C}. \tag{2.1}$$

Here we use the following branch of the square root on the complex plane: $(re^{i\varphi})^{1/2} = r^{1/2} e^{i\varphi/2}$, where $r \geq 0$ and $\varphi \in (-\pi, \pi]$. It is easy to check that H is a conformal mapping which sends the interior of D onto the open upper half-plane $\mathbb{R}^2_+ := \mathbb{R} \times (0, \infty)$. Next, let $\mathcal{U} = \mathcal{U}^{p,a} : \mathbb{R}^2_+ \to \mathbb{R}$ be given by the Poisson integral

$$\mathcal{U}(\alpha, \beta) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\beta}{(t-\alpha)^2 + \beta^2} \left(\left| \frac{1}{\pi} \ln \left| \frac{e^{\pi a} t^2 - 1}{e^{\pi a} - t^2} \right| \right|^p - C \chi_{\{|\ln|t|| \leq \pi a/2\}} \right) \mathrm{d}t,$$

where

$$C = C_{p,a} = (4 \sinh(\pi a/2))^{-1} \int_{\mathbb{R}} \left(\left| \frac{1}{\pi} \ln \left| \frac{e^{\pi a} t^2 - 1}{e^{\pi a} - t^2} \right| \right|^p - a^p \right) \mathrm{d}t.$$

Obviously, the function $\mathcal{U}$ is harmonic and satisfies the boundary behavior

$$\lim_{\beta \downarrow 0} \mathcal{U}(\alpha, \beta) = \left| \frac{1}{\pi} \ln \left| \frac{e^{\pi a} \alpha^2 - 1}{e^{\pi a} - \alpha^2} \right| \right|^p - C \chi_{\{|\ln|\alpha|| \leq \pi a/2\}} \tag{2.2}$$

for $\alpha \in \mathbb{R} \setminus \{\pm e^{\pm \pi a/2}, 0\}$. Finally, let $U = U^{p,a}$ be a function defined on the interior of D by the formula

$$U(x,y) = \mathcal{U}(H(x,y)).$$

Then U is harmonic, being the composition of a harmonic function with a conformal mapping. Furthermore, by (2.2), we have

$$\lim_{(x,y) \to (\pm 1, u)} U(x,y) = |u|^p - C$$

and

$$\lim_{(x,y)\to(0,u)} U(x,y) = |u|^p \qquad \text{for } |u| \geq a.$$

In other words, U is the continuous solution to the Dirichlet problem

$$\begin{cases} \Delta U = 0 & \text{inside } D, \\ U(x,y) = |y|^p - C|x| & \text{for } (x,y) \in \partial D. \end{cases}$$

In particular, the function U satisfies the symmetry condition

$$U(x,y) = U(|x|,|y|) \qquad \text{for } (x,y) \in D \tag{2.3}$$

(this can be also proved directly, by performing appropriate substitutions in the integral defining $\mathcal{U}$).

The function U is of fundamental importance to our considerations. The remaining part of this section is devoted to the study of the properties of U which will be needed later. We start with a technical lemma.

Lemma 2.1. *We have*

$$\lim_{\beta\to\infty} \frac{\beta}{\pi} \int_{\mathbb{R}} \frac{t^4 + 3t^2\beta^2}{(t^2+\beta^2)^2} \left(\left| \frac{1}{\pi} \ln \left| \frac{e^{\pi a}t^2 - 1}{e^{\pi a} - t^2} \right| \right|^p - a^p \right) dt = \frac{2pa^{p-1}(e^{\pi a} - e^{-\pi a})}{\pi}.$$

Proof. We will use twice the following simple property of the Poisson integral: if $f : \mathbb{R} \to \mathbb{R}$ is a locally integrable function satisfying $\lim_{x\to\pm\infty} f(x) = M$, then

$$\lim_{\beta\to\infty} \frac{1}{\pi} \int_{\mathbb{R}} \frac{\beta f(t)}{t^2+\beta^2} \mathrm{d}t = M. \tag{2.4}$$

This implies

$$\begin{aligned} &\lim_{\beta\to\infty} \frac{\beta}{\pi} \int_{\mathbb{R}} \frac{3t^2}{t^2+\beta^2} \left(\left| \frac{1}{\pi} \ln \left| \frac{e^{\pi a}t^2 - 1}{e^{\pi a} - t^2} \right| \right|^p - a^p \right) \mathrm{d}t \\ &\qquad = \lim_{t\to\infty} 3t^2 \left(\left| \frac{1}{\pi} \ln \left| \frac{e^{\pi a}t^2 - 1}{e^{\pi a} - t^2} \right| \right|^p - a^p \right). \end{aligned} \tag{2.5}$$

Furthermore, integrating by parts and applying (2.4) again, we obtain

$$\begin{aligned} &- \lim_{\beta\to\infty} \frac{2\beta}{\pi} \int_{\mathbb{R}} \frac{t^4}{(t^2+\beta^2)^2} \left(\left| \frac{1}{\pi} \ln \left| \frac{e^{\pi a}t^2 - 1}{e^{\pi a} - t^2} \right| \right|^p - a^p \right) \mathrm{d}t \\ &\quad = - \lim_{\beta\to\infty} \frac{\beta}{\pi} \int_{\mathbb{R}} \frac{1}{t^2+\beta^2} \left\{ t^3 \left(\left| \frac{1}{\pi} \ln \left| \frac{e^{\pi a}t^2 - 1}{e^{\pi a} - t^2} \right| \right|^p - a^p \right) \right\}' \mathrm{d}t \\ &\quad = - \lim_{t\to\infty} \left\{ t^3 \left(\left| \frac{1}{\pi} \ln \left| \frac{e^{\pi a}t^2 - 1}{e^{\pi a} - t^2} \right| \right|^p - a^p \right) \right\}', \end{aligned}$$

which added to (2.5) gives

$$\begin{aligned} &\lim_{\beta\to\infty} \frac{\beta}{\pi} \int_{\mathbb{R}} \frac{t^4 + 3t^2\beta^2}{(t^2+\beta^2)^2} \left(\left| \frac{1}{\pi} \ln \left| \frac{e^{\pi a}t^2 - 1}{e^{\pi a} - t^2} \right| \right|^p - a^p \right) \mathrm{d}t \\ &= - \lim_{t\to\infty} t^3 \left(\left| \frac{1}{\pi} \ln \left| \frac{e^{\pi a}t^2 - 1}{e^{\pi a} - t^2} \right| \right|^p - a^p \right)' = \frac{2pa^{p-1}(e^{\pi a} - e^{-\pi a})}{\pi}. \qquad \square \end{aligned}$$

In the lemma below, we establish an appropriate "smooth-fit" property at the point $(0,a)$.

Lemma 2.2. *We have* $\lim_{y\uparrow a} U_y(0,y) = pa^{p-1}$.

Proof. The equality follows from the definition of C. Observe that $U(0,y) = \mathcal{U}(H(0,y))$ and

$$H(0,y) = i\left(\frac{e^{\pi a+\pi y}-1}{e^{\pi a}-e^{\pi y}}\right)^{1/2}$$

is purely imaginary. Consequently,

$$U_y(0,y) = \mathcal{U}_\beta\left(0,\left(\frac{e^{\pi a+\pi y}-1}{e^{\pi a}-e^{\pi y}}\right)^{1/2}\right)\cdot H_y(0,y).$$

For brevity, let us denote $\left(\frac{e^{\pi a+\pi y}-1}{e^{\pi a}-e^{\pi y}}\right)^{1/2}$ by β. Clearly, when y increases to a, then β tends to infinity. Furthermore, we compute directly that

$$H_y(0,y) \sim \frac{\pi}{2}\frac{(e^{\pi a+\pi y}-1)^{1/2}}{(e^{\pi a}-e^{\pi y})^{3/2}}\cdot e^{\pi y} \sim \frac{\pi e^{\pi a}}{2(e^{2\pi a}-1)}\beta^3,$$

where the symbol $\sim$ above means that the ratio of the expressions on both sides of it tends to 1 as $y\uparrow a$. Consequently, we see that

$$\lim_{y\uparrow a} U_y(0,y) = \frac{\pi e^{\pi a}}{2(e^{2\pi a}-1)}\lim_{\beta\to\infty} U_\beta(0,\beta)\beta^3.$$

By the definition of $\mathcal{U}$, we compute that $\mathcal{U}(0,\beta)$ equals

$$\frac{1}{\pi}\int_{\mathbb{R}}\frac{\beta}{t^2+\beta^2}\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p \mathrm{d}t - \frac{2C}{\pi}\left[\arctan(e^{\pi a/2}/\beta)-\arctan(e^{-\pi a/2}/\beta)\right].$$

A direct differentiation with respect to β yields

$$\Big(\arctan(e^{\pi a/2}/\beta)\Big)' = -\frac{e^{\pi a/2}}{e^{\pi a}+\beta^2} = -\frac{e^{\pi a/2}}{\beta^2}+O(\beta^{-4})$$

and

$$\Big(\arctan(e^{-\pi a/2}/\beta)\Big)' = -\frac{e^{-\pi a/2}}{e^{-\pi a}+\beta^2} = -\frac{e^{-\pi a/2}}{\beta^2}+O(\beta^{-4}),$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\left\{\frac{2C}{\pi}\left[\arctan(e^{\pi a/2}/\beta)-\arctan(e^{-\pi a/2}/\beta)\right]\right\}$$
$$= \frac{2C(-e^{\pi a/2}+e^{-\pi a/2})}{\pi\beta^2}+O(\beta^{-4}).$$

Furthermore,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\beta}\left\{\frac{1}{\pi}\int_{\mathbb{R}}\frac{\beta}{t^2+\beta^2}\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p \mathrm{d}t\right\}\\
&=\frac{\mathrm{d}}{\mathrm{d}\beta}\left\{\frac{1}{\pi}\int_{\mathbb{R}}\frac{\beta}{t^2+\beta^2}\left(\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p - a^p\right)\mathrm{d}t\right\}\\
&=\frac{1}{\pi}\int_{\mathbb{R}}\frac{t^2-\beta^2}{(t^2+\beta^2)^2}\left(\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p - a^p\right)\mathrm{d}t = I_1+I_2,
\end{aligned}$$

where

$$I_1 = -\frac{1}{\beta^2}\cdot\frac{1}{\pi}\int_{\mathbb{R}}\left(\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p - a^p\right)\mathrm{d}t = \frac{2C(-e^{\pi a/2}+e^{-\pi a/2})}{\pi\beta^2}$$

and

$$I_2=\frac{1}{\beta^2}\cdot\frac{1}{\pi}\int_{\mathbb{R}}\frac{t^4+3t^2\beta^2}{(t^2+\beta^2)^2}\left(\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p-a^p\right)\mathrm{d}t.$$

Putting the above facts together, we obtain

$$\lim_{\beta\to\infty}\mathcal{U}_\beta(0,\beta)\beta^3=\lim_{\beta\to\infty}\frac{\beta}{\pi}\int_{\mathbb{R}}\frac{t^4+3t^2\beta^2}{(t^2+\beta^2)^2}\left(\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p-a^p\right)\mathrm{d}t.$$

It remains to use the previous lemma to get the claim. □

Lemma 2.3. *We have $U_y(x,y)\leq py^{p-1}$ for $x\in[-1,1]$ and $y\geq 0$.*

Proof. Fix an arbitrary point (x,y) belonging to the half-strip $(0,1)\times\mathbb{R}$. The function U is continuous on $[0,1]\times\mathbb{R}$, so we have

$$U(x,y)=\int_{\{0,1\}\times\mathbb{R}}U(u,v)\mathrm{d}\mu_{x,y}(u,v),$$

where $\mu_{x,y}$ is the harmonic measure on $\{0,1\}\times\mathbb{R}$ with respect to the point (x,y). Since $[0,1]\times\mathbb{R}$ is invariant with respect to vertical translations, we also have

$$U(x,y+h)=\int_{\{0,1\}\times\mathbb{R}}U(u,v+h)\mathrm{d}\mu_{x,y}(u,v)$$

and hence, by Lebesgue's dominated convergence theorem,

$$U_y(x,y)=\lim_{h\to 0}\frac{U(x,y+h)-U(x,y)}{h}=\int_{\{0,1\}\times\mathbb{R}}U_y(u,v)\mathrm{d}\mu_{x,y}(u,v).$$

Since $y\mapsto U_y(0,y)$ and $y\mapsto U_y(1,y)$ are continuous, we conclude that U_y extends to a continuous function on $[0,1]\times\mathbb{R}$, and hence also to a continuous function on $[-1,1]\times\mathbb{R}$. Now, consider the upper half-strip $S^+=((-1,1)\times(0,\infty))\setminus\{(0,y):y\geq a\}$. The crucial observation is that on the boundary of S^+, U_y coincides with the function $W(x,y)=py^{p-1}$, which is superharmonic in the interior of S^+. Indeed, the equalities $U_y(\pm 1,y)=py^{p-1}$ and $U_y(0,y)=py^{p-1}$ for $y\geq a$ are obvious, while $U_y(x,0)=0$ follows from the symmetry condition (2.3). Since U_y is continuous on S^+, we obtain $U_y\leq W$ on S^+, which completes the proof. □

Lemma 2.4. *We have $U_{yy}\geq 0$ in the interior of D and $\lim_{y\downarrow a}U_x(0+,y)=U_x(0,a)=0$.*

Proof. Fix arbitrary (x,y), $(x,y+\delta)\in D$, where $\delta\in(0,a)$ is a small positive number. Consider the auxiliary domain $\mathcal{D}_{a,\delta}=((-1,1)\times\mathbb{R})\setminus\{(0,v):v\geq a-\delta\text{ or }v\leq -a\}$. Since $\mathcal{D}_{a,\delta}$ and its translation $i\delta+\mathcal{D}_{a,\delta}$ are contained in the interior of D, we may write

$$U_y(x,y)=\int_{\partial\mathcal{D}_{a,\delta}}U_y(u,v)\mathrm{d}\mu_{x,y}^{\mathcal{D}_{a,\delta}}(u,v)$$

and

$$U_y(x,y+\delta)=\int_{\partial\mathcal{D}_{a,\delta}}U_y(u,v+\delta)\mathrm{d}\mu_{x,y}^{\mathcal{D}_{a,\delta}}(u,v),$$

so

$$U_y(x,y+\delta)-U_y(x,y)=\int_{\partial\mathcal{D}_{a,\delta}}\big(U_y(u,v+\delta)-U_y(u,v)\big)\mathrm{d}\mu_{x,y}^{\mathcal{D}_{a,\delta}}(u,v).$$

But the integrand is nonnegative. Indeed, if $u = \pm 1$, or $u = 0$ and $|v+\delta|, |v| \geq a$, then $U_y(u, v+\delta) - U_y(u,v) = p|v+\delta|^{p-1}\operatorname{sgn}(v+\delta) - p|v|^{p-1}\operatorname{sgn}(v) \geq 0$. If $u = 0$ and $v + \delta \geq a > v$, then $v \geq 0$ (here we use the assumption $\delta < a$) and by the previous lemma,

$$U_y(u, v+\delta) - U_y(u,v) \geq p|v+\delta|^{p-1}\operatorname{sgn}(v+\delta) - p|v|^{p-1}\operatorname{sgn}(v) \geq 0. \tag{2.6}$$

Finally, if $v + \delta > -a \geq v$, then by the symmetry of U we have $U_y(u, v+\delta) - U_y(u,v) = U_y(u,-v) - U_y(u, -v-\delta) \geq 0$, by (2.6). Consequently, we have shown that for each $x \in [-1,1]$, the function $U_y(x,\cdot)$ is nondecreasing, which yields the first part of the claim. To handle the second part, note that $U_{xx} = -U_{yy} \leq 0$ in the interior of D. Furthermore, by the symmetry condition (2.3), we have $U_x(0,y) = 0$ for $|y| < a$. These two facts imply $U(x,y) \leq U(0,y)$ for all $x \in [-1,1]$ and $y \in (-a,a)$, which, by the continuity of U, is also true for $y = a$. This gives $U_x(0+,a) \leq 0$; to see that both sides are equal, note that if $U(\cdot,a)$ had a concave cusp at $x = 0$, then we would have $\lim_{y\uparrow a} U_y(0,y) = \infty$, by elementary facts about harmonic functions. This proves that $U_x(0,a) = 0$.

Next, we will show that the function $y \mapsto U_x(0+,y)$ is nonincreasing on $[a,\infty)$. Pick $y' > y > a$. By the previous lemma we may write, for any $x \in (0,1)$,

$$\begin{aligned}
&\frac{U(x,y') - U(0,y')}{x} \\
&\quad = \frac{U(x,y') - U(x,y) + U(x,y) - U(0,y) + U(0,y) - U(0,y')}{x} \\
&\quad \leq \frac{\int_y^{y'} ps^{p-1}\mathrm{d}s + U(x,y) - U(0,y) + y^p - (y')^p}{x} \\
&\quad = \frac{U(x,y) - U(0,y)}{x}.
\end{aligned}$$

Hence, letting $x \downarrow 0$ gives the desired monotonicity $U_x(0+,y') \leq U_x(0,y)$; in particular, this shows that the limit $\lim_{y\downarrow a} U_x(0+,y)$ exists and is at most zero. However, if we had $\lim_{y\downarrow a} U_x(0+,y) = M < 0$, then we would have $U_x(0+,y) \leq M$ for all $y > a$ and the estimate $U_{xx} \leq 0$ would imply that

$$U(x,y) \leq U(0,y) + U_x(0+,y)x \leq y^p + Mx$$

for all $y > a$ and $x \in (0,1)$. Letting $y \downarrow a$, we would obtain $U_x(0+,a) \leq M$, a contradiction. □

Remark 2.5. The second half of the above lemma can be computed directly. Let us briefly outline the proof. We start from the observation that if $x \downarrow 0$, then

$$H(x,a) \sim \left(\frac{(e^{2\pi a}-1)^2}{2e^{2\pi a}(1-\cos \pi x)}\right)^{1/4}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \sim (\beta,\beta),$$

for $\beta = \left(\frac{\sinh a}{\pi x}\right)^{1/2} \to \infty$. A direct differentiation shows that both the real and the imaginary parts of $H_x(x,a)$ are equal and behave like β^3, up to a universal multiplicative constant. Consequently, it is enough to show that

$$\lim_{\beta\to\infty} (\mathcal{U}_x(\beta,\beta) + \mathcal{U}_y(\beta,\beta))\beta^3 = 0. \tag{2.7}$$

For simplicity, denote $g(t) = \left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p - a^p$ and note that $K = \lim_{t\to\infty} t^2 g(t)$ is finite. Some tedious, but rather straightforward computations reveal that

$$\begin{aligned}\mathcal{U}_x(\beta,\beta)+\mathcal{U}_y(\beta,\beta) &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{(\beta-t)^2-\beta^2+2\beta(t-\beta)}{((\beta-t)^2+\beta^2)^2} g(t)\mathrm{d}t \\ &\quad - \frac{C}{\pi}\Bigg[-\frac{e^{\pi a/2}}{\beta^2+(\beta-e^{\pi a/2})^2} + \frac{e^{-\pi a/2}}{\beta^2+(\beta-e^{-\pi a/2})^2} \\ &\qquad + \frac{e^{-\pi a/2}}{\beta^2+(\beta+e^{-\pi a/2})^2} - \frac{e^{\pi a/2}}{\beta^2+(\beta+e^{\pi a/2})^2}\Bigg] \\ &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{t^2-2\beta^2}{((\beta-t)^2+\beta^2)^2} g(t)\mathrm{d}t + \frac{C}{\pi}\left[2\beta^{-2}\sinh\frac{\pi a}{2} + O(\beta^{-4})\right]\end{aligned}$$

and hence, by the very definition of C, the limit in (2.7) equals

$$\begin{aligned}&\lim_{\beta\to\infty}\frac{\beta^3}{\pi}\int_{\mathbb{R}}\left(\frac{t^2-2\beta^2}{((\beta-t)^2+\beta^2)^2}+\frac{1}{2\beta^2}\right)g(t)\mathrm{d}t \\ &= \lim_{\beta\to\infty}\frac{\beta}{\pi}\int_{\mathbb{R}}\frac{t^4-4\beta t^3+10\beta^2t^2}{((\beta-t)^2+\beta^2)^2}g(t)\mathrm{d}t - \lim_{\beta\to\infty}\frac{\beta}{\pi}\int_{\mathbb{R}}\frac{8t\beta^3}{((\beta-t)^2+\beta^2)^2}g(t)\mathrm{d}t = I_1 - I_2.\end{aligned}$$

As for the I_1, the calculations are simple and similar to Lemma 2.1. The substitution $t=\beta s$ yields

$$\begin{aligned}\frac{\beta}{\pi}\int_{\mathbb{R}}\frac{t^4-4\beta t^3+10\beta^2t^2}{((\beta-t)^2+\beta^2)^2}g(t)\mathrm{d}t &= \frac{\beta}{\pi}\int_{\mathbb{R}}\frac{t^2-4\beta t+10\beta^2}{((\beta-t)^2+\beta^2)^2}\left[t^2g(t)\right]\mathrm{d}t \\ &= \frac{1}{\pi}\int_{\mathbb{R}}\frac{s^2-4s+10}{((1-s)^2+1)^2}\left[(s\beta)^2g(s\beta)\right]\mathrm{d}s.\end{aligned}$$

It is not difficult to see that we can pull the limit inside the integral, obtaining $I_1 = K\frac{1}{\pi}\int_{\mathbb{R}}\frac{s^2-4s+10}{((1-s)^2+1)^2}\mathrm{d}s = 4K$. Next, we rewrite the expression I_2 in the form

$$I_2 = \lim_{\beta\to\infty}\frac{\beta}{\pi}\int_{\mathbb{R}}\frac{8\beta^3}{t((\beta-t)^2+\beta^2)^2}\left[t^2g(t)\right]\mathrm{d}t.$$

If we bound the integral away from the singularity point 0, then we perform calculations similar to those above, obtaining

$$\begin{aligned}&\lim_{\beta\to\infty}\frac{\beta}{\pi}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}\frac{8\beta^3}{t((\beta-t)^2+\beta^2)^2}\left[t^2g(t)\right]\mathrm{d}t \\ &= \lim_{\beta\to\infty}\frac{1}{\pi}\int_{\mathbb{R}\setminus[-\varepsilon/\beta,\varepsilon/\beta]}\frac{8}{s((1-s)^2+1)^2}\left[(s\beta)^2g(s\beta)\right]\mathrm{d}s \\ &= \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}}\frac{8K}{s((1-s)^2+1)^2}\mathrm{d}s = 4K.\end{aligned}$$

Near the singularity point, it is enough to notice that $t^2g(t) = O(t^4)$ (as $t\to 0$), so

$$\lim_{\beta\to\infty}\frac{\beta}{\pi}\int_{-\varepsilon}^{\varepsilon}\frac{8\beta^3}{t((\beta-t)^2+\beta^2)^2}\left[t^2g(t)\right]\mathrm{d}t = 0.$$

Hence, subtracting I_1 from I_2, we obtain that the limit in (2.7) equals $4K-4K=0$.

The following statement is the main result of the section.

Theorem 2.6. *The function U is a superharmonic majorant of the function $V : D \to \mathbb{R}$ given by $V(x,y) = |y|^p - C|x|$.*

Proof. To show the superharmonicity, it is convenient to use some basic facts from stochastic analysis. Fix an arbitrary ball $K \subset [-1,1]\times\mathbb{R}$ of center (x,y) and radius r. Let $B = (B^{(1)}, B^{(2)})$ be a two-dimensional Brownian motion started at (x,y) and stopped upon reaching the boundary of K. The function U is of class C^2 on $(0,1)\times\mathbb{R}$, of class C^1 on $[0,1]\times\mathbb{R}$ and satisfies (2.3), so Itô's formula gives

$$U(B_t) = U(|B_t^{(1)}|, B_t^{(2)}) = U(|B_0^{(1)}|, B_0^{(2)}) + I_1 + \frac{1}{2}I_2, \tag{2.8}$$

where

$$I_1 = \int_0^t U_x(|B_s^{(1)}|, B_s^{(2)})\mathrm{d}|B^{(1)}|_s + \int_0^t U_y(|B_s^{(1)}|, B_s^{(2)})\mathrm{d}B_s^{(2)},$$

and

$$I_2 = \int_0^t \Delta U(|B_s^{(1)}|, B_s^{(2)})\mathrm{d}s.$$

Note that $\mathrm{d}|B^{(1)}|_s = \operatorname{sgn}(B_s^{(1)})\mathrm{d}B_s^{(1)} + \mathrm{d}\ell_s$, where ℓ is the local time of $B^{(1)}$ at zero. Since the local time is a monotone process which increases on the set $\{t : B_t^{(1)} = 0\}$ and $U_x(0+,y) \le 0$ for all y, we have

$$\begin{aligned}\int_0^t U_x(|B_s^{(1)}|, B_s^{(2)})\mathrm{d}|B^{(1)}|_s &= \int_0^t U_x(|B_s^{(1)}|, B_s^{(2)}) \operatorname{sgn}(B_s^{(1)})\mathrm{d}B_s^{(1)} + \int_0^t U_x(0, B_s^{(2)})\mathrm{d}\ell_s \\ &\le \int_0^t U_x(|B_s^{(1)}|, B_s^{(2)}) \operatorname{sgn}(B_s^{(1)})\mathrm{d}B_s^{(1)}.\end{aligned}$$

But the latter integral, as well as the second integral in I_1, has zero expectation: this follows at once from the properties of stochastic integrals. Finally, I_2 vanishes, since U is harmonic inside $[0,1]\times\mathbb{R}$. Thus, taking the expectation in (2.8), we obtain $\mathbb{E}U(B_t) \le U(|B_0^{(1)}|, B_0^{(2)}) = U(x,y)$. Letting $t \to \infty$ yields the superharmonicity, since the random variable B_∞ is uniformly distributed at the boundary of K.

Concerning the majorization $U(x,y) \ge V(x,y)$, let us first show it for $x \in \{0,1\}$ and $y \ge 0$. We have $U(1,y) = V(1,y)$ for all y, and $U(0,y) = V(0,y)$ for $|y| \ge a$. The estimate $U(0,y) \ge V(0,y)$, for $y \in [0,a]$, follows at once from the equality $U_y(0,a) = V_y(0,a)$ (see Lemma 2.2) and the estimate $U_y(0,y) \le py^{p-1}$ proved in Lemma 2.3. Now we extend the majorization to $x \in \{0,1\}$ and $y \in \mathbb{R}$, using the symmetry of U and V. Since U is harmonic on $[0,1]\times\mathbb{R}$ and V is subharmonic on this strip, we deduce the estimate $U \ge V$ on $[0,1]\times\mathbb{R}$; finally, using the symmetry with respect to the variable x, we obtain the majorization on the full range. □

3. Proof of Theorem 1.2

3.1. **Proof of** (1.5)**.** If $1 < p \le 2$, then $\|\mathcal{H}\|_{L^p(\mathbb{T})\to L^{q,\infty}(\mathbb{T})} \le \|\mathcal{H}\|_{L^p(\mathbb{T})\to L^{p,\infty}(\mathbb{T})} \le K_{p,q}$, where the last estimate was established in [2]. Therefore, we may restrict ourselves to the case $p > 2$. Now we split the argument into two parts.

Step 1. An auxiliary estimate. First we will show that for an arbitrary measurable function f bounded in absolute value by 1 we have

$$\|\mathcal{H}f\|_{L^{p'}(\mathbb{T})}^{p'} \le \big(\omega_{p'}\big(\|f\|_{L^1(\mathbb{T})}\big)\big)^{p'}. \tag{3.1}$$

To this end, let u, v be the harmonic extensions of f and $\mathcal{H}f$ to the unit disc $\mathbb{D}$, obtained by integration against the appropriate Poisson kernel; then $v(0) = 0$ and $u + iv$ is a holomorphic function on $\mathbb{D}$. Let $U = U^{p',a}$ be the special function constructed in the previous section (note that the conjugate exponent p' is used, the parameter a will be specified in a moment). Then the composition $U(u, v)$ is superharmonic and hence

$$\int_{\mathbb{T}} U(u,v)\mathrm{d}m_{\mathbb{T}} \leq U(u(0),v(0)) = U(u(0),0).$$

By Theorem 2.6, the left-hand side is not smaller than $\int_{\mathbb{T}} V(u,v)\mathrm{d}m_{\mathbb{T}}$. Furthermore, Lemma 2.4 combined with the superharmonicity of U gives that the function $x \mapsto U(x, 0)$ is concave on $[-1, 1]$. Applying (2.3), the maximal value of this function is $U(0, 0)$. Putting all the above facts together, we obtain

$$\int_{\mathbb{T}} |v|^{p'} - C_{p',a}|u|\mathrm{d}m_{\mathbb{T}} \leq U(0,0),$$

or $\|\mathcal{H}f\|^{p'}_{L^{p'}(\mathbb{T})} \leq U(0, 0) + C_{p',a}\|f\|_{L^1(\mathbb{T})}$. It is high time to specify a: we plug $a = \frac{2}{\pi}\ln\left(\tan\left(\frac{\pi}{4}\left(\|f\|_{L^1(\mathbb{T})} + 1\right)\right)\right)$, obtaining

$$\begin{aligned} U(0,0) &= \mathcal{U}(0,1) \\ &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{t^2+1}\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^{p'}\mathrm{d}t - \frac{2C_{p',a}}{\pi}\left[\arctan e^{\pi a/2} - \arctan e^{-\pi a/2}\right] \\ &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{t^2+1}\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^{p'}\mathrm{d}t - C_{p',a}\|f\|_{L^1(\mathbb{T})}. \end{aligned}$$

Inserting this into the previous bound, we see that

$$\|\mathcal{H}f\|^{p'}_{L^{p'}(\mathbb{T})} \leq \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{t^2+1}\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^{p'}\mathrm{d}t = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}\tan^2 s-1}{e^{\pi a}-\tan^2 s}\right|\right|^{p'}\mathrm{d}s.$$

But by the definition of a, we have $e^{\pi a/2} = \tan\left(\frac{\pi}{4}\left(\|f\|_{L^1(\mathbb{T})} + 1\right)\right)$, so

$$\begin{aligned} \frac{e^{\pi a}\tan^2 s-1}{e^{\pi a}-\tan^2 s} &= \frac{e^{\pi a/2}\tan s-1}{e^{\pi a/2}-\tan s}\cdot\frac{e^{\pi a/2}\tan s+1}{e^{\pi a/2}+\tan s} \\ &= \tan\left(\frac{\pi}{4}(\|f\|_{L^1(\mathbb{T})}+1)+s\right)\cdot\tan\left(\frac{\pi}{4}(\|f\|_{L^1(\mathbb{T})}+1)-s\right). \end{aligned}$$

The latter expression, considered as a function of s, is π-periodic. Therefore, plugging it above and substituting $s := s + \frac{\pi}{4}$ in the integral, we get

$$\|\mathcal{H}f\|^{p'}_{L^{p'}(\mathbb{T})} \leq \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left|\Psi\left(s+\frac{\pi}{4}\|f\|_{L^1(\mathbb{T})}\right) - \Psi\left(s-\frac{\pi}{4}\|f\|_{L^1(\mathbb{T})}\right)\right|^{p'}\mathrm{d}s,$$

where $\Psi(u) = \frac{1}{\pi}\ln|\tan u|$. By a simple change of variables in the latter expression, we obtain (3.1).

Step 2. Proof of (1.5). Let $q \in (1, p)$, pick an arbitrary function $f \in L^p(\mathbb{T})$ and let E be a measurable subset of $\mathbb{T}$ satisfying $m_{\mathbb{T}}(E) > 0$. Since the adjoint of $\mathcal{H}$ is equal to $-\mathcal{H}$, we have

$$\int_E |\mathcal{H}f|\mathrm{d}m_{\mathbb{T}} = \int_{\mathbb{T}} \mathcal{H}f\,\mathrm{sgn}(\mathcal{H}f)\chi_E \mathrm{d}m_{\mathbb{T}} = -\int_{\mathbb{T}} f\mathcal{H}\big(\mathrm{sgn}(\mathcal{H}f)\chi_E\big)\mathrm{d}m_{\mathbb{T}}$$
$$\leq \|f\|_{L^p(\mathbb{T})} \left\|\mathcal{H}\big(\mathrm{sgn}(\mathcal{H}f)\chi_E\big)\right\|_{L^{p'}(\mathbb{T})},$$

where we use the convention $\mathrm{sgn}(0) = 1$. Now we apply the estimate (3.1) for the function $\mathrm{sgn}(\mathcal{H}f)\chi_E$: note that the function takes values in $[-1,1]$, so the application is permitted. The L^1-norm of this function is not bigger than $m_{\mathbb{T}}(E)$, so we obtain

$$\frac{1}{m_{\mathbb{T}}(E)^{1-1/q}} \int_E |\mathcal{H}f|\mathrm{d}m_{\mathbb{T}} \leq \|f\|_{L^p(\mathbb{T})} \cdot m_{\mathbb{T}}(E)^{-1/q'}\omega_{p'}(m_{\mathbb{T}}(E)) \leq K_{p,q}\|f\|_{L^p(\mathbb{T})}.$$

This is the desired weak-type inequality.

3.2. **Proof of $\|\mathcal{H}\|_{L^p(\mathbb{T})\to L^{q,\infty}(\mathbb{T})} \geq K_{p,q}$, the case $p \leq 2$.** Consider the conformal map $F : \mathbb{D} \to [-1,1]\times\mathbb{R}$, given by

$$F(z) = \frac{2i}{\pi}\log\left[\frac{iz-1}{z-i}\right] + 1.$$

Then F maps the unit circle onto the boundary $\{-1,1\}\times\mathbb{R}$. We easily check the following explicit formulas on $\mathbb{T}$:

$$\varphi(e^{it}) := \mathrm{Re}\,F(e^{it}) = -\chi_{\{|t|\leq\pi/2\}} + \chi_{\{|t|>\pi/2\}}$$

and

$$\mathcal{H}\varphi(e^{it}) = \mathrm{Im}\,F(e^{it}) = \frac{2}{\pi}\ln\left|\frac{1+\sin t}{\cos t}\right|.$$

Set $f = -|\mathcal{H}\varphi|^{p'-2}\mathcal{H}\varphi$. Since φ takes values in $\{-1,1\}$, we have

$$\|\mathcal{H}f\|_{L^{q,\infty}(\mathbb{T})} \geq \frac{1}{m_{\mathbb{T}}(\mathbb{T})}\int_{\mathbb{T}} \mathcal{H}f\varphi\mathrm{d}m_{\mathbb{T}} = -\int_{\mathbb{T}} f\mathcal{H}\varphi\mathrm{d}m_{\mathbb{T}} = \int_{\mathbb{T}} |\mathcal{H}\varphi|^{p'}\mathrm{d}m_{\mathbb{T}}.$$

However, we compute that

$$\begin{aligned}
\|\mathcal{H}\varphi\|_{L^{p'}(\mathbb{T})}^{p'} &= \int_{-\pi}^{\pi}\left|\frac{2}{\pi}\ln\left|\frac{1+\sin t}{\cos t}\right|\right|^{p'}\frac{\mathrm{d}t}{2\pi} = \frac{1}{\pi}\int_{\mathbb{R}}\frac{\left|\frac{2}{\pi}\log|t|\right|^{p'}}{t^2+1}\mathrm{d}t \\
&= \frac{2^{p'+1}}{\pi^{p'+1}}\int_0^\infty \frac{|\log t|^{p'}}{t^2+1}\mathrm{d}t = \frac{2^{p'+1}}{\pi^{p'+1}}\int_{-\infty}^\infty \frac{|s|^{p'}e^s}{e^{2s}+1}\mathrm{d}s \\
&= \frac{2^{p'+2}}{\pi^{p'+1}}\int_0^\infty s^{p'}e^{-s}\sum_{k=0}^\infty(-e^{-2s})^k\mathrm{d}s = \frac{2^{p'+2}}{\pi^{p'+1}}\Gamma(p'+1)\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)^{p'+1}} \\
&= K_{p,q}^{p'}.
\end{aligned}$$

Combining this with the preceding estimate, we obtain

$$\|\mathcal{H}f\|_{L^{q,\infty}(\mathbb{T})} \geq \|\mathcal{H}\varphi\|_{L^{p'}(\mathbb{T})} \cdot \|\mathcal{H}\varphi\|_{L^{p'}(\mathbb{T})}^{p'-1} = K_{p,q}\|f\|_{L^p(\mathbb{T})}.$$

Hence the constant $K_{p,q}$ in (1.5) cannot be improved.

3.3. **Proof of $\|\mathcal{H}\|_{L^p(\mathbb{T})\to L^{q,\infty}(\mathbb{T})} \geq K_{p,q}$, the case $p > 2$.** Fix an arbitrary parameter T belonging to $(0,1]$ and set $a = \frac{2}{\pi}\ln\left(\tan\frac{\pi(T+1)}{4}\right) > 0$. Let G be a conformal mapping which sends the unit disc $\mathbb{D}$ onto the set D and satisfies $G(0) = 0$. Finally, put $\varphi = \operatorname{Re} G|_{\mathbb{T}}$, $E = \{\varphi \neq 0\}$ and $f = -|\mathcal{H}\varphi|^{p'-2}\mathcal{H}\varphi$. Note that $\varphi \in \{0, \pm 1\}$, which gives

$$\int_E |\mathcal{H}f|\mathrm{d}m_{\mathbb{T}} \geq \int_{\mathbb{T}} \mathcal{H}f\varphi \mathrm{d}m_{\mathbb{T}} = -\int_{\mathbb{T}} f\mathcal{H}\varphi \mathrm{d}m_{\mathbb{T}} = \int_{\mathbb{T}} |\mathcal{H}\varphi|^{p'}\mathrm{d}m_{\mathbb{T}}.$$

To evaluate the latter integral, we apply appropriate conformal changes of variables. First, note that

$$\int_{\mathbb{T}} |\mathcal{H}\varphi|^{p'}\mathrm{d}m_{\mathbb{T}} = \int_{\partial D} |v|^{p'}\mathrm{d}\mu_{(0,0)}(u,v).$$

Next, recall the mapping H defined in (2.1). It sends D onto the upper halfplane $\mathbb{R}^2_+$ and 0 to i. Since $\frac{dt}{\pi(1+t^2)}$ is the harmonic measure on $\partial\mathbb{R}^2_+$ with respect to i, the latter integral equals

$$\int_{\partial\mathbb{R}^2_+} |\operatorname{Im}(H^{-1}(0,t))|^{p'}\frac{\mathrm{d}t}{\pi(1+t^2)} = \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{1+t^2}\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{t^2-e^{\pi a}}\right|\right|^{p'}\mathrm{d}t = \big(\omega_{p'}(T)\big)^{p'}.$$

(To see the last equality, repeat the calculations appearing in §3.1.) A similar reasoning reveals that

$$m_{\mathbb{T}}(E) = \int_{\mathbb{T}} |\varphi|\mathrm{d}m_{\mathbb{T}} = \frac{4}{\pi}\arctan e^{\pi a/2} - 1 = T.$$

Putting all the above facts together, we obtain

$$\begin{aligned}\|\mathcal{H}f\|_{L^{q,\infty}(\mathbb{T})} &\geq \frac{1}{m_{\mathbb{T}}(E)^{1-1/q}}\int_E |\mathcal{H}f|\mathrm{d}m_{\mathbb{T}} \geq \frac{\|\mathcal{H}\varphi\|_{L^{p'}(\mathbb{T})}}{m_{\mathbb{T}}(E)^{1-1/q}} \cdot \|\mathcal{H}\varphi\|^{p'-1}_{L^{p'}(\mathbb{T})} \\ &\geq T^{-1/q'}\omega_{p'}(T)\|f\|_{L^p(\mathbb{T})}.\end{aligned}$$

Taking the supremum over all T, we see that the constant in (1.5) is indeed the best possible.

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