SHARP INEQUALITY FOR BOUNDED SUBMARTINGALES AND THEIR DIFFERENTIAL SUBORDINATES

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Abstract. For any fixed $\alpha \in [0, 1]$ and $\lambda > 0$ we determine the optimal function $V_{\alpha, \lambda}$ satisfying

$$P(\max_n |g_n| \geq \lambda) \leq \mathbb{E}V_{\alpha, \lambda}(f_0, g_0)$$

for any submartingale $f = (f_n)$ bounded in absolute value by 1 and any process $g = (g_n)$ which is real-valued, adapted, integrable and satisfying

$$|dg_n| \leq |df_n| \quad \text{and} \quad |\mathbb{E}(dg_n|\mathcal{F}_{n-1})| \leq \alpha|\mathbb{E}(df_n|\mathcal{F}_{n-1})|, \quad n = 1, 2, \ldots,$$

with probability 1. As a corollary, a sharp exponential inequality for the distribution function of $\max_n |g_n|$ is established.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, equipped with a discrete filtration $(\mathcal{F}_n)$. Let $f = (f_n)_{n=0}^\infty, g = (g_n)_{n=0}^\infty$ be adapted integrable processes taking values in a certain separable Hilbert space $\mathcal{H}$. The difference sequences $df = (df_n), dg = (dg_n)$ of these processes are given by

$$df_0 = f_0, \quad df_n = f_n - f_{n-1}, \quad dg_0 = g_0, \quad dg_n = g_n - g_{n-1}, \quad n = 1, 2, \ldots.$$

Let $g^*$ stand for the maximal function of $g$, that is, $g^* = \max_n |g_n|$. The following notion of differential subordination is due to Burkholder. The process $g$ is differentially subordinate to $f$ (or, in short, subordinate to $f$) if for any nonnegative integer $n$ we have, almost surely,

$$|dg_n| \leq |df_n|.$$

We will slightly change this definition and say that $g$ is differentially subordinate to $f$, if the above inequality for the differences holds for any positive integer $n$.

Let $\alpha$ be a fixed nonnegative number. Then $g$ is $\alpha$-differentially subordinate to $f$ (or, in short, $\alpha$-subordinate to $f$), if it is subordinate to $f$ and for any positive integer $n$ we have

$$|\mathbb{E}(dg_n|\mathcal{F}_{n-1})| \leq \alpha|\mathbb{E}(df_n|\mathcal{F}_{n-1})|.$$

This concept was introduced by Burkholder in [2] in the special case $\alpha = 1$. In general form, it first appeared in the paper by Choi [3].

In the sequel it will sometimes be convenient to work with simple processes. A process $f$ is called simple, if for any $n$ the variable $f_n$ is simple and there exists $N$ such that $f_N = f_{N+1} = f_{N+2} = \ldots$. Given such a process, we will identify it with the finite sequence $(f_n)_{n=0}^N$.

Assume that the processes $f$ and $g$ are real-valued and fix $\alpha \in [0, 1]$. The objective of this paper is to establish a sharp exponential inequality for the distribution function of $g^*$ under the assumption that $f$ is a submartingale satisfying $||f||_\infty \leq 1$.
and \( g \) is \( \alpha \)-subordinate to \( f \). To be more precise, for any \( \lambda > 0 \) define the function \( V_{\alpha, \lambda} : [-1, 1] \times \mathbb{R} \to \mathbb{R} \) by the formula

\[
V_{\alpha, \lambda}(x_0, y_0) = \sup \mathbb{P}(g^* \geq \lambda).
\]

Here the supremum is taken over all pairs \((f, g)\) of integrable adapted processes, such that \((f_0, y_0) \equiv (x_0, y_0)\) almost surely, \( f \) is a submartingale satisfying \( \|f\|_\infty \leq 1 \) and \( g \) is \( \alpha \)-subordinate to \( f \). The filtration must also vary, as well as the probability space, unless it is nonatomic. Our main result is an explicit formula for the functions \( V_{\alpha, \lambda}, \lambda > 0 \). Usually we will omit the index \( \alpha \) and write \( V_{\lambda} \) instead of \( V_{\alpha, \lambda} \).

Let us discuss some related results which appeared in the literature. In [1] Burkholder studied the analogous question in the case of \( f, g \) being Hilbert space-valued martingales. The paper [1] contains also a related one-sided sharp exponential inequality for real martingales. This work was later extended by Hammack [4], who established a similar (two-sided) inequality under the assumption that \( f \) is a submartingale bounded by 1 and \( g \) is \( R^\nu \)-valued, \( \nu \geq 1 \), and strongly 1-subordinate to \( f \). Both papers present applications to stochastic integrals.

The paper is organized as follows. In the next section we introduce a family of special functions \( U_{\lambda}, \lambda > 0 \) and study their properties. This enables us to establish the inequality \( V_{\lambda} \leq U_{\lambda} \) in Section 3. Then we prove the reverse inequality in the last section.

Throughout the paper, \( \alpha \) is a fixed number from the interval \([0, 1]\). All the considered processes are assumed to be real valued.

## 2. The Explicit Formulas

Let \( S \) be the strip \([-1, 1] \times \mathbb{R} \). Consider the following subsets of \( S \): for \( 0 < \lambda \leq 2 \),

\[
\begin{align*}
A_\lambda &= \{(x, y) \in S : |y| \geq x + \lambda - 1\}, \\
B_\lambda &= \{(x, y) \in S : 1 - x \leq |y| < x + \lambda - 1\}, \\
C_\lambda &= \{(x, y) \in S : |y| < 1 - x \text{ and } |y| < x + \lambda - 1\}.
\end{align*}
\]

For \( \lambda \in (2, 4) \), define

\[
\begin{align*}
A_\lambda &= \{(x, y) \in S : |y| \geq \alpha x + \lambda - \alpha\}, \\
B_\lambda &= \{(x, y) \in S : \alpha x + \lambda - \alpha > |y| \geq x - 1 + \lambda\}, \\
C_\lambda &= \{(x, y) \in S : x - 1 + \lambda > |y| \geq 1 - x\}, \\
D_\lambda &= \{(x, y) \in S : 1 - x > |y| \geq -x - 3 + \lambda \text{ and } |y| < x - 1 + \lambda\}, \\
E_\lambda &= \{(x, y) \in S : -x - 3 + \lambda > |y|\}.
\end{align*}
\]

Finally, for \( \lambda \geq 4 \), let

\[
\begin{align*}
A_\lambda &= \{(x, y) \in S : |y| \geq \alpha x + \lambda - \alpha\}, \\
B_\lambda &= \{(x, y) \in S : \alpha x + \lambda - \alpha > |y| \geq x - 1 + \lambda\}, \\
C_\lambda &= \{(x, y) \in S : x - 1 + \lambda > |y| \geq -x - 3 + \lambda\}, \\
D_\lambda &= \{(x, y) \in S : -x - 3 + \lambda > |y| \geq 1 - x\}, \\
E_\lambda &= \{(x, y) \in S : 1 - x > |y|\}.
\end{align*}
\]

Let \( H : S \times (-1, \infty) \to \mathbb{R} \) be a function given by

\[
H(x, y, z) = \frac{1}{\alpha + 2} \left[ 1 + \frac{(x + 1 + |y|)^{(\alpha+1)}}{(1+z)^{(\alpha+2)}} \right].
\]

Now we will define the special functions \( U_{\lambda} : S \to \mathbb{R} \). For \( 0 < \lambda \leq 2 \), let

\[
U_{\lambda}(x, y) = \begin{cases} 
1 & \text{if } (x, y) \in A_\lambda, \\
\frac{2 - 2z}{1 + \lambda - x - |y|} & \text{if } (x, y) \in B_\lambda, \\
\frac{1}{\lambda^2 - (\lambda - 1 + x + |y|)} & \text{if } (x, y) \in C_\lambda.
\end{cases}
\]
For $2 < \lambda < 4$, set
\begin{equation}
U_{\lambda}(x, y) = \begin{cases} 
1 & \text{if } (x, y) \in A_{\lambda}, \\
1 - (\alpha(x - 1) - |y| + \lambda) \cdot \frac{2\lambda - 4}{\lambda^2} & \text{if } (x, y) \in B_{\lambda}, \\
\frac{2 - 2\alpha}{\lambda} - \frac{2(1-\alpha)(1-\lambda)(1-\lambda/2)}{\lambda^2} & \text{if } (x, y) \in C_{\lambda}, \\
\frac{2}{\alpha}(1-\alpha)(1-\lambda) + \frac{2(1-x)^2|y|^2}{\lambda^2} & \text{if } (x, y) \in D_{\lambda}, \\
a_{\lambda}H(x, y, \lambda - 3) + b_{\lambda} & \text{if } (x, y) \in E_{\lambda},
\end{cases}
\end{equation}

where
\begin{equation}
a_{\lambda} = -b_{\lambda} = \frac{(1 + \alpha)(\lambda - 2)}{2\lambda} \exp\left(\frac{4 - \lambda}{2(\alpha + 2)}\right).
\end{equation}

For $\lambda \geq 4$, set
\begin{equation}
U_{\lambda}(x, y) = \begin{cases} 
1 & \text{if } (x, y) \in A_{\lambda}, \\
1 - \frac{\alpha(x-1)-|y|+\lambda}{4} & \text{if } (x, y) \in B_{\lambda}, \\
\frac{2 - 2\alpha}{\lambda} - \frac{(1-\alpha)(1-\lambda)}{4} & \text{if } (x, y) \in C_{\lambda}, \\
\frac{(1-x)(1-\alpha)}{4} \exp\left(\frac{3+x+|y|-\lambda}{2(\alpha+1)}\right) & \text{if } (x, y) \in D_{\lambda}, \\
a_{\lambda}H(x, y, 1) + b_{\lambda} & \text{if } (x, y) \in E_{\lambda},
\end{cases}
\end{equation}

where
\begin{equation}
a_{\lambda} = -b_{\lambda} = \frac{(1 + \alpha)}{2} \exp\left(\frac{4 - \lambda}{2(\alpha + 2)}\right).
\end{equation}

For $\alpha = 1$, the formulas (2.2), (2.3), (2.5) give the special functions constructed by Hammack [4]. The key properties of $U_{\lambda}$ are described in the two lemmas below.

**Lemma 2.1.** For $\lambda > 2$, let $\phi_{\lambda}$, $\psi_{\lambda}$ denote the partial derivatives of $U_{\lambda}$ with respect to $x$, $y$ on the interiors of $A_{\lambda}$, $B_{\lambda}$, $C_{\lambda}$, $D_{\lambda}$, $E_{\lambda}$, extended continuously to the whole of these sets. The following statements hold.

(i) The functions $U_{\lambda}$, $\lambda > 2$, are continuous on $S \setminus \{(1, \pm\lambda)\}$.

(ii) Let
\[ S_{\lambda} = \{(x, y) \in [-1, 1] \times \mathbb{R} : |y| \neq \alpha x + \lambda - \alpha \text{ and } |y| \neq x + \lambda - 1\}. \]

Then
\begin{equation}
\phi_{\lambda}, \psi_{\lambda}, \lambda > 2, \text{ are continuous on } S_{\lambda}.
\end{equation}

(iii) For any $(x, y) \in S$, the function $\lambda \mapsto U_{\lambda}(x, y)$, $\lambda > 0$, is left-continuous.

(iv) For any $\lambda > 2$ we have the inequality
\begin{equation}
\phi_{\lambda} \leq -\alpha|\psi_{\lambda}|.
\end{equation}

(v) For $\lambda > 2$ and any $(x, y) \in S$ we have $\chi_{\{|y| \geq \lambda\}} \leq U_{\lambda}(x, y) \leq 1.$

**Proof.** We start with computing the derivatives. Let $y' = y/|y|$ stand for the sign of $y$, with $y' = 0$ if $y = 0$. For $\lambda \in (2, 4)$ we have
\begin{equation}
\phi_{\lambda}(x, y) = \begin{cases} 
0 & \text{if } (x, y) \in A_{\lambda}, \\
-\frac{(2\lambda-4)x}{2\lambda - 2|y|} + \frac{(2\lambda-4)(1-\alpha)}{\lambda^2} & \text{if } (x, y) \in B_{\lambda}, \\
\frac{2}{\lambda} \left(1 - \frac{(1-\alpha)(1-\lambda)}{\lambda}\right) & \text{if } (x, y) \in C_{\lambda}, \\
\frac{x - (1-\alpha)(1-\lambda)}{\lambda} + \frac{2(1-x)}{\lambda^2} & \text{if } (x, y) \in D_{\lambda}, \\
-c_{\lambda}(x + |y| + 1)^{-\alpha/(\alpha+1)}(x + 1 + \frac{\alpha}{\alpha+1}|y|) & \text{if } (x, y) \in E_{\lambda},
\end{cases}
\end{equation}
\[
\psi_\lambda(x, y) = \begin{cases} 
0 & \text{if } (x, y) \in A_\lambda, \\
\frac{2\lambda-4y'}{\lambda^2} & \text{if } (x, y) \in B_\lambda, \\
\frac{1+2\lambda-2|x|y'}{1+\lambda-x-|y|} & \text{if } (x, y) \in C_\lambda, \\
\frac{2y}{\lambda^2} & \text{if } (x, y) \in D_\lambda, \\
c_\lambda(x + |y| + 1)^{-\alpha/(\alpha+1)} & \text{if } (x, y) \in E_\lambda, 
\end{cases}
\]

where
\[c_\lambda = 2(1+\alpha)(\lambda - 2)^{\alpha/(\alpha+1)}\lambda^{-2}.
\]

Finally, for \(\lambda \geq 4\), set
\[
\phi_\lambda(x, y) = \begin{cases} 
0 & \text{if } (x, y) \in A_\lambda, \\
\frac{2\lambda-4|x|}{\lambda^2} + \frac{1-\alpha}{4} & \text{if } (x, y) \in B_\lambda, \\
\frac{x+1+2\alpha}{8} \exp \left( \frac{x+|y|+3-\lambda}{2(\alpha+1)} \right) & \text{if } (x, y) \in C_\lambda, \\
\frac{1-\alpha}{8} \exp \left( \frac{x+|y|+3-\lambda}{2(\alpha+1)} \right) & \text{if } (x, y) \in D_\lambda, \\
c_\lambda(x + |y| + 1)^{-\alpha/(\alpha+1)}(x + 1 + \frac{\alpha}{\alpha+1}|y|) & \text{if } (x, y) \in E_\lambda, 
\end{cases}
\]

and
\[
\psi_\lambda(x, y) = \begin{cases} 
0 & \text{if } (x, y) \in A_\lambda, \\
\frac{y'}{4} & \text{if } (x, y) \in B_\lambda, \\
\frac{2+2|x|y'}{1+\lambda-x-|y|} & \text{if } (x, y) \in C_\lambda, \\
\frac{1-\alpha}{8} \exp \left( \frac{x+|y|+3-\lambda}{2(\alpha+1)} \right) & \text{if } (x, y) \in D_\lambda, \\
c_\lambda(x + |y| + 1)^{-\alpha/(\alpha+1)} & \text{if } (x, y) \in E_\lambda, 
\end{cases}
\]

where
\[c_\lambda = (1+\alpha)2^{-(2\alpha+3)/(\alpha+1)}\exp \left( \frac{4-\lambda}{2(\alpha+1)} \right).
\]

Now the properties (i), (ii), (iii) follow by straightforward computation. To prove (iv), note first that for any \(\lambda \geq 2\) the condition (2.8) is clearly satisfied on the sets \(A_\lambda\) and \(B_\lambda\). Suppose \((x, y) \in C_\lambda\). Then \(\lambda - |y| \in [0, 4]\), \(1-x \leq \min\{\lambda - |y|, 4 - \lambda + |y|\}\) and (2.8) takes form
\[2\alpha(1-x) - \alpha |y| + (2\lambda - 4) - (1-\alpha)(1-x - \lambda - |y|)^2 + (\lambda - |y|)(1-\alpha) \leq 0,
\]

or
\[2\alpha(1-x) - \alpha |y| + (1-\alpha) - (1-x - \lambda - |y|)^2 + (\lambda - |y|)(1-\alpha) \leq 0,
\]

depending on whether \(\lambda < 4\) or \(\lambda \geq 4\). As \((2\lambda - 4)/\lambda^2 \leq \frac{1}{4}\), it suffices to show (2.9). If \(\lambda - |y| \leq 2\), then, as \(1-x \leq \lambda - |y|\), the left-hand side does not exceed
\[-2(\lambda - |y|) + (1-\alpha)(\lambda - |y|)^2 + 2\alpha(1-x) \leq 0,
\]

or
\[-2(\lambda - |y|) + 1 - \alpha \cdot (1-x - \lambda - |y|)^2 + 2\alpha(1-x) \leq 0,
\]

Similarly, if \(\lambda - |y| \in (2, 4]\), then we use the bound \(1-x \leq 4 - \lambda + |y|\) and conclude that the left-hand side of (2.9) is not greater than
\[-2(\lambda - |y|) + (1-\alpha) + 2\alpha(4 - \lambda + |y|) = -2(\lambda - |y| - 2)(1+\alpha) \leq 0
\]

and we are done with the case \((x, y) \in C_\lambda\).

Assume that \((x, y) \in D_\lambda\). For \(\lambda \in (2, 4]\), the inequality (2.8) is equivalent to
\[\frac{2}{\lambda} \left[ 1 - \frac{(1-\alpha)(\lambda - 2)}{\lambda} \right] + \frac{2 - 2x}{\lambda^2} \leq -\frac{2\alpha|y|}{\lambda^2},
\]

or, after some simplifications, \(\alpha|y| + 1 - x \leq 2 + \alpha\lambda - 2\alpha\). It is easy to check that \(\alpha|y| + 1 - x\) attains its maximum for \(x = -1\) and \(|y| = \lambda - 2\) and then we have the equality. If \((x, y) \in D_\lambda\) and \(\lambda \geq 4\), then (2.8) takes form
\[-2(\alpha+1+x) \leq -\alpha(1-x),
\]

or \((x + 1)(\alpha + 1) \geq 0\). Finally, on the set \(E_\lambda\), the inequality (2.8) is obvious.
(v) By (2.8), we have $\phi_\lambda \leq 0$, so $U_\lambda(x, y) \geq U_\lambda(1, y) = \chi_{|y| \geq \lambda}$. Furthermore, as $U_\lambda(x, y) = 1$ for $|y| \geq \lambda$ and $\psi_\lambda(x, y)y' \geq 0$ on $S_\lambda$, the second estimate follows. □

**Lemma 2.2.** Let $x, h, y, k$ be fixed real numbers, satisfying $x, x + h \in [-1, 1]$ and $|k| \leq |h|$. Then for any $\lambda > 2$ and $\alpha \in [0, 1)$,

$$
U_\lambda(x + h, y + k) \leq U_\lambda(x, y) + \phi_\lambda(x, y)h + \psi_\lambda(x, y)k.
$$

We will need the following fact, proved by Burkholder; see page 17 of [1].

**Lemma 2.3.** Let $x, h, y, k, z$ be real numbers satisfying $|k| \leq |h|$ and $z > -1$. Then the function

$$
F(t) = H(x + th, y + tk, z),
$$
defined on $\{t : |x + th| \leq 1\}$, is convex.

**Proof of the Lemma 2.2.** Consider the function

$$
G(t) = G_{x,y,h,k}(t) = U_\lambda(x + th, y + tk),
$$
defined on the set $\{t : |x + th| \leq 1\}$. It is easy to check that $G$ is continuous. As explained in [1], the inequality (2.10) follows once the concavity of $G$ is established. This will be done by proving the inequality $G'' \leq 0$ at the points, where $G$ is twice differentiable and checking the inequality $G''(t) \leq G''(t)$ for those $t$, for which $G$ is not differentiable (even once). Note that we may assume $t = 0$, by a translation argument $G''_{y,h,k}(t) = G''_{x,y,h,k}(0)$, with analogous equalities for one-sided derivatives. Clearly, we may assume that $h \geq 0$, changing the signs of both $h, k$, if necessary. Due to the symmetry of $U_\lambda$, we are allowed to consider $y \geq 0$ only.

We start from the observation that $G''(0) = 0$ on the interior of $A_\lambda$ and $G''(0) \leq G''(0)$ on $A_\lambda \cap B_\lambda$. The latter inequality holds since $U_\lambda \equiv 1$ on $A_\lambda$ and $U_\lambda \leq 1$ on $B_\lambda$. For the remaining inequalities, we consider the cases $\lambda \in (2, 4)$, $\lambda \geq 4$ separately.

The case $\lambda \in (2, 4)$. The inequality $G''(0) \leq 0$ is clear for $(x, y)$ lying in the interior of $B_\lambda$. On $C_\lambda$, we have

$$
G''(0) = \frac{-4(h + k)(h(\lambda - y) - k(1 - x))}{(1 - x - y + \lambda)^2} \leq 0,
$$
which follows from $|k| \leq h$ and the fact that $\lambda - y \geq 1 - x$. For $(x, y)$ in the interior of $D_\lambda$,

$$
G''(0) = \frac{-h^2 + k^2}{\lambda^2} \leq 0,
$$
as $|k| \leq h$. Finally, on $E_\lambda$, the concavity follows by Lemma 2.3.

It remains to check the inequalities for one-sided derivatives. By Lemma 2.1 (ii), the points $(x, y)$, for which $G$ is not differentiable at 0, do not belong to $S_\lambda$. Since we excluded the set $A_\lambda \cap B_\lambda$, they lie on the line $y = x + 1 + \lambda$. For such points $(x, y)$, the left derivative equals

$$
G'_-(0) = -\frac{2\lambda - 4}{\lambda^2}(\alpha h - k),
$$
while the right one is given by

$$
G'_+(0) = \frac{-h + k}{2(\lambda - y)} + \frac{(2\lambda - 4)(1 - \alpha)h}{\lambda^2},
$$
or

$$
G'_+(0) = -\frac{2h}{\lambda} \left[1 - \frac{(1 - \alpha)(\lambda - 2)}{\lambda}\right] + \frac{2(1 - x)h + 2yk}{\lambda^2}.
$$
depending on whether \(y \geq 1 - x\) or \(y < 1 - x\). In the first case, the inequality \(G_+^\prime(0) \leq G_-^\prime(0)\) reduces to
\[
(h - k)\left(\frac{1}{2(\lambda - y)} - \frac{2(\lambda - 2)}{\lambda^2}\right) \geq 0,
\]
while in the remaining one,
\[
\frac{2}{\lambda^2}(h - k)(y - (\lambda - 2)) \geq 0.
\]
Both inequalities follow from the estimate \(\lambda - y \leq 2\) and the condition \(|k| \leq h\).

The case \(\lambda \geq 4\). On the set \(B_\lambda\) the concavity is clear. For \(C_\lambda\), we have that the formula (2.11) holds. If \((x, y)\) since
\[
\lambda
\]
indeed, note that \(U\) an obvious one, as \(\lambda\) depends on whether \(k\)
\[
2\lambda\]
\(f\) is a submartingale satisfying \(|f|_\infty \leq 1\) and \(g\) is an adapted process which is \(\alpha\)-subordinate to \(f\). Then for all \(\lambda > 0\) we have
\[
P(g^\ast \geq \lambda) \leq EU_\lambda(f_0, g_0).
\]
Proof. If \(\lambda \leq 2\), then this follows immediately from the result of Hammack [4]; indeed, note that \(U_\lambda\) coincides with Hammack’s special function and, furthermore, since \(g\) is \(\alpha\)-subordinate to \(f\), it is also 1-subordinate to \(f\).

Fix \(\lambda > 2\). We may assume \(\alpha < 1\). It suffices to show that for any nonnegative integer \(n\),
\[
P(|g_n| \geq \lambda) \leq EU_\lambda(f_0, g_0).
\]
To see that this implies (3.1), fix \(\varepsilon > 0\) and consider a stopping time \(\tau = \inf\{k : |g_k| \geq \lambda - \varepsilon\}\). The process \(f^\tau = (f_{\tau \wedge n})\), by Doob’s optional sampling theorem, is a submartingale. Furthermore, we obviously have that \(|f^\tau|_\infty \leq 1\) and the process \(g^\tau = (g_{\tau \wedge n})\) is \(\alpha\)-subordinate to \(f^\tau\). Therefore, by (3.2),
\[
P(|g_n^\ast| \geq \lambda - \varepsilon) \leq EU_{\lambda - \varepsilon}(f_0^\tau, g_0^\tau) = EU_{\lambda - \varepsilon}(f_0, g_0).
\]
Now if we let \(n \to \infty\), we obtain \(P(g^\ast \geq \lambda) \leq EU_{\lambda - \varepsilon}(f_0, g_0)\) and by left-continuity of \(U_\lambda\) as a function of \(\lambda\), (3.1) follows.

Thus it remains to establish (3.2). By Lemma 2.1 (v), \(P(|g_n| \geq \lambda) \leq EU_\lambda(f_n, g_n)\) and it suffices to show that for all \(1 \leq j \leq n\) we have
\[
EU_\lambda(f_j, g_j) \leq EU_\lambda(f_{j-1}, g_{j-1}).
\]
To do this, note that, since \(|dg_j| \leq |df_j|\) almost surely, the inequality (2.10) yields
\[
U_\lambda(f_j, g_j) \leq U_\lambda(f_{j-1}, g_{j-1}) + \phi_\lambda(f_{j-1}, g_{j-1})df_j + \psi_\lambda(f_{j-1}, g_{j-1})dg_j
\]

3. The main theorem

Now we may state and prove the main result of the paper.

**Theorem 3.1.** Suppose \(f\) is a submartingale satisfying \(|f|_\infty \leq 1\) and \(g\) is an adapted process which is \(\alpha\)-subordinate to \(f\). Then for all \(\lambda > 0\) we have
\[
P(g^\ast \geq \lambda) \leq EU_\lambda(f_0, g_0).
\]
with probability 1. Assume for now that \( \phi_\lambda(f_j-1, g_j-1)df_j, \psi_\lambda(f_j-1, g_j-1)dg_j \) are integrable. By \( \alpha \)-subordination, the condition (2.8) and the submartingale property \( \mathbb{E}(d_j|\mathcal{F}_{j-1}) \geq 0 \), we have
\[
\mathbb{E}[\phi_\lambda(f_j-1, g_j-1)df_j + \psi_\lambda(f_j-1, g_j-1)dg_j|\mathcal{F}_{j-1}] \\
\leq \phi_\lambda(f_j-1, g_j-1)\mathbb{E}(df_j|\mathcal{F}_{j-1}) + |\psi_\lambda(f_j-1, g_j-1)| \cdot |\mathbb{E}(dg_j|\mathcal{F}_{j-1})| \\
\leq [\phi_\lambda(f_j-1, g_j-1) + \alpha|\psi_\lambda(f_j-1, g_j-1)|]\mathbb{E}(df_j|\mathcal{F}_{j-1}) \leq 0.
\]
Therefore, it suffices to take the expectation of both sides of (3.4) to obtain (3.3).

Thus the proof will be complete if we show the integrability of \( \phi_\lambda(f_j-1, g_j-1)df_j \) and \( \psi_\lambda(f_j-1, g_j-1)dg_j \). In both the cases \( \lambda \in (2, 4), \lambda \geq 4 \), all we need is that the variables
\[
\frac{2\lambda - 2|g_j-1|}{(1 - f_j-1 - |g_j-1| + \lambda)^2} df_j \quad \text{and} \quad \frac{2 - 2f_j-1}{(1 - f_j-1 - |g_j-1| + \lambda)^2} dg_j
\]
are integrable on the set \( K = \{|g_j-1| < f_j-1 + \lambda - 1, \quad |g_j-1| \geq \lambda - 1\} \), since outside it the derivatives \( \phi_\lambda, \psi_\lambda \) are bounded by a constant depending only on \( \alpha, \lambda \) and \( |df_j|, |dg_j| \) do not exceed 2. The integrability is proved exactly in the same manner as in [4]. We omit the details. \( \square \)

We will now establish the following sharp exponential inequality.

**Theorem 3.2.** Suppose \( f \) is a submartingale satisfying \( ||f||_\infty \leq 1 \) and \( g \) is an adapted process which is \( \alpha \)-subordinate to \( f \). In addition, assume that \( |g_0| \leq |f_0| \) with probability 1. Then for \( \lambda \geq 4 \) we have
\[
P(g^* \geq \lambda) \leq \gamma e^{-\lambda/(2\alpha+2)},
\]
where
\[
\gamma = \frac{1 + \alpha}{2\alpha + 4} \left( \alpha + 1 + 2^{-\frac{2+2}{\alpha+1}} \right) \exp \left( \frac{2}{\alpha + 1} \right).
\]
The inequality is sharp.

This should be compared to Burkholder’s estimate (Theorem 8.1 in [1])
\[
P(g^* \geq \lambda) \leq \frac{e^2}{4} \cdot e^{-\lambda}, \quad \lambda \geq 2,
\]
in the case when \( f, g \) are Hilbert space-valued martingales and \( g \) is subordinate to \( f \). For \( \alpha = 1 \), we obtain the inequality of Hammack [4],
\[
P(g^* \geq \lambda) \leq \frac{(8 + \sqrt{2})e}{12} \cdot e^{-\lambda/4}, \quad \lambda \geq 4.
\]

*Proof of the inequality (3.6).* We will prove that the maximum of \( U_\lambda \) on the set \( K = \{(x, y) \in S : |y| \leq |x|\} \) is given by the right hand side of (3.6). This, together with the inequality (3.1) and the assumption \( P((f_0, g_0) \in K) = 1 \), will imply the desired estimate. Clearly, by symmetry, we may restrict ourselves to the set \( K^+ = K \cap \{ y \geq 0 \} \). If \( (x, y) \in K^+ \) and \( x \geq 0 \), then it is easy to check that
\[
U_\lambda(x, y) \leq U_\lambda((x + y)/2, (x + y)/2).
\]
Furthermore, a straightforward computation shows that the function \( F : [0, 1] \to \mathbb{R} \) given by \( F(s) = U_\lambda(s, s) \) is nonincreasing. Thus we have \( U_\lambda(x, y) \leq U_\lambda(0, 0) \). On the other hand, if \( (x, y) \in K^+ \) and \( x \leq 0 \), then it is easy to prove that \( U_\lambda(x, y) \leq U_\lambda(-1, x + y + 1) \) and the function \( G : [0, 1] \to \mathbb{R} \) given by \( G(s) = U_\lambda(-1, s) \) is nondecreasing. Combining all these facts we have that for any \( (x, y) \in K^+ \),
\[
U_\lambda(x, y) \leq U_\lambda(-1, 1) = \gamma e^{-\lambda/(2\alpha+2)}.
\]
Thus (3.6) holds. The sharpness will be shown in the next section. \( \square \)
4. Sharpness

Recall the function $V_\lambda = V_{\alpha, \lambda}$ defined by (1.1) in the introduction. The main result in this section is Theorem 4.1 below, which, combined with Theorem 3.1, implies that the functions $U_\lambda$ and $V_\lambda$ coincide. If we apply this at the point $(-1, 1)$ and use the equality appearing in (3.7), we obtain that the inequality (3.6) is sharp.

**Theorem 4.1.** For any $\lambda > 0$ we have

\[
U_\lambda \leq V_\lambda.
\]

The main tool in the proof is the following ,,splicing” argument. Assume that the underlying probability space is the interval $[0, 1]$ with the Lebesgue measure.

**Lemma 4.1.** Fix $(x_0, y_0) \in [-1, 1] \times \mathbb{R}$. Suppose there exists a filtration and a pair $(f, g)$ of simple adapted processes, starting from $(x_0, y_0)$, such that $f$ is a submartingale satisfying $||f||_\infty \leq 1$ and $g$ is $\alpha$-subordinate to $f$. Then $V_\lambda(x_0, y_0) \geq \mathbb{E}V_\lambda(f_\infty, g_\infty)$ for $\lambda > 0$.

**Proof.** Let $N$ be such that $(f_N, g_N) = (f_\infty, g_\infty)$ and fix $\varepsilon > 0$. With no loss of generality, we may assume that $\sigma$-field generated by $f$, $g$ is generated by the family of intervals $[a_i, a_{i+1}) : i = 1, 2, \ldots, M - 1$, $0 = a_1 < a_2 < \ldots < a_M = 1$. For any $i \in \{1, 2, \ldots, M - 1\}$, denote $f^*_0 = f_N(a_i)$, $y^*_0 = g_N(a_i)$. There exists a filtration and a pair $(f', g')$ of adapted processes, with $f'$ being a submartingale bounded in absolute value by 1 and $g'$ being $\alpha$-subordinate to $f'$, which satisfy $f^*_0 = f^*_0(x_i^0, 1)$, $y^*_0 = g^*_0(x_i^0)$ and $\mathbb{P}(\langle f' \rangle^* \geq \lambda) > \mathbb{E}V_\lambda(f^*_0, g^*_0) - \varepsilon$. Define the processes $F$, $G$ by $F_k = f_k$, $G_k = g_k$ if $k \leq N$ and

\[
F_k(\omega) = \sum_{i=1}^{M-1} f^*_{k-N}(\omega - a_i)/(a_{i+1} - a_i) \chi_{[a_i, a_{i+1})}(\omega),
\]

\[
G_k(\omega) = \sum_{i=1}^{M-1} g^*_{k-N}(\omega - a_i)/(a_{i+1} - a_i) \chi_{[a_i, a_{i+1})}(\omega)
\]

for $k > N$. It is easy to check that there exists a filtration, relative to which the process $F$ is a submartingale satisfying $||F||_\infty \leq 1$ and $G$ is an adapted process which is $\alpha$-subordinate to $F$. Furthermore, we have

\[
\mathbb{P}(G^* \geq \lambda) \geq \sum_{i=1}^{M-1} (a_{i+1} - a_i) \mathbb{P}(\langle g' \rangle^* \geq \lambda) = \sum_{i=1}^{M-1} (a_{i+1} - a_i) \mathbb{E}V_\lambda(f^*_{0}, g^*_{0}) - \varepsilon = \mathbb{E}V_\lambda(f_\infty, g_\infty) - \varepsilon.
\]

Since $\varepsilon$ was arbitrary, the result follows. \[\square\]

**Proof of Theorem 4.1.** First note the following obvious properties of the functions $V_\lambda$, $\lambda > 0$: we have $V_\lambda \in [0, 1]$ and $V_\lambda(x, y) = V_\lambda(x, -y)$. The second equality is an immediate consequence of the fact that if $g$ is $\alpha$-subordinate to $f$, then so is $-g$.

In the proof of Theorem 4.1 we repeat several times the following procedure. Having fixed a point $(x_0, y_0)$ from the strip $S$, we construct certain simple finite processes $f$, $g$ starting from $(x_0, y_0)$, take their natural filtration $(\mathcal{F}_n)$, apply Lemma 4.1 and thus obtain a bound for $V_\lambda(x_0, y_0)$. All the constructed processes appearing in the proof below are easily checked to satisfy the conditions of this lemma: the condition $||f||_\infty \leq 1$ is straightforward, while the $\alpha$-subordination and the fact that $f$ is a submartingale are implied by the following. For any $n \geq 1$, either $df_n$ satisfies $\mathbb{E}(df_n|\mathcal{F}_{n-1}) = 0$ and $dg_n = \pm df_n$, or $df_n \geq 0$ and $dg_n = \pm \alpha df_n$. 


We will consider the cases $\lambda \leq 2$, $2 < \lambda < 4$, $\lambda \geq 4$ separately. Note that by symmetry, it suffices to establish (4.1) on $S \cap \{ y \geq 0 \}$.

The case $\lambda \leq 2$. Assume $(x_0, y_0) \in A_\lambda$. If $y_0 \geq \lambda$, then $g^* \geq \lambda$ almost surely, so $V_\lambda(x_0, y_0) \geq 1 = U_\lambda(x_0, y_0)$. If $\lambda > y_0 \geq \alpha x_0 - \alpha + \lambda$, then let $(f_0, g_0) \equiv (x_0, y_0)$,

$$df_1 = \frac{y_0 - x_0 + 1 - \lambda}{1 - \alpha} \chi(x_0, y_0) \quad \text{and} \quad dg_1 = \alpha df_1,$$

Then we have $g_1 = y_0 + \alpha - \alpha x_0 \geq \lambda$, which implies $g^* \geq \lambda$ almost surely and (4.1) follows. Now suppose $(x_0, y_0) \in A_\lambda$ and $y_0 < \alpha x_0 - \alpha + \lambda$. Let $(f, g) \equiv (x_0, y_0)$,

$$df_1 = \frac{y_0 - x_0 + 1 - \lambda}{1 - \alpha} \chi(x_0, y_0), \quad dg_1 = \alpha df_1$$

and

$$df_2 = dg_2 = \beta \chi(\lambda^2, 1) + (\beta - 2) \chi(1, 1),$$

where

$$\beta = \frac{\alpha x_0 - y_0 - \alpha + \lambda}{1 - \alpha} \in [0, 2].$$

Then $(f_2, g_2)$ takes values $(-1, \lambda - 2), (1, \lambda)$ with probabilities $\beta/2, 1 - \beta/2$, respectively, so, by Lemma 4.1,

$$V_\lambda(x_0, y_0) \geq \frac{\beta}{2} V_\lambda(-1, \lambda - 2) + (1 - \frac{\beta}{2}) V_\lambda(1, \lambda) = \frac{\beta}{2} V_\lambda(-1, \lambda - 2) + 1 - \frac{\beta}{2}.$$

Note that $(-1, \lambda - 2) \in A_\lambda$. If $2 - \lambda \geq \alpha \cdot (-1) - \alpha + \lambda$, then, as already proved, $V_\lambda(-1, \lambda - 2) = 1$ and $V_\lambda(x_0, y_0) \geq 1 = U_\lambda(x_0, y_0)$. If the converse inequality holds, i.e., $2 - \lambda < -2\alpha + \lambda$, then we may apply (4.6) to $x_0 = -1$, $y_0 = 2 - \lambda$ to get

$$V_\lambda(-1, \lambda - 2) \geq \frac{\beta}{2} V_\lambda(-1, \lambda - 2) + 1 - \frac{\beta}{2},$$

or $V_\lambda(-1, \lambda - 2) \geq 1$. Thus we established $V_\lambda(x_0, y_0) = 1$ for any $(x_0, y_0) \in A_\lambda$.

Suppose then, that $(x_0, y_0) \in B_\lambda$. Let

$$\beta = \frac{2(1 - x_0)}{1 - x_0 - y_0 + \lambda} \in [0, 1]$$

and consider a pair $(f, g)$ starting from $(x_0, y_0)$ and satisfying

$$df_1 = -dg_1 = \frac{x_0 - y_0 - 1 + \lambda}{2} \chi(x_0, y_0) + (1 - x_0) \chi(\beta, 1).$$

On $[0, \beta]$, the pair $(f_1, g_1)$ lies in $A_\lambda$; Lemma 4.1 implies $V_\lambda(x_0, y_0) \geq \beta = U_\lambda(x_0, y_0)$.

Finally, for $(x_0, y_0) \in C_\lambda$, let $(f, g)$ start from $(x_0, y_0)$ and

$$df_1 = -dg_1 = \frac{x_0 - \lambda + 1 + y_0}{2} \chi(x_0, y_0) + \frac{y_0 - x_0 + 1}{2} \chi(\gamma, 1),$$

where

$$\gamma = \frac{y_0 - x_0 + 1}{\lambda} \in [0, 1].$$

On $[0, \gamma]$, the pair $(f_1, g_1)$ lies in $A_\lambda$, while on $[\gamma, 1]$ we have $(f_1, g_1) = ((x_0 + y_0 + 1)/2, (x_0 + y_0 - 1)/2) \in B_\lambda$. Hence

$$V_\lambda(x_0, y_0) \geq \gamma \cdot 1 + (1 - \gamma) \cdot \frac{1 - x_0 - y_0}{\lambda} = U_\lambda(x_0, y_0).$$

The case $2 < \lambda < 4$. For $(x_0, y_0) \in A_\lambda$ we prove (4.1) using the same processes as in the previous case, i.e. the constant ones if $y_0 \geq \lambda$ and the ones given by (4.2) otherwise. The next step is to establish the inequality

$$V_\lambda(-1, \lambda - 2) \geq U_\lambda(-1, \lambda - 2) = \frac{1 + \alpha}{2} + \frac{1 - \alpha}{2} \cdot \left( \frac{4 - \lambda}{\lambda} \right)^2.$$
To do this, fix $\delta \in (0, 1]$ and set

$$\beta = \frac{\delta(1 - \alpha)}{\lambda}, \quad \kappa = \frac{4 - \lambda - \delta(1 + \alpha)}{\lambda}, \quad \beta, \gamma = \beta + (1 - \beta) \cdot \frac{\delta(1 + \alpha)}{4}, \quad \nu = \kappa \cdot \frac{\lambda}{4}$$

We have $0 \leq \nu \leq \kappa \leq \beta \leq \gamma \leq 1$. Consider processes $f$, $g$ given by $(f_0, g_0) \equiv (-1, \lambda - 2)$, $(df_1, dg_1) \equiv (\delta, \alpha \delta)$,

$$df_2 = -dg_2 = \frac{\lambda - \delta(1 - \alpha)}{2} \chi_{[0, \beta]} - \frac{\delta(1 - \alpha)}{2} \chi_{[\beta, 1]},$$

$$df_3 = dg_3 = -\left(\lambda - 2 + \frac{\delta(1 + \alpha)}{2}\right) \chi_{[0, \kappa]} + \left(2 - \frac{\delta(1 + \alpha)}{2}\right) \chi_{[\kappa, \beta]} + \left(2 - \frac{\delta(1 + \alpha)}{2}\right) \chi_{[\beta, \gamma]} - \frac{\delta(1 + \alpha)}{2} \chi_{[\gamma, 1]},$$

$$df_4 = -dg_4 = \left(-2 + \frac{\lambda}{2}\right) \chi_{[0, \kappa]} + \frac{\lambda}{2} \chi_{[\nu, \kappa]}.$$  

As $(f_4, |g_4|)$ takes values $(1, \lambda)$, $(1, 0)$ and $(-1, \lambda - 2)$ with probabilities $(\gamma - \beta) + (\kappa - \nu), \beta - \kappa$ and $1 - \gamma + \nu$, respectively, we have

$$V_\lambda(-1, \lambda - 2) \geq \gamma - \beta + \kappa - \nu + (1 - \gamma + \nu) V_\lambda(-1, \lambda - 2),$$

or

$$V_\lambda(-1, \lambda - 2) \geq \frac{\gamma - \beta + \kappa - \nu}{\gamma - \nu} = \frac{1 + \alpha}{2} + \frac{1 - \alpha}{2} \cdot \frac{(4 - \lambda)(\lambda - 2)}{\lambda^2} = \frac{\delta(1 - \alpha)^2}{\lambda^2}.$$

As $\delta$ is arbitrary, we obtain (4.9). Now suppose $(x_0, y_0) \in B_\lambda$ and recall the pair $(f, g)$ starting from $(x_0, y_0)$ given by (4.3) and (4.4) (with $\beta$ defined in (4.5)). As previously, it leads to (4.6), which takes form

$$V_\lambda(x_0, y_0) \geq \frac{\beta}{2} \left[\frac{1 + \alpha}{2} + \frac{1 - \alpha}{2} \cdot \frac{(4 - \lambda)(\lambda - 2)}{\lambda^2}\right] + \frac{\beta}{2}$$

$$= \frac{\beta(1 - \alpha)}{4} \left[\frac{4 - \lambda}{\lambda}\right]^2 - 1 + 1 = \frac{(\alpha x_0 - \alpha - y_0 + \lambda)(4 - 2\lambda)}{\lambda^2} + 1 = U_\lambda(x_0, y_0).$$

For $(x_0, y_0) \in C_\lambda$, consider a pair $(f, g)$, starting from $(x_0, y_0)$ defined by (4.8) (with $\beta$ given by (4.7)). On $[0, \beta]$ we have $(f_1, g_1) = ((x_0 + y_0 + 1 - \lambda)/2, (x_0 + y_0 + 1 + \lambda)/2) \in B_\lambda$, so Lemma 4.1 yields

$$V_\lambda(x_0, y_0) \geq \beta V_\lambda\left(\frac{x_0 + y_0 + 1 - \lambda}{2}, \frac{x_0 + y_0 - 1 + \lambda}{2}\right)$$

$$= \frac{2(1 - x_0)}{1 + \lambda - x_0 - y_0} \cdot \left\{1 - \left[\alpha\left(\frac{x_0 + y_0 - 1 - \lambda}{2}\right) - \frac{x_0 + y_0 - 1 - \lambda}{2}\right] \cdot \frac{2\lambda - 4}{\lambda^2}\right\} = U_\lambda(x_0, y_0).$$

For $(x_0, y_0) \in D_\lambda$, set $\beta = (y_0 - x_0 + 1)/\lambda \in [0, 1]$ and let a pair $(f, g)$ be given by $(f_0, g_0) \equiv (x_0, y_0)$ and

$$df_1 = -dg_1 = -\frac{x_0 + y_0 + 1 - \lambda}{2} \chi_{[0, \beta]} + \frac{-x_0 + y_0 + 1 - \lambda}{2} \chi_{[\beta, 1]}.$$

As $(f_1, g_1)$ takes values

$$\left(\frac{x_0 + y_0 + 1 - \lambda}{2}, \frac{x_0 + y_0 + 1 - \lambda}{2}\right) \in B_\lambda \quad \text{and} \quad \left(\frac{x_0 + y_0 + 1}{2}, \frac{x_0 + y_0 - 1}{2}\right) \in C_\lambda$$

with probabilities $\beta$ and $1 - \beta$, respectively, we obtain $V_\lambda(x_0, y_0)$ is not smaller than

$$\beta V_\lambda\left(\frac{x_0 + y_0 + 1 - \lambda}{2}, \frac{x_0 + y_0 - 1 + \lambda}{2}\right) + (1 - \beta)V_\lambda\left(\frac{x_0 + y_0 + 1}{2}, \frac{x_0 + y_0 - 1}{2}\right)$$

$$= \frac{y_0 - x_0 + 1}{\lambda} \cdot \left\{1 - \left[\alpha\left(\frac{x_0 + y_0 - 1 - \lambda}{2}\right) - \frac{x_0 + y_0 - 1 - \lambda}{2}\right] \cdot \frac{2\lambda - 4}{\lambda^2}\right\}$$

$$+ \frac{\lambda - y_0 + x_0 - 1}{\lambda} \left[\frac{1 - x_0 - y_0}{\lambda} - \frac{(1 - x_0 - y_0)(1 - \alpha)(\lambda - 2)}{\lambda^2}\right].$$
Furthermore, set
\[ dg = \frac{y_0 - x_0 + 1}{\lambda} + \frac{(\lambda - y_0 + x_0 - 1)(1 - x_0 - y_0)}{\lambda^2} = 2(1 - x_0) - \frac{(1 - x_0)^2 - y_0^2}{\lambda^2} \]
and
\[
I + III = \frac{(1 - \alpha)(\lambda - 2)}{\lambda^3} \left[ (y_0 - x_0 + 1)(y_0 + x_0 - 1 - \lambda) - (1 - x_0 - y_0)(\lambda - y_0 + x_0 - 1) \right] = \frac{(1 - \alpha)(\lambda - 2)}{\lambda^3} \cdot \lambda(2 - 2x_0).
\]
Combining these facts, we obtain \( V_\lambda(x_0, y_0) \geq U_\lambda(x_0, y_0) \).

Finally, the variable \( f \) satisfies (4.10) for \( j = 1, 2, \ldots, N + 1 \), and we set
\[ p_1^N = (1 + x_0)/(1 + x_0 + y_0), \]
\[ (4.10) \quad p_{j+1}^N = \frac{(1 + x_N^j)(1 + x_N^j + \delta(\alpha + 1))}{(1 + x_N^{j+1})(1 + x_N^{j+1} + \delta(\alpha + 1))} + \delta \frac{1 + x_N^{j+1}}{1 + x_N^{j+1}}, \quad j = 1, 2, \ldots, N. \]
We construct a process \((f, g)\) starting from \((x_0, y_0)\) such that for \( j = 1, 2, \ldots, N + 1 \), the variable \((f_N, g_N)\) takes values \((x_N^j, 0)\) and \((-1, 1 + x_N^j)\) with probabilities \(p_j^N\) and \(1 - p_j^N\), respectively.

We do this by induction. Let
\[ df_1 = -dg_1 = y_0 \chi_{[0, x_N^1]} + (-1 - x_0) \chi_{[x_N^1, 1]}, \quad df_2 = dg_2 = df_3 = dg_3 = 0. \]
Note that (4.11) is satisfied for \( j = 1 \). Now suppose we have a pair \((f, g)\), which satisfies (4.11) for \( j = 1, 2, \ldots, n, n \leq N \). Let us describe \( f_k \) and \( g_k \) for \( k = 3n + 1, 3n + 2, 3n + 3 \). The difference \( df_{3n+1} \) is determined by the following three conditions: it is a martingale difference, i.e., satisfies \( E(df_{3n+1}|\mathcal{F}_{3n}) = 0 \); conditionally on \( \{f_{3n} = x_N^n\} \), it takes values in \( \{-1, x_N^n, \delta(\alpha + 1)/2\} \); and vanishes on \( \{f_{3n} \neq x_N^n\} \). Furthermore, set \( dg_{3n+2} = df_{3n+1} \). Moreover,
\[ df_{3n+2} = \delta \chi_{(f_{3n+1} = -1)}, \quad dg_{3n+2} = \frac{g_{3n+1}}{g_{3n+1} + 1} \alpha \cdot df_{3n+2}. \]
Finally, the variable \( df_{3n+3} \) satisfies \( E(df_{3n+3}|\mathcal{F}_{3n+2}) = 0 \), and, in addition, the variable \( f_{3n+3} \) takes values in \( \{-1, x_N^n + \delta(\alpha + 1)\} = \{-1, x_N^{n+1}\} \). The description is completed by
\[ df_{3n+3} = \frac{g_{3n+2}}{g_{3n+2}} df_{3n+3}. \]
One easily checks that \((f_{3n+3}, g_{3n+3})\) takes values in \( \{(x_N^{n+1}, 0), (1 - 1 + x_N^{n+1})\} \); moreover, since
\[
Ef_{3n+3} = Ef_{3n} + Edf_{3n+2} = x_N^n p_n^N - (1 - p_n^N) + \delta \mathbb{P}(f_{3n+1} = -1),
\]
\[
= x_N^n p_n^N - (1 - p_n^N) + \delta \left(1 - p_n^N + p_n^N \frac{\delta(\alpha + 1)}{2(1 + x_N^n) + \delta(\alpha + 1)}\right)
\]
\[
= p_n^N \left(1 - x_N^n + \delta(\alpha + 1)/2\right) + \delta - 1,
\]
we see that \( \mathbb{P}(f_{n+1} = x_N^{n+1}) = p_{n+1}^N \) and the pair \((f, g)\) satisfies (4.10) for \( j = n + 1 \). Thus there exists \((f, g)\) satisfying (4.10) for \( j = 1, 2, \ldots, N + 1 \). In particular,
$(f_{3(N+1)}, g_{3(N+1)})$ takes values $(w, 0), (−1, w + 1) \in D_\lambda$ with probabilities $p_N^{N_0}, p_{N+1}^{N_0}$. By Lemma 4.1, 
\begin{equation}
V(x_0, y_0) \geq p_{N+1}^{N_0}V_{\lambda}(w, 0) + (1 - p_{N+1}^{N_0})V_{\lambda}(−1, w + 1).
\end{equation}
Recall the function $H$ defined by (2.1). The function $h: [x_0 + y_0, w] \to \mathbb{R}$ given by $h(t) = H(x_0, y_0, t)$, satisfies the differential equation
$$h'(t) + \frac{\alpha + 2}{\alpha + 1} \cdot \frac{h(t)}{1 + t} = \frac{1}{(\alpha + 1) (1 + t)}.$$ 
As we assumed $x_0 + y_0 > −1$, the expression $(h(x + \delta) − h(x))/\delta$ converges uniformly to $h'(x)$ on $[x_0 + y_0, λ − 3]$. Therefore there exist constants $\varepsilon_N$, which depend only on $N$ and $x_0 + y_0$ satisfying $\lim_{N \to \infty} \varepsilon_N = 0$ and for $1 \leq j \leq N$,
$$\frac{|h(x_j^{N+1}) - h(x_j^{N})|}{(\alpha + 1) \delta_N} + \left[ \frac{\alpha + 2}{\alpha + 1}(1 + x_j^{N}) - \frac{\delta_N (\alpha + 1)}{2} \right] \frac{h(x_j^{N})}{(1 + x_j^{N})(1 + x_j^{N} + \frac{\delta_N (\alpha + 1)}{2})} \leq \frac{1}{(\alpha + 1) (1 + x_j^{N+1})} \leq \varepsilon_N,$$
or, equivalently,
$$\frac{|h(x_j^{N+1}) - h(x_j^{N})|}{(\alpha + 1) \delta_N} + \left[ \frac{\alpha + 2}{\alpha + 1}(1 + x_j^{N}) - \frac{\delta_N (\alpha + 1)}{2} \right] \frac{h(x_j^{N})}{(1 + x_j^{N})(1 + x_j^{N} + \frac{\delta_N (\alpha + 1)}{2})} \leq \frac{\delta_N (\alpha + 1)}{1 + x_j^{N+1}} \leq \varepsilon_N.$$ 
Together with (4.10), this leads to
$$|h(x_j^{N+1}) - p_j^{N+1}| \leq \left[ \frac{1 + x_j^{N}}{(1 + x_j^{N})(1 + x_j^{N} + \frac{\delta_N (\alpha + 1)}{2})} \right] |h(x_j^{N}) - p_j^{N}| + (\alpha + 1) \delta_N \varepsilon_N.$$ 
Since $p_j^{N} = h(x_j^{N})$, we have
$$|h(w) - p_N^{N+1}| \leq (\alpha + 1) N \delta_N \varepsilon_N = (\lambda - 3 - x_0 - y_0) \varepsilon_N$$
and hence $\lim_{N \to \infty} p_N^{N+1} = h(w)$. Combining this with (4.12), we obtain
$$V_\lambda(x_0, y_0) \geq h(w)(V_\lambda(w, 0) - V_\lambda(-1, w + 1)) + V_\lambda(-1, w + 1).$$
As $w = \lambda - 3$, it suffices to check that we have
$$a_\lambda = V_\lambda(\lambda - 3, 0) - V_\lambda(-1, \lambda - 2))$$
and $b_\lambda = V_\lambda(-1, \lambda - 2)$,
where $a_\lambda$, $b_\lambda$ were defined in (2.4). Finally, if $(x_0, y_0) = (−1, 0)$, then considering a pair $(f, g)$ starting from $(x_0, y_0)$ and satisfying $df_1 \equiv \delta$, $dg_1 \equiv \alpha \delta$, we get
\begin{equation}
V(-1, 0) \geq V(-1, -\delta, \alpha \delta).
\end{equation}
Now let $\delta \to 0$ to obtain $V(-1, 0) \geq U(-1, 0)$.

The case $\lambda \geq 4$. We proceed as in previous case. We deal with $(x_0, y_0) \in A_\lambda$ exactly in the same manner. Then we establish the analogue of (4.9), which is
\begin{equation}
V(-1, \lambda - 2) \geq U(-1, \lambda - 2) = \frac{1 + \alpha}{2}.
\end{equation}
To do this, fix $\delta \in (0, 1)$ and set
$$\beta = \frac{4 - 2\delta}{4 - \delta}(1 + \alpha), \quad \gamma = \beta \cdot \left(1 - \frac{\delta(\alpha + 1)}{4}\right).$$ 
Now let a pair $(f, g)$ be defined by $(f_0, g_0) \equiv (−1, \lambda - 2)$, $(df_1, dg_1) \equiv (\delta, \alpha \delta)$,
$$df_2 = -dg_2 = -\frac{\delta(\alpha - 1)}{2} \chi_{[0, \beta)} + (2 - \delta) \chi_{[\beta, 1]},$$
$$df_3 = dg_3 = -\frac{\delta(\alpha + 1)}{2} \chi_{[0, \gamma)} + (2 - \delta(\alpha + 1)) \chi_{[\gamma, \beta)}.$$
Then \((f_\lambda, g_\lambda)\) takes values \((-1, \lambda - 2), (1, \lambda)\) and \((1, \lambda - 4 + \delta(\alpha + 1))\) with probabilities \(\gamma, \beta - \gamma\) and \(1 - \beta\), respectively, and Lemma 4.1 yields
\[
V(-1, \lambda - 2) \geq \gamma V(-1, \lambda - 2) + (\beta - \gamma) V(1, \lambda),
\]
or
\[
V(-1, \lambda - 2) \geq \frac{\beta - \gamma}{1 - \gamma} = \frac{(\alpha + 1)(2 - \delta)}{4 - \delta(\alpha + 1)}.
\]
It suffices to let \(\delta \to 0\) to obtain (4.14). The cases \((x_0, y_0) \in B_\lambda, C_\lambda\) are dealt with using the same processes as in the case \(\lambda \in (2, 4)\). If \((x_0, y_0) \in D_\lambda\), then Lemma 4.1, applied to the pair \((f, g)\) given by \((f_0, y_0) \equiv (x_0, y_0), df_1 = -dg_1 = -(1 + x_0)\chi_{[0, (1 - x_0)/2]} + (1 - x_0)\chi_{[(1 - x_0)/2, 1]},\) yields
\[
V(x_0, y_0) \geq \frac{1 - x_0}{2} V(-1, x_0 + y_0 + 1).
\]
Furthermore, for any number \(y\) and any \(\delta \in (0, 1)\), we have
\[
V(-1, y) \geq V(-1 + \delta, y + \alpha \delta),
\]
which is proved in the same manner as (4.13). Hence, for large \(N\), if we set \(\delta = (\lambda - 3 - x_0 - y_0) / (N(\alpha + 1))\), the inequalities (4.15) and (4.16) give
\[
V(x_0, y_0) \geq \frac{1 - x_0}{2} V(-1, x_0 + y_0 + 1) \geq \frac{1 - x_0}{2} V(-1 + \delta, x_0 + y_0 + 1 + \alpha \delta)
\]
\[
\geq \frac{1 - x_0}{2} \left(1 - \frac{\delta}{2}\right) V(-1, x_0 + y_0 + 1 + (\alpha + 1)\delta)
\]
\[
\geq \frac{1 - x_0}{2} \left(1 - \frac{\delta}{2}\right)^N V(-1, x_0 + y_0 + 1 + N(\alpha + 1)\delta)
\]
\[
= \frac{1 - x_0}{2} \left(1 - \frac{\lambda - 3 - x_0 - y_0}{2N(\alpha + 1)}\right)^N V(-1, \lambda - 2)
\]
\[
= \frac{(1 - x_0)(1 + \alpha)}{4} \left(1 - \frac{\lambda - 3 - x_0 - y_0}{2N(\alpha + 1)}\right)^N.
\]
Now take \(N \to \infty\) to obtain \(V_\lambda(x_0, y_0) \geq U_\lambda(x_0, y_0)\).

Finally, if \((x_0, y_0) \in E_\lambda\) we use the pair \((f, g)\) used in the proof of the case \((x_0, y_0) \in E_\lambda, \lambda \in (2, 4)\), with \(\omega = 1\). Then the process \((f, |g|)\) ends at the points \((1, 0)\) and \((-1, 2)\) with probabilities, which can be made arbitrarily close to \(H(x_0, y_0, 1)\) and \(1 - H(x_0, y_0, 1)\), respectively. It suffices to apply Lemma 4.1 and check that it gives \(V_\lambda(x_0, y_0) \geq U_\lambda(x_0, y_0)\). \(\Box\)

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**References**


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