# SHARP INEQUALITY FOR BOUNDED SUBMARTINGALES AND THEIR DIFFERENTIAL SUBORDINATES

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Abstract. For any fixed $\alpha \in [0,1]$ and $\lambda > 0$ we determine the optimal function $V_{\alpha,\lambda}$ satisfying

$$\mathbb{P}(\max_n |g_n| \geq \lambda) \leq \mathbb{E}V_{\alpha,\lambda}(f_0, g_0)$$

for any submartingale $f = (f_n)$ bounded in absolute value by 1 and any process $g = (g_n)$ which is real-valued, adapted, integrable and satisfying

$$|dg_n| \leq |df_n| \quad \text{and} \quad |\mathbb{E}(dg_n|\mathcal{F}_{n-1})| \leq \alpha\mathbb{E}(df_n|\mathcal{F}_{n-1}), \qquad n = 1, 2 \ldots,$$

with probability 1. As a corollary, a sharp exponential inequality for the distribution function of $\max_n |g_n|$ is established.

## 1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, equipped with a discrete filtration $(\mathcal{F}_n)$. Let $f = (f_n)_{n=0}^\infty$, $g = (g_n)_{n=0}^\infty$ be adapted integrable processes taking values in a certain separable Hilbert space $\mathcal{H}$. The difference sequences $df = (df_n)$, $dg = (dg_n)$ of these processes are given by

$$df_0 = f_0, \quad df_n = f_n - f_{n-1}, \quad dg_0 = g_0, \quad dg_n = g_n - g_{n-1}, \quad n = 1,\ 2,\ \ldots.$$

Let $g^*$ stand for the maximal function of $g$, that is, $g^* = \max_n |g_n|$.

The following notion of differential subordination is due to Burkholder. The process $g$ is differentially subordinate to $f$ (or, in short, subordinate to $f$) if for any nonnegative integer $n$ we have, almost surely,

$$|dg_n| \leq |df_n|.$$

We will slightly change this definition and say that $g$ is differentially subordinate to $f$, if the above inequality for the differences holds for any *positive* integer $n$.

Let $\alpha$ be a fixed nonnegative number. Then $g$ is $\alpha$-differentially subordinate to $f$ (or, in short, $\alpha$-subordinate to $f$), if it is subordinate to $f$ and for any positive integer $n$ we have

$$|\mathbb{E}(dg_n|\mathcal{F}_{n-1})| \leq \alpha|\mathbb{E}(df_n|\mathcal{F}_{n-1})|.$$

This concept was introduced by Burkholder in [2] in the special case $\alpha = 1$. In general form, it first appeared in the paper by Choi [3].

In the sequel it will sometimes be convenient to work with simple processes. A process $f$ is called simple, if for any $n$ the variable $f_n$ is simple and there exists $N$ such that $f_N = f_{N+1} = f_{N+2} = \ldots$. Given such a process, we will identify it with the finite sequence $(f_n)_{n=0}^N$.

Assume that the processes $f$ and $g$ are real-valued and fix $\alpha \in [0,1]$. The objective of this paper is to establish a sharp exponential inequality for the distribution function of $g^*$ under the assumption that $f$ is a submartingale satisfying $||f||_\infty \leq 1$

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and $g$ is $\alpha$-subordinate to $f$. To be more precise, for any $\lambda > 0$ define the function $V_{\alpha,\lambda} : [-1, 1] \times \mathbb{R} \to \mathbb{R}$ by the formula

$$V_{\alpha,\lambda}(x_0, y_0) = \sup \mathbb{P}(g^* \geq \lambda). \tag{1.1}$$

Here the supremum is taken over all pairs $(f, g)$ of integrable adapted processes, such that $(f_0, g_0) \equiv (x_0, y_0)$ almost surely, $f$ is a submartingale satisfying $||f||_\infty \leq 1$ and $g$ is $\alpha$-subordinate to $f$. The filtration must also vary, as well as the probability space, unless it is nonatomic. Our main result is an explicit formula for the functions $V_{\alpha,\lambda}$, $\lambda > 0$. Usually we will omit the index $\alpha$ and write $V_\lambda$ instead of $V_{\alpha,\lambda}$.

Let us discuss some related results which appeared in the literature. In [1] Burkholder studied the analogous question in the case of $f$, $g$ being Hilbert space-valued martingales. The paper [1] contains also a related one-sided sharp exponential inequality for real martingales. This work was later extended by Hammack [4], who established a similar (two-sided) inequality under the assumption that $f$ is a submartingale bounded by 1 and $g$ is $R^\nu$-valued, $\nu \geq 1$, and strongly 1-subordinate to $f$. Both papers present applications to stochastic integrals.

The paper is organized as follows. In the next section we introduce a family of special functions $U_\lambda$, $\lambda > 0$ and study their properties. This enables us to establish the inequality $V_\lambda \leq U_\lambda$ in Section 3. Then we prove the reverse inequality in the last section.

Throughout the paper, $\alpha$ is a fixed number from the interval $[0, 1]$. All the considered processes are assumed to be real valued.

## 2. The explicit formulas

Let $S$ be the strip $[-1, 1] \times \mathbb{R}$. Consider the following subsets of $S$: for $0 < \lambda \leq 2$,

$$\begin{aligned} A_\lambda &= \{(x, y) \in S : |y| \geq x + \lambda - 1\}, \\ B_\lambda &= \{(x, y) \in S : 1 - x \leq |y| < x + \lambda - 1\}, \\ C_\lambda &= \{(x, y) \in S : |y| < 1 - x \text{ and } |y| < x + \lambda - 1\}. \end{aligned}$$

For $\lambda \in (2, 4)$, define

$$\begin{aligned} A_\lambda &= \{(x, y) \in S : |y| \geq \alpha x + \lambda - \alpha\}, \\ B_\lambda &= \{(x, y) \in S : \alpha x + \lambda - \alpha > |y| \geq x - 1 + \lambda\}, \\ C_\lambda &= \{(x, y) \in S : x - 1 + \lambda > |y| \geq 1 - x\}, \\ D_\lambda &= \{(x, y) \in S : 1 - x > |y| \geq -x - 3 + \lambda \text{ and } |y| < x - 1 + \lambda\}, \\ E_\lambda &= \{(x, y) \in S : -x - 3 + \lambda > |y|\}. \end{aligned}$$

Finally, for $\lambda \geq 4$, let

$$\begin{aligned} A_\lambda &= \{(x, y) \in S : |y| \geq \alpha x + \lambda - \alpha\}, \\ B_\lambda &= \{(x, y) \in S : \alpha x + \lambda - \alpha > |y| \geq x - 1 + \lambda\}, \\ C_\lambda &= \{(x, y) \in S : x - 1 + \lambda > |y| \geq -x - 3 + \lambda\}, \\ D_\lambda &= \{(x, y) \in S : -x - 3 + \lambda > |y| \geq 1 - x\}, \\ E_\lambda &= \{(x, y) \in S : 1 - x > |y|\}. \end{aligned}$$

Let $H : S \times (-1, \infty) \to \mathbb{R}$ be a function given by

$$H(x, y, z) = \frac{1}{\alpha + 2}\Big[1 + \frac{(x + 1 + |y|)^{1/(\alpha+1)}((\alpha + 1)(x + 1) - |y|)}{(1 + z)^{(\alpha+2)/(\alpha+1)}}\Big]. \tag{2.1}$$

Now we will define the special functions $U_\lambda : S \to \mathbb{R}$. For $0 < \lambda \leq 2$, let

$$U_\lambda(x, y) = \begin{cases} 1 & \text{if } (x, y) \in A_\lambda, \\ \frac{2-2x}{1+\lambda-x-|y|} & \text{if } (x, y) \in B_\lambda, \\ 1 - \frac{(\lambda-1+x-|y|)(\lambda-1+x+|y|)}{\lambda^2} & \text{if } (x, y) \in C_\lambda. \end{cases} \tag{2.2}$$

For $2 < \lambda < 4$, set

$$U_\lambda(x,y) = \begin{cases} 1 & \text{if } (x,y) \in A_\lambda, \\ 1 - (\alpha(x-1) - |y| + \lambda) \cdot \frac{2\lambda-4}{\lambda^2} & \text{if } (x,y) \in B_\lambda, \\ \frac{2-2x}{1+\lambda-x-|y|} - \frac{2(1-x)(1-\alpha)(\lambda-2)}{\lambda^2} & \text{if } (x,y) \in C_\lambda, \\ \frac{2(1-x)}{\lambda}\left[1 - \frac{(1-\alpha)(\lambda-2)}{\lambda}\right] - \frac{(1-x)^2-|y|^2}{\lambda^2} & \text{if } (x,y) \in D_\lambda, \\ a_\lambda H(x,y,\lambda-3) + b_\lambda & \text{if } (x,y) \in E_\lambda, \end{cases} \tag{2.3}$$

where

$$a_\lambda = -\frac{2(1+\alpha)(\lambda-2)^2}{\lambda^2}, \quad b_\lambda = 1 - \frac{4(\lambda-2)(1-\alpha)}{\lambda^2}. \tag{2.4}$$

For $\lambda \geq 4$, set

$$U_\lambda(x,y) = \begin{cases} 1 & \text{if } (x,y) \in A_\lambda, \\ 1 - \frac{\alpha(x-1)-|y|+\lambda}{4} & \text{if } (x,y) \in B_\lambda, \\ \frac{2-2x}{1+\lambda-x-|y|} - \frac{(1-x)(1-\alpha)}{4} & \text{if } (x,y) \in C_\lambda, \\ \frac{(1-x)(1+\alpha)}{4} \exp\left(\frac{3+x+|y|-\lambda}{2(\alpha+1)}\right) & \text{if } (x,y) \in D_\lambda, \\ a_\lambda H(x,y,1) + b_\lambda & \text{if } (x,y) \in E_\lambda, \end{cases} \tag{2.5}$$

where

$$a_\lambda = -b_\lambda = -\frac{(1+\alpha)}{2} \exp\left(\frac{4-\lambda}{2\alpha+2}\right). \tag{2.6}$$

For $\alpha = 1$, the formulas (2.2), (2.3), (2.5) give the special functions constructed by Hammack [4]. The key properties of $U_\lambda$ are described in the two lemmas below.

**Lemma 2.1.** *For $\lambda > 2$, let $\phi_\lambda$, $\psi_\lambda$ denote the partial derivatives of $U_\lambda$ with respect to $x$, $y$ on the interiors of $A_\lambda$, $B_\lambda$, $C_\lambda$, $D_\lambda$, $E_\lambda$, extended continuously to the whole of these sets. The following statements hold.*

*(i) The functions $U_\lambda$, $\lambda > 2$, are continuous on $S \setminus \{(1, \pm\lambda)\}$ .*

*(ii) Let*

$$S_\lambda = \{(x,y) \in [-1,1] \times \mathbb{R} : |y| \neq \alpha x + \lambda - \alpha \text{ and } |y| \neq x + \lambda - 1\}.$$

*Then*

$$\phi_\lambda,\ \psi_\lambda,\ \lambda > 2, \text{ are continuous on } S_\lambda. \tag{2.7}$$

*(iii) For any $(x,y) \in S$, the function $\lambda \mapsto U_\lambda(x,y)$, $\lambda > 0$, is left-continuous.*

*(iv) For any $\lambda > 2$ we have the inequality*

$$\phi_\lambda \leq -\alpha|\psi_\lambda|. \tag{2.8}$$

*(v) For $\lambda > 2$ and any $(x,y) \in S$ we have $\chi_{\{|y| \geq \lambda\}} \leq U_\lambda(x,y) \leq 1$.*

*Proof.* We start with computing the derivatives. Let $y' = y/|y|$ stand for the sign of $y$, with $y' = 0$ if $y = 0$. For $\lambda \in (2,4)$ we have

$$\phi_\lambda(x,y) = \begin{cases} 0 & \text{if } (x,y) \in A_\lambda, \\ -\frac{(2\lambda-4)\alpha}{\lambda^2} & \text{if } (x,y) \in B_\lambda, \\ -\frac{2\lambda-2|y|}{(1+\lambda-x-|y|)^2} + \frac{(2\lambda-4)(1-\alpha)}{\lambda^2} & \text{if } (x,y) \in C_\lambda, \\ -\frac{2}{\lambda}\left[1 - \frac{(1-\alpha)(\lambda-2)}{\lambda}\right] + \frac{2(1-x)}{\lambda^2} & \text{if } (x,y) \in D_\lambda, \\ -c_\lambda(x+|y|+1)^{-\alpha/(\alpha+1)}(x+1+\frac{\alpha}{\alpha+1}|y|) & \text{if } (x,y) \in E_\lambda, \end{cases}$$

$$\psi_\lambda(x,y)=\begin{cases} 0 & \text{if } (x,y)\in A_\lambda,\\ \frac{2\lambda-4}{\lambda^2}y' & \text{if } (x,y)\in B_\lambda,\\ \frac{2-2x}{(1+\lambda-x-|y|)^2}y' & \text{if } (x,y)\in C_\lambda,\\ \frac{2y}{\lambda^2} & \text{if } (x,y)\in D_\lambda,\\ c_\lambda(x+|y|+1)^{-\alpha/(\alpha+1)}\frac{y}{1+\alpha} & \text{if } (x,y)\in E_\lambda, \end{cases}$$

where

$$c_\lambda = 2(1+\alpha)(\lambda-2)^{\alpha/(\alpha+1)}\lambda^{-2}.$$

Finally, for $\lambda\geq 4$, set

$$\phi_\lambda(x,y)=\begin{cases} 0 & \text{if } (x,y)\in A_\lambda,\\ -\frac{\alpha}{4} & \text{if } (x,y)\in B_\lambda,\\ -\frac{2\lambda-2|y|}{(1+\lambda-x-|y|)^2}+\frac{1-\alpha}{4} & \text{if } (x,y)\in C_\lambda,\\ -\frac{x+1+2\alpha}{8}\exp\left(\frac{x+|y|+3-\lambda}{2(\alpha+1)}\right) & \text{if } (x,y)\in D_\lambda,\\ -c_\lambda(x+|y|+1)^{-\alpha/(\alpha+1)}(x+1+\frac{\alpha}{\alpha+1}|y|) & \text{if } (x,y)\in E_\lambda, \end{cases}$$

$$\psi_\lambda(x,y)=\begin{cases} 0 & \text{if } (x,y)\in A_\lambda,\\ \frac{1}{4}y' & \text{if } (x,y)\in B_\lambda,\\ \frac{2-2x}{(1+\lambda-x-|y|)^2}y' & \text{if } (x,y)\in C_\lambda,\\ \frac{(1-x)}{8}\exp\left(\frac{x+|y|+3-\lambda}{2(\alpha+1)}\right)y' & \text{if } (x,y)\in D_\lambda,\\ c_\lambda(x+|y|+1)^{-\alpha/(\alpha+1)}\frac{y}{1+\alpha} & \text{if } (x,y)\in E_\lambda, \end{cases}$$

where

$$c_\lambda=(1+\alpha)2^{-(2\alpha+3)/(\alpha+1)}\exp\left(\frac{4-\lambda}{2(\alpha+1)}\right).$$

Now the properties (i), (ii), (iii) follow by straightforward computation. To prove (iv), note first that for any $\lambda>2$ the condition (2.8) is clearly satisfied on the sets $A_\lambda$ and $B_\lambda$. Suppose $(x,y)\in C_\lambda$. Then $\lambda-|y|\in[0,4]$, $1-x\leq\min\{\lambda-|y|,4-\lambda+|y|\}$ and (2.8) takes form

$$-2(\lambda-|y|)+\frac{2\lambda-4}{\lambda^2}(1-\alpha)(1-x+\lambda-|y|)^2+2\alpha(1-x)\leq 0,$$

or

$$-2(\lambda-|y|)+\frac{1-\alpha}{4}\cdot(1-x+\lambda-|y|)^2+2\alpha(1-x)\leq 0, \tag{2.9}$$

depending on whether $\lambda<4$ or $\lambda\geq 4$. As $(2\lambda-4)/\lambda^2\leq\frac{1}{4}$, it suffices to show (2.9). If $\lambda-|y|\leq 2$, then, as $1-x\leq\lambda-|y|$, the left-hand side does not exceed

$$\begin{aligned}-2(\lambda-|y|)+(1-\alpha)(\lambda-|y|)^2+2\alpha(\lambda-|y|)&=(\lambda-|y|)(-2+(1-\alpha)(\lambda-|y|)+2\alpha)\\&\leq(\lambda-|y|)(-2+2(1-\alpha)+2\alpha)=0.\end{aligned}$$

Similarly, if $\lambda-|y|\in(2,4]$, then we use the bound $1-x\leq 4-\lambda+|y|$ and conclude that the left-hand side of (2.9) is not greater than

$$-2(\lambda-|y|)+4(1-\alpha)+2\alpha(4-\lambda+|y|)=-2(\lambda-|y|-2)(1+\alpha)\leq 0$$

and we are done with the case $(x,y)\in C_\lambda$.

Assume that $(x,y)\in D_\lambda$. For $\lambda\in(2,4)$, the inequality (2.8) is equivalent to

$$-\frac{2}{\lambda}\Big[1-\frac{(1-\alpha)(\lambda-2)}{\lambda}\Big]+\frac{2-2x}{\lambda^2}\leq-\frac{2\alpha|y|}{\lambda^2},$$

or, after some simplifications, $\alpha|y|+1-x\leq 2+\alpha\lambda-2\alpha$. It is easy to check that $\alpha|y|+1-x$ attains its maximum for $x=-1$ and $|y|=\lambda-2$ and then we have the equality. If $(x,y)\in D_\lambda$ and $\lambda\geq 4$, then (2.8) takes form $-(2\alpha+1+x)\leq-\alpha(1-x)$, or $(x+1)(\alpha+1)\geq 0$. Finally, on the set $E_\lambda$, the inequality (2.8) is obvious.

(v) By (2.8), we have $\phi_\lambda \leq 0$, so $U_\lambda(x,y) \geq U_\lambda(1,y) = \chi_{\{|y|\geq\lambda\}}$. Furthermore, as $U_\lambda(x,y) = 1$ for $|y| \geq \lambda$ and $\psi_\lambda(x,y)y' \geq 0$ on $S_\lambda$, the second estimate follows. □

**Lemma 2.2.** *Let $x,\ h,\ y,\ k$ be fixed real numbers, satisfying $x,\ x+h \in [-1,1]$ and $|k| \leq |h|$. Then for any $\lambda > 2$ and $\alpha \in [0,1)$,*

$$U_\lambda(x+h, y+k) \leq U_\lambda(x,y) + \phi_\lambda(x,y)h + \psi_\lambda(x,y)k. \tag{2.10}$$

We will need the following fact, proved by Burkholder; see page 17 of [1].

**Lemma 2.3.** *Let $x,\ h,\ y,\ k,\ z$ be real numbers satisfying $|k| \leq |h|$ and $z > -1$. Then the function*

$$F(t) = H(x+th, y+tk, z),$$

*defined on $\{t : |x+th| \leq 1\}$, is convex.*

*Proof of the Lemma 2.2.* Consider the function

$$G(t) = G_{x,y,h,k}(t) = U_\lambda(x+th, y+tk),$$

defined on the set $\{t : |x+th| \leq 1\}$. It is easy to check that $G$ is continuous. As explained in [1], the inequality (2.10) follows once the concavity of $G$ is established. This will be done by proving the inequality $G'' \leq 0$ at the points, where $G$ is twice differentiable and checking the inequality $G'_+(t) \leq G'_-(t)$ for those $t$, for which $G$ is not differentiable (even once). Note that we may assume $t = 0$, by a translation argument $G''_{x,y,h,k}(t) = G''_{x+th,y+tk,h,k}(0)$, with analogous equalities for one-sided derivatives. Clearly, we may assume that $h \geq 0$, changing the signs of both $h$, $k$, if necessary. Due to the symmetry of $U_\lambda$, we are allowed to consider $y \geq 0$ only.

We start from the observation that $G''(0) = 0$ on the interior of $A_\lambda$ and $G'_+(0) \leq G'_-(0)$ for $(x,y) \in A_\lambda \cap \overline{B}_\lambda$. The latter inequality holds since $U_\lambda \equiv 1$ on $A_\lambda$ and $U_\lambda \leq 1$ on $B_\lambda$. For the remaining inequalities, we consider the cases $\lambda \in (2,4)$, $\lambda \geq 4$ separately.

*The case* $\lambda \in (2,4)$. The inequality $G''(0) \leq 0$ is clear for $(x,y)$ lying in the interior of $B_\lambda$. On $C_\lambda$, we have

$$G''(0) = -\frac{4(h+k)(h(\lambda-y) - k(1-x))}{(1-x-y+\lambda)^3} \leq 0, \tag{2.11}$$

which follows from $|k| \leq h$ and the fact that $\lambda - y \geq 1 - x$. For $(x,y)$ in the interior of $D_\lambda$,

$$G''(0) = \frac{-h^2+k^2}{\lambda^2} \leq 0,$$

as $|k| \leq h$. Finally, on $E_\lambda$, the concavity follows by Lemma 2.3.

It remains to check the inequalities for one-sided derivatives. By Lemma 2.1 (ii), the points $(x,y)$, for which $G$ is not differentiable at 0, do not belong to $S_\lambda$. Since we excluded the set $A_\lambda \cap \overline{B}_\lambda$, they lie on the line $y = x - 1 + \lambda$. For such points $(x,y)$, the left derivative equals

$$G'_-(0) = -\frac{2\lambda-4}{\lambda^2}(\alpha h - k),$$

while the right one is given by

$$G'_+(0) = \frac{-h+k}{2(\lambda-y)} + \frac{(2\lambda-4)(1-\alpha)h}{\lambda^2},$$

or

$$G'_+(0) = -\frac{2h}{\lambda}\Big[1 - \frac{(1-\alpha)(\lambda-2)}{\lambda}\Big] + \frac{2(1-x)h+2yk}{\lambda^2},$$

depending on whether $y \geq 1-x$ or $y < 1-x$. In the first case, the inequality $G'_+(0) \leq G'_-(0)$ reduces to

$$(h-k)\Big(\frac{1}{2(\lambda-y)} - \frac{2(\lambda-2)}{\lambda^2}\Big) \geq 0,$$

while in the remaining one,

$$\frac{2}{\lambda^2}(h-k)(y-(\lambda-2)) \geq 0.$$

Both inequalities follow from the estimate $\lambda - y \leq 2$ and the condition $|k| \leq h$.

*The case* $\lambda \geq 4$. On the set $B_\lambda$ the concavity is clear. For $C_\lambda$, we have that the formula (2.11) holds. If $(x,y)$ lies in the interior of $D_\lambda$, then

$$G''(0) = \frac{1}{8}\exp\Big(\frac{3+x+y-\lambda}{2(\alpha+1)}\Big)\Big[\frac{1-x}{2(\alpha+1)}\cdot(-h^2+k^2) - \Big(2-\frac{1-x}{\alpha+1}\Big)(h^2+hk)\Big] \leq 0,$$

since $|k| \leq h$ and $(1-x)/(\alpha+1) \leq 2$. The concavity on $E_\lambda$ is a consequence of Lemma 2.3. It remains to check the inequality for one-sided derivatives. By Lemma 2.1 (ii), we may assume $y = x + \lambda - 1$, and the inequality $G'_+(0) \leq G'_-(0)$ reads

$$\frac{1}{2}(h-k)\Big(\frac{1}{\lambda-y} - \frac{1}{2}\Big) \geq 0,$$

an obvious one, as $\lambda - y \leq 2$. □

## 3. The main theorem

Now we may state and prove the main result of the paper.

**Theorem 3.1.** *Suppose $f$ is a submartingale satisfying $||f||_\infty \leq 1$ and $g$ is an adapted process which is $\alpha$-subordinate to $f$. Then for all $\lambda > 0$ we have*

$$\mathbb{P}(g^* \geq \lambda) \leq \mathbb{E}U_\lambda(f_0, g_0). \tag{3.1}$$

*Proof.* If $\lambda \leq 2$, then this follows immediately from the result of Hammack [4]; indeed, note that $U_\lambda$ coincides with Hammack's special function and, furthermore, since $g$ is $\alpha$-subordinate to $f$, it is also 1-subordinate to $f$.

Fix $\lambda > 2$. We may assume $\alpha < 1$. It suffices to show that for any nonnegative integer $n$,

$$\mathbb{P}(|g_n| \geq \lambda) \leq \mathbb{E}U_\lambda(f_0, g_0). \tag{3.2}$$

To see that this implies (3.1), fix $\varepsilon > 0$ and consider a stopping time $\tau = \inf\{k : |g_k| \geq \lambda - \varepsilon\}$. The process $f^\tau = (f_{\tau\wedge n})$, by Doob's optional sampling theorem, is a submartingale. Furthermore, we obviously have that $||f^\tau||_\infty \leq 1$ and the process $g^\tau = (g_{\tau\wedge n})$ is $\alpha$-subordinate to $f^\tau$. Therefore, by (3.2),

$$\mathbb{P}(|g_n^\tau| \geq \lambda - \varepsilon) \leq \mathbb{E}U_{\lambda-\varepsilon}(f_0^\tau, g_0^\tau) = \mathbb{E}U_{\lambda-\varepsilon}(f_0, g_0).$$

Now if we let $n \to \infty$, we obtain $\mathbb{P}(g^* \geq \lambda) \leq \mathbb{E}U_{\lambda-\varepsilon}(f_0, g_0)$ and by left-continuity of $U_\lambda$ as a function of $\lambda$, (3.1) follows.

Thus it remains to establish (3.2). By Lemma 2.1 (v), $\mathbb{P}(|g_n| \geq \lambda) \leq \mathbb{E}U_\lambda(f_n, g_n)$ and it suffices to show that for all $1 \leq j \leq n$ we have

$$\mathbb{E}U_\lambda(f_j, g_j) \leq \mathbb{E}U_\lambda(f_{j-1}, g_{j-1}). \tag{3.3}$$

To do this, note that, since $|dg_j| \leq |df_j|$ almost surely, the inequality (2.10) yields

$$U_\lambda(f_j, g_j) \leq U_\lambda(f_{j-1}, g_{j-1}) + \phi_\lambda(f_{j-1}, g_{j-1})df_j + \psi_\lambda(f_{j-1}, g_{j-1})dg_j \tag{3.4}$$

with probability 1. Assume for now that $\phi_\lambda(f_{j-1},g_{j-1})df_j$, $\psi_\lambda(f_{j-1},g_{j-1})dg_j$ are integrable. By $\alpha$-subordination, the condition (2.8) and the submartingale property $\mathbb{E}(d_j|\mathcal{F}_{j-1})\geq 0$, we have

$$\begin{aligned}&\mathbb{E}\big[\phi_\lambda(f_{j-1},g_{j-1})df_j+\psi_\lambda(f_{j-1},g_{j-1})dg_j|\mathcal{F}_{j-1}\big]\\&\qquad\leq \phi_\lambda(f_{j-1},g_{j-1})\mathbb{E}(df_j|\mathcal{F}_{j-1})+\big|\psi_\lambda(f_{j-1},g_{j-1})\big|\cdot\big|\mathbb{E}(dg_j|\mathcal{F}_{j-1})\big|\\&\qquad\leq\big[\phi_\lambda(f_{j-1},g_{j-1})+\alpha|\psi_\lambda(f_{j-1},g_{j-1})|\big]\mathbb{E}(df_j|\mathcal{F}_{j-1})\leq 0.\end{aligned}$$

Therefore, it suffices to take the expectation of both sides of (3.4) to obtain (3.3).

Thus the proof will be complete if we show the integrability of $\phi_\lambda(f_{j-1},g_{j-1})df_j$ and $\psi_\lambda(f_{j-1},g_{j-1})dg_j$. In both the cases $\lambda\in(2,4)$, $\lambda\geq 4$, all we need is that the variables

$$\frac{2\lambda-2|g_{j-1}|}{(1-f_{j-1}-|g_{j-1}|+\lambda)^2}df_j\quad\text{and}\quad\frac{2-2f_{j-1}}{(1-f_{j-1}-|g_{j-1}|+\lambda)^2}dg_j \tag{3.5}$$

are integrable on the set $K=\{|g_{j-1}|<f_{j-1}+\lambda-1,\ \ |g_{j-1}|\geq\lambda-1\}$, since outside it the derivatives $\phi_\lambda$, $\psi_\lambda$ are bounded by a constant depending only on $\alpha$, $\lambda$ and $|df_j|$, $|dg_j|$ do not exceed 2. The integrability is proved exactly in the same manner as in [4]. We omit the details. □

We will now establish the following sharp exponential inequality.

**Theorem 3.2.** *Suppose $f$ is a submartingale satisfying $||f||_\infty\leq 1$ and $g$ is an adapted process which is $\alpha$-subordinate to $f$. In addition, assume that $|g_0|\leq|f_0|$ with probability 1. Then for $\lambda\geq 4$ we have*

$$\mathbb{P}(g^*\geq\lambda)\leq\gamma e^{-\lambda/(2\alpha+2)}, \tag{3.6}$$

*where*

$$\gamma=\frac{1+\alpha}{2\alpha+4}\big(\alpha+1+2^{-\frac{\alpha+2}{\alpha+1}}\big)\exp\Big(\frac{2}{\alpha+1}\Big).$$

*The inequality is sharp.*

This should be compared to Burkholder's estimate (Theorem 8.1 in [1])

$$\mathbb{P}(g^*\geq\lambda)\leq\frac{e^2}{4}\cdot e^{-\lambda},\quad\lambda\geq 2,$$

in the case when $f$, $g$ are Hilbert space-valued martingales and $g$ is subordinate to $f$. For $\alpha=1$, we obtain the inequality of Hammack [4],

$$\mathbb{P}(g^*\geq\lambda)\leq\frac{(8+\sqrt{2})e}{12}\cdot e^{-\lambda/4},\quad\lambda\geq 4.$$

*Proof of the inequality (3.6).* We will prove that the maximum of $U_\lambda$ on the set $K=\{(x,y)\in S:|y|\leq|x|\}$ is given by the right hand side of (3.6). This, together with the inequality (3.1) and the assumption $\mathbb{P}((f_0,g_0)\in K)=1$, will imply the desired estimate. Clearly, by symmetry, we may restrict ourselves to the set $K^+=K\cap\{y\geq 0\}$. If $(x,y)\in K^+$ and $x\geq 0$, then it is easy to check that

$$U_\lambda(x,y)\leq U_\lambda((x+y)/2,(x+y)/2).$$

Furthermore, a straightforward computation shows that the function $F:[0,1]\to\mathbb{R}$ given by $F(s)=U_\lambda(s,s)$ is nonincreasing. Thus we have $U_\lambda(x,y)\leq U_\lambda(0,0)$. On the other hand, if $(x,y)\in K^+$ and $x\leq 0$, then it is easy to prove that $U_\lambda(x,y)\leq U_\lambda(-1,x+y+1)$ and the function $G:[0,1]\to\mathbb{R}$ given by $G(s)=U_\lambda(-1,s)$ is nondecreasing. Combining all these facts we have that for any $(x,y)\in K^+$,

$$U_\lambda(x,y)\leq U_\lambda(-1,1)=\gamma e^{-\lambda/(2\alpha+2)}. \tag{3.7}$$

Thus (3.6) holds. The sharpness will be shown in the next section. □

## 4. Sharpness

Recall the function $V_\lambda = V_{\alpha,\lambda}$ defined by (1.1) in the introduction. The main result in this section is Theorem 4.1 below, which, combined with Theorem 3.1, implies that the functions $U_\lambda$ and $V_\lambda$ coincide. If we apply this at the point $(-1, 1)$ and use the equality appearing in (3.7), we obtain that the inequality (3.6) is sharp.

**Theorem 4.1.** *For any $\lambda > 0$ we have*

$$U_\lambda \le V_\lambda. \tag{4.1}$$

The main tool in the proof is the following „splicing" argument. Assume that the underlying probability space is the interval $[0, 1]$ with the Lebesgue measure.

**Lemma 4.1.** *Fix $(x_0, y_0) \in [-1, 1] \times \mathbb{R}$. Suppose there exists a filtration and a pair $(f, g)$ of simple adapted processes, starting from $(x_0, y_0)$, such that $f$ is a submartingale satisfying $||f||_\infty \le 1$ and $g$ is $\alpha$-subordinate to $f$. Then $V_\lambda(x_0, y_0) \ge \mathbb{E}V_\lambda(f_\infty, g_\infty)$ for $\lambda > 0$.*

*Proof.* Let $N$ be such that $(f_N, g_N) = (f_\infty, g_\infty)$ and fix $\varepsilon > 0$. With no loss of generality, we may assume that $\sigma$-field generated by $f,\ g$ is generated by the family of intervals $\{[a_i, a_{i+1}) : i = 1,\ 2,\ \dots,\ M-1\}$, $0 = a_1 < a_2 < \dots < a_M = 1$. For any $i \in \{1,\ 2,\ \dots,\ M-1\}$, denote $x_0^i = f_N(a_i),\ y_0^i = g_N(a_i)$. There exists a filtration and a pair $(f^i, g^i)$ of adapted processes, with $f$ being a submartingale bounded in absolute value by 1 and $g$ being $\alpha$-subordinate to $f$, which satisfy $f_0^i = x_0^i\chi_{[0,1)}$, $g_0^i = y_0^i\chi_{[0,1)}$ and $\mathbb{P}((g^i)^* \ge \lambda) > \mathbb{E}V_\lambda(f_0^i, g_0^i) - \varepsilon$. Define the processes $F,\ G$ by $F_k = f_k,\ G_k = g_k$ if $k \le N$ and

$$F_k(\omega) = \sum_{i=1}^{M-1} f_{k-N}^i((\omega - a_i)/(a_{i+1} - a_i))\chi_{[a_i,a_{i+1})}(\omega),$$
$$G_k(\omega) = \sum_{i=1}^{M-1} g_{k-N}^i((\omega - a_i)/(a_{i+1} - a_i))\chi_{[a_i,a_{i+1})}(\omega)$$

for $k > N$. It is easy to check that there exists a filtration, relative to which the process $F$ is a submartingale satisfying $||F||_\infty \le 1$ and $G$ is an adapted process which is $\alpha$-subordinate to $F$. Furthermore, we have

$$\mathbb{P}(G^* \ge \lambda) \ge \sum_{i=1}^{M-1} (a_{i+1} - a_i)\mathbb{P}((g^i)^* \ge \lambda)$$
$$> \sum_{i=1}^{M-1} (a_{i+1} - a_i)\big(\mathbb{E}V_\lambda(f_0^i, g_0^i) - \varepsilon\big) = \mathbb{E}V_\lambda(f_\infty, g_\infty) - \varepsilon.$$

Since $\varepsilon$ was arbitrary, the result follows. □

*Proof of Theorem 4.1.* First note the following obvious properties of the functions $V_\lambda$, $\lambda > 0$: we have $V_\lambda \in [0, 1]$ and $V_\lambda(x, y) = V_\lambda(x, -y)$. The second equality is an immediate consequence of the fact that if $g$ is $\alpha$-subordinate to $f$, then so is $-g$.

In the proof of Theorem 4.1 we repeat several times the following procedure. Having fixed a point $(x_0, y_0)$ from the strip $S$, we construct certain simple finite processes $f, g$ starting from $(x_0, y_0)$, take their natural filtration $(\mathcal{F}_n)$, apply Lemma 4.1 and thus obtain a bound for $V_\lambda(x_0, y_0)$. All the constructed processes appearing in the proof below are easily checked to satisfy the conditions of this lemma: the condition $||f||_\infty \le 1$ is straightforward, while the $\alpha$-subordination and the fact that $f$ is a submartingale are implied by the following. For any $n \ge 1$, either $df_n$ satisfies $\mathbb{E}(df_n|\mathcal{F}_{n-1}) = 0$ and $dg_n = \pm df_n$, or $df_n \ge 0$ and $dg_n = \pm\alpha df_n$.

We will consider the cases $\lambda \leq 2$, $2 < \lambda < 4$, $\lambda \geq 4$ separately. Note that by symmetry, it suffices to establish (4.1) on $S \cap \{y \geq 0\}$.

*The case* $\lambda \leq 2$. Assume $(x_0, y_0) \in A_\lambda$. If $y_0 \geq \lambda$, then $g^* \geq \lambda$ almost surely, so $V_\lambda(x_0, y_0) \geq 1 = U_\lambda(x_0, y_0)$. If $\lambda > y_0 \geq \alpha x_0 - \alpha + \lambda$, then let $(f_0, g_0) \equiv (x_0, y_0)$,

$$df_1 = (1 - x_0)\chi_{[0,1]} \quad \text{and} \quad dg_1 = \alpha df_1. \tag{4.2}$$

Then we have $g_1 = y_0 + \alpha - \alpha x_0 \geq \lambda$, which implies $g^* \geq \lambda$ almost surely and (4.1) follows. Now suppose $(x_0, y_0) \in A_\lambda$ and $y_0 < \alpha x_0 - \alpha + \lambda$. Let $(f, g) \equiv (x_0, y_0)$,

$$df_1 = \frac{y_0 - x_0 + 1 - \lambda}{1 - \alpha}\chi_{[0,1]}, \ dg_1 = \alpha df_1 \tag{4.3}$$

and

$$df_2 = dg_2 = \beta\chi_{[0,1-\beta/2)} + (\beta - 2)\chi_{[1-\beta/2,1]}, \tag{4.4}$$

where

$$\beta = \frac{\alpha x_0 - y_0 - \alpha + \lambda}{1 - \alpha} \in [0, 2]. \tag{4.5}$$

Then $(f_2, g_2)$ takes values $(-1, \lambda-2)$, $(1, \lambda)$ with probabilities $\beta/2$, $1-\beta/2$, respectively, so, by Lemma 4.1,

$$V_\lambda(x_0, y_0) \geq \frac{\beta}{2}V_\lambda(-1, \lambda - 2) + \big(1 - \frac{\beta}{2}\big)V_\lambda(1, \lambda) = \frac{\beta}{2}V_\lambda(-1, 2 - \lambda) + 1 - \frac{\beta}{2}. \tag{4.6}$$

Note that $(-1, 2 - \lambda) \in A_\lambda$. If $2 - \lambda \geq \alpha \cdot (-1) - \alpha + \lambda$, then, as already proved, $V_\lambda(-1, 2-\lambda) = 1$ and $V_\lambda(x_0, y_0) \geq 1 = U_\lambda(x_0, y_0)$. If the converse inequality holds, i.e., $2 - \lambda < -2\alpha + \lambda$, then we may apply (4.6) to $x_0 = -1$, $y_0 = 2 - \lambda$ to get

$$V_\lambda(-1, 2 - \lambda) \geq \frac{\beta}{2}V_\lambda(-1, 2 - \lambda) + 1 - \frac{\beta}{2},$$

or $V_\lambda(-1, 2 - \lambda) \geq 1$. Thus we established $V_\lambda(x_0, y_0) = 1$ for any $(x_0, y_0) \in A_\lambda$.

Suppose then, that $(x_0, y_0) \in B_\lambda$. Let

$$\beta = \frac{2(1 - x_0)}{1 - x_0 - y_0 + \lambda} \in [0, 1] \tag{4.7}$$

and consider a pair $(f, g)$ starting from $(x_0, y_0)$ and satisfying

$$df_1 = -dg_1 = -\frac{x_0 - y_0 - 1 + \lambda}{2}\chi_{[0,\beta)} + (1 - x_0)\chi_{[\beta,1]}. \tag{4.8}$$

On $[0, \beta)$, the pair $(f_1, g_1)$ lies in $A_\lambda$; Lemma 4.1 implies $V_\lambda(x_0, y_0) \geq \beta = U_\lambda(x_0, y_0)$.

Finally, for $(x_0, y_0) \in C_\lambda$, let $(f, g)$ start from $(x_0, y_0)$ and

$$df_1 = -dg_1 = \frac{-x_0 - \lambda + 1 + y_0}{2}\chi_{[0,\gamma)} + \frac{y_0 - x_0 + 1}{2}\chi_{[\gamma,1]},$$

where

$$\gamma = \frac{y_0 - x_0 + 1}{\lambda} \in [0, 1].$$

On $[0, \gamma)$, the pair $(f_1, g_1)$ lies in $A_\lambda$, while on $[\gamma, 1]$ we have $(f_1, g_1) = ((x_0 + y_0 + 1)/2, (x_0 + y_0 - 1)/2) \in B_\lambda$. Hence

$$V_\lambda(x_0, y_0) \geq \gamma \cdot 1 + (1 - \gamma) \cdot \frac{1 - x_0 - y_0}{\lambda} = U_\lambda(x_0, y_0).$$

*The case* $2 < \lambda < 4$. For $(x_0, y_0) \in A_\lambda$ we prove (4.1) using the same processes as in the previous case, i.e. the constant ones if $y_0 \geq \lambda$ and the ones given by (4.2) otherwise. The next step is to establish the inequality

$$V_\lambda(-1, \lambda - 2) \geq U_\lambda(-1, \lambda - 2) = \frac{1 + \alpha}{2} + \frac{1 - \alpha}{2} \cdot \Big(\frac{4 - \lambda}{\lambda}\Big)^2. \tag{4.9}$$

To do this, fix $\delta \in (0,1]$ and set

$$\beta = \frac{\delta(1-\alpha)}{\lambda},\ \kappa = \frac{4-\lambda-\delta(1+\alpha)}{\lambda}\cdot\beta,\ \gamma = \beta + (1-\beta)\cdot\frac{\delta(1+\alpha)}{4},\ \nu = \kappa\cdot\frac{\lambda}{4}.$$

We have $0 \le \nu \le \kappa \le \beta \le \gamma \le 1$. Consider processes $f$, $g$ given by $(f_0, g_0) \equiv (-1,\ \lambda-2)$, $(df_1, dg_1) \equiv (\delta,\ \alpha\delta)$,

$$df_2 = -dg_2 = \frac{\lambda-\delta(1-\alpha)}{2}\chi_{[0,\beta)} - \frac{\delta(1-\alpha)}{2}\chi_{[\beta,1]},$$

$$df_3 = dg_3 = -\big(\lambda-2+\frac{\delta(1+\alpha)}{2}\big)\chi_{[0,\kappa)} + \big(2-\frac{\lambda+\delta(1+\alpha)}{2}\big)\big)\chi_{[\kappa,\beta)}$$
$$+\big(2-\frac{\delta(1+\alpha)}{2}\big)\chi_{[\beta,\gamma)} - \frac{\delta(1+\alpha)}{2}\chi_{[\gamma,1)},$$

$$df_4 = -dg_4 = \big(-2+\frac{\lambda}{2}\big)\chi_{[0,\nu)} + \frac{\lambda}{2}\chi_{[\nu,\kappa)}.$$

As $(f_4, |g_4|)$ takes values $(1,\lambda)$, $(1,0)$ and $(-1,\lambda-2)$ with probabilities $(\gamma-\beta)+(\kappa-\nu)$, $\beta-\kappa$ and $1-\gamma+\nu$, respectively, we have

$$V_\lambda(-1,\lambda-2) \ge \gamma-\beta+\kappa-\nu+(1-\gamma+\nu)V_\lambda(-1,\lambda-2),$$

or

$$V_\lambda(-1,\lambda-2) \ge \frac{\gamma-\beta+\kappa-\nu}{\gamma-\nu} = \frac{1+\alpha}{2} + \frac{1-\alpha}{2}\cdot\Big(\frac{4-\lambda}{\lambda}\Big)^2 - \frac{\delta(1-\alpha^2)}{\lambda^2}.$$

As $\delta$ is arbitrary, we obtain (4.9). Now suppose $(x_0, y_0) \in B_\lambda$ and recall the pair $(f, g)$ starting from $(x_0, y_0)$ given by (4.3) and (4.4) (with $\beta$ defined in (4.5)). As previously, it leads to (4.6), which takes form

$$V_\lambda(x_0,y_0) \ge \frac{\beta}{2}\Big[\frac{1+\alpha}{2} + \frac{1-\alpha}{2}\cdot\Big(\frac{4-\lambda}{\lambda}\Big)^2\Big] + 1 - \frac{\beta}{2}$$
$$= \frac{\beta(1-\alpha)}{4}\Big[\Big(\frac{4-\lambda}{\lambda}\Big)^2 - 1\Big] + 1 = \frac{(\alpha x_0 - \alpha - y_0 + \lambda)(4-2\lambda)}{\lambda^2} + 1 = U_\lambda(x_0,y_0).$$

For $(x_0, y_0) \in C_\lambda$, consider a pair $(f, g)$, starting from $(x_0, y_0)$ defined by (4.8) (with $\beta$ given by (4.7)). On $[0, \beta)$ we have $(f_1, g_1) = ((x_0 + y_0 + 1 - \lambda)/2, (x_0 + y_0 - 1 + \lambda)/2) \in B_\lambda$, so Lemma 4.1 yields

$$V_\lambda(x_0,y_0) \ge \beta V_\lambda\Big(\frac{x_0+y_0+1-\lambda}{2}, \frac{x_0+y_0-1+\lambda}{2}\Big)$$
$$= \frac{2(1-x_0)}{1+\lambda-x_0-y_0}\cdot\Big\{1 - \Big[\alpha\Big(\frac{x_0+y_0-1-\lambda}{2}\Big) - \frac{x_0+y_0-1-\lambda}{2}\Big]\cdot\frac{2\lambda-4}{\lambda^2}\Big\}$$
$$= U_\lambda(x_0,y_0).$$

For $(x_0, y_0) \in D_\lambda$, set $\beta = (y_0 - x_0 + 1)/\lambda \in [0, 1]$ and let a pair $(f, g)$ be given by $(f_0, g_0) \equiv (x_0, y_0)$ and

$$df_1 = -dg_1 = \frac{-x_0+y_0+1-\lambda}{2}\chi_{[0,\beta)} + \frac{-x_0+y_0+1}{2}\chi_{[\beta,1]}.$$

As $(f_1, g_1)$ takes values

$$\Big(\frac{x_0+y_0+1-\lambda}{2}, \frac{x_0+y_0-1+\lambda}{2}\Big) \in B_\lambda \text{ and } \Big(\frac{x_0+y_0+1}{2}, \frac{x_0+y_0-1}{2}\Big) \in C_\lambda$$

with probabilites $\beta$ and $1-\beta$, respectively, we obtain $V_\lambda(x_0, y_0)$ is not smaller than

$$\beta V_\lambda\Big(\frac{x_0+y_0+1-\lambda}{2}, \frac{x_0+y_0-1+\lambda}{2}\Big) + (1-\beta)V_\lambda\Big(\frac{x_0+y_0+1}{2}, \frac{x_0+y_0-1}{2}\Big)$$
$$= \frac{y_0-x_0+1}{\lambda}\cdot\Big\{1 - \Big[\alpha\Big(\frac{x_0+y_0-1-\lambda}{2}\Big) - \frac{x_0+y_0-1-\lambda}{2}\Big]\cdot\frac{2\lambda-4}{\lambda^2}\Big\}$$
$$+\frac{\lambda-y_0+x_0-1}{\lambda}\Big[\frac{1-x_0-y_0}{\lambda} - \frac{(1-x_0-y_0)(1-\alpha)(\lambda-2)}{\lambda^2}\Big]$$

$$= I + II + III + IV,$$

where

$$I+III = \frac{y_0 - x_0 + 1}{\lambda} + \frac{(\lambda - y_0 + x_0 - 1)(1 - x_0 - y_0)}{\lambda^2} = \frac{2(1-x_0)}{\lambda} - \frac{(1-x_0)^2 - y_0^2}{\lambda^2}$$

and

$$II+IV = \frac{(1-\alpha)(\lambda-2)}{\lambda^3}\big[(y_0-x_0+1)(y_0+x_0-1-\lambda)-(1-x_0-y_0)(\lambda-y_0+x_0-1)\big]$$
$$= -\frac{(1-\alpha)(\lambda-2)}{\lambda^3}\cdot\lambda(2-2x_0).$$

Combining these facts, we obtain $V_\lambda(x_0,y_0) \geq U_\lambda(x_0,y_0)$.

For $(x_0,y_0) \in E_\lambda$ with $(x_0,y_0) \neq (-1,0)$, the following contruction will turn to be useful. Denote $w = \lambda - 3$, so, as $(x_0,y_0) \in E_\lambda$, we have $x_0 + y_0 < w$. Fix positive integer $N$ and set $\delta = \delta_N = (w - x_0 - y_0)/[N(\alpha+1)]$. Consider sequences $(x_j^N)_{j=1}^{N+1}$, $(p_j)_{j=1}^{N+1}$, defined by

$$x_j^N = x_0 + y_0 + (j-1)\delta(\alpha+1),\ j = 1,\ 2,\ \ldots, N+1,$$

and $p_1^N = (1+x_0)/(1+x_0+y_0)$,

$$p_{j+1}^N = \frac{(1+x_j^N)\big(1+x_j^N+\frac{\delta(\alpha-1)}{2}\big)p_j^N}{(1+x_{j+1}^N)\big(1+x_j^N+\frac{\delta(\alpha+1)}{2}\big)} + \frac{\delta}{1+x_{j+1}^N},\ j=1,\ 2,\ \ldots,\ N. \tag{4.10}$$

We construct a process $(f,g)$ starting from $(x_0,y_0)$ such that for $j = 1, 2, \ldots, N+1$,

(4.11) the variable $(f_{3j}, |g_{3j}|)$ takes values $(x_j^N, 0)$ and $(-1, 1+x_j^N)$ with probabilities $p_j^N$ and $1-p_j^N$, respectively.

We do this by induction. Let

$$df_1 = -dg_1 = y_0\chi_{[0,p_1^N)} + (-1-x_0)\chi_{[p_1^N,1]},\ df_2 = dg_2 = df_3 = dg_3 = 0.$$

Note that (4.11) is satisfied for $j = 1$. Now suppose we have a pair $(f,g)$, which satisfies (4.11) for $j = 1,\ 2,\ \ldots,\ n$, $n \leq N$. Let us describe $f_k$ and $g_k$ for $k = 3n+1$, $3n+2$, $3n+3$. The difference $df_{3n+1}$ is determined by the following three conditions: it is a martingale difference, i.e., satisfies $\mathbb{E}(df_{3n+1}|\mathcal{F}_{3n}) = 0$; conditionally on $\{f_{3n} = x_n^N\}$, it takes values in $\{-1-x_n^N,\ \delta(\alpha+1)/2\}$; and vanishes on $\{f_{3n} \neq x_n^N\}$. Furthermore, set $dg_{3n+1} = df_{3n+1}$. Moreover,

$$df_{3n+2} = \delta\chi_{\{f_{3n+1}=-1\}},\quad dg_{3n+2} = \frac{g_{3n+1}}{|g_{3n+1}|}\alpha\cdot df_{3n+2}.$$

Finally, the variable $df_{3n+3}$ satisfies $\mathbb{E}(df_{3n+3}|\mathcal{F}_{3n+2}) = 0$, and, in addition, the variable $f_{3n+3}$ takes values in $\{-1, x_n^N + \delta(\alpha+1)\} = \{-1, x_n^{N+1}\}$. The description is completed by

$$dg_{3n+3} = -\frac{g_{3n+2}}{|g_{3n+2}|}df_{3n+3}.$$

One easily checks that $(f_{3n+3}, |g_{3n+3}|)$ takes values in $\{(x_{n+1}^N, 0), (-1, 1+x_{n+1}^N)\}$; moreover, since

$$\begin{aligned}\mathbb{E}f_{3n+3} &= \mathbb{E}f_{3n} + \mathbb{E}df_{3n+2} = x_n^N p_n^N - (1-p_n^N) + \delta\mathbb{P}(f_{3n+1} = -1)\\ &= x_n^N p_n^N - (1-p_n^N) + \delta\left(1 - p_n^N + p_n^N\frac{\delta(\alpha+1)}{2(1+x_n^N)+\delta(\alpha+1)}\right)\\ &= p_n^N\cdot\frac{(x_n^N+1)(1+x_n^N+\delta(\alpha-1)/2)}{1+x_n^N+\delta(\alpha+1)/2} + \delta - 1,\end{aligned}$$

we see that $\mathbb{P}(f_{3n+3} = x_{n+1}^N) = p_{n+1}^N$ and the pair $(f,g)$ satisfies (4.10) for $j = n+1$. Thus there exists $(f,g)$ satisfying (4.10) for $j = 1,\ 2,\ \ldots, N+1$. In particular,

$(f_{3(N+1)}, |g_{3(N+1)}|)$ takes values $(w, 0)$, $(-1, w+1) \in D_\lambda$ with probabilities $p^N_{N+1}$, $1 - p^N_{N+1}$. By Lemma 4.1,

$$V_\lambda(x_0, y_0) \geq p^N_{N+1} V_\lambda(w, 0) + (1 - p^N_{N+1}) V_\lambda(-1, w+1). \tag{4.12}$$

Recall the function $H$ defined by (2.1). The function $h : [x_0 + y_0, w] \to \mathbb{R}$ given by $h(t) = H(x_0, y_0, t)$, satisfies the differential equation

$$h'(t) + \frac{\alpha+2}{\alpha+1} \cdot \frac{h(t)}{1+t} = \frac{1}{(\alpha+1)(1+t)}.$$

As we assumed $x_0 + y_0 > -1$, the expression $(h(x+\delta) - h(x))/\delta$ converges uniformly to $h'(x)$ on $[x_0 + y_0, \lambda - 3]$. Therefore there exist constants $\varepsilon_N$, which depend only on $N$ and $x_0 + y_0$ satisfying $\lim_{N\to\infty} \varepsilon_N = 0$ and for $1 \leq j \leq N$,

$$\left| \frac{h(x^N_{j+1}) - h(x^N_j)}{(\alpha+1)\delta_N} + \frac{\left[\frac{\alpha+2}{\alpha+1}(1 + x^N_j) - \frac{\delta_N(\alpha+1)}{2}\right] h(x^N_j)}{(1 + x^N_{j+1})\left(1 + x^N_j + \frac{\delta_N(\alpha+1)}{2}\right)} - \frac{1}{(\alpha+1)(1 + x^N_{j+1})} \right| \leq \varepsilon_N,$$

or, equivalently,

$$\left| h(x^N_{j+1}) - \frac{(1 + x^N_j)\left(1 + x^N_j + \frac{\delta_N(\alpha-1)}{2}\right) h(x^N_j)}{(1 + x^N_{j+1})\left(1 + x^N_j + \frac{\delta_N(\alpha+1)}{2}\right)} - \frac{\delta_N}{1 + x^N_{j+1}} \right| \leq (\alpha+1)\delta_N \varepsilon_N.$$

Together with (4.10), this leads to

$$|h(x^N_{j+1}) - p^N_{j+1}| \leq \frac{(1 + x^N_j)\left(1 + x^N_j + \frac{\delta_N(\alpha-1)}{2}\right)}{(1 + x^N_{j+1})\left(1 + x^N_j + \frac{\delta_N(\alpha+1)}{2}\right)} |h(x^N_j) - p^N_j| + (\alpha+1)\delta_N \varepsilon_N.$$

Since $p^N_1 = h(x^N_1)$, we have

$$|h(w) - p^N_{N+1}| \leq (\alpha+1) N \delta_N \varepsilon_N = (\lambda - 3 - x_0 - y_0)\varepsilon_N$$

and hence $\lim_{N\to\infty} p^N_{N+1} = h(w)$. Combining this with (4.12), we obtain

$$V_\lambda(x_0, y_0) \geq h(w)(V_\lambda(w, 0) - V_\lambda(-1, w+1)) + V_\lambda(-1, w+1).$$

As $w = \lambda - 3$, it suffices to check that we have

$$a_\lambda = V_\lambda(\lambda - 3, 0) - V_\lambda(-1, \lambda - 2)) \text{ and } b_\lambda = V_\lambda(-1, \lambda - 2),$$

where $a_\lambda$, $b_\lambda$ were defined in (2.4). Finally, if $(x_0, y_0) = (-1, 0)$, then considering a pair $(f, g)$ starting from $(x_0, y_0)$ and satisfying $df_1 \equiv \delta$, $dg_1 \equiv \alpha\delta$, we get

$$V(-1, 0) \geq V(-1 + \delta, \alpha\delta). \tag{4.13}$$

Now let $\delta \to 0$ to obtain $V(-1, 0) \geq U(-1, 0)$.

*The case* $\lambda \geq 4$. We proceed as in previous case. We deal with $(x_0, y_0) \in A_\lambda$ exactly in the same manner. Then we establish the analogue of (4.9), which is

$$V(-1, \lambda - 2) \geq U_\lambda(-1, \lambda - 2) = \frac{1+\alpha}{2}. \tag{4.14}$$

To do this, fix $\delta \in (0, 1)$ and set

$$\beta = \frac{4 - 2\delta}{4 - \delta(1+\alpha)}, \quad \gamma = \beta \cdot \Big(1 - \frac{\delta(\alpha+1)}{4}\Big).$$

Now let a pair $(f, g)$ be defined by $(f_0, g_0) \equiv (-1, \lambda - 2)$, $(df_1, dg_1) \equiv (\delta, \alpha\delta)$,

$$df_2 = -dg_2 = -\frac{\delta(1-\alpha)}{2} \chi_{[0,\beta)} + (2 - \delta)\chi_{[\beta,1]},$$

$$df_3 = dg_3 = -\frac{\delta(1+\alpha)}{2} \chi_{[0,\gamma)} + \Big(2 - \frac{\delta(1+\alpha)}{2}\Big)\chi_{[\gamma,\beta)}.$$

Then $(f_3, g_3)$ takes values $(-1, \lambda-2)$, $(1, \lambda)$ and $(1, \lambda-4+\delta(\alpha+1))$ with probabilities $\gamma$, $\beta - \gamma$ and $1 - \beta$, respectively, and Lemma 4.1 yields

$$V(-1, \lambda - 2) \geq \gamma V(-1, \lambda - 2) + (\beta - \gamma)V(1, \lambda),$$

or

$$V(-1, \lambda - 2) \geq \frac{\beta - \gamma}{1 - \gamma} = \frac{(\alpha + 1)(2 - \delta)}{4 - \delta(\alpha + 1)}.$$

It suffices to let $\delta \to 0$ to obtain (4.14). The cases $(x_0, y_0) \in B_\lambda$, $C_\lambda$ are dealt with using the same processes as in the case $\lambda \in (2, 4)$. If $(x_0, y_0) \in D_\lambda$, then Lemma 4.1, applied to the pair $(f, g)$ given by $(f_0, g_0) \equiv (x_0, y_0)$, $df_1 = -dg_1 = -(1 + x_0)\chi_{[0,(1-x_0)/2)} + (1 - x_0)\chi_{[(1-x_0)/2,1]}$, yields

$$V(x_0, y_0) \geq \frac{1 - x_0}{2} V(-1, x_0 + y_0 + 1). \tag{4.15}$$

Furthermore, for any number $y$ and any $\delta \in (0, 1)$, we have

$$V(-1, y) \geq V(-1 + \delta, y + \alpha\delta), \tag{4.16}$$

which is proved in the same manner as (4.13). Hence, for large $N$, if we set $\delta = (\lambda - 3 - x_0 - y_0)/(N(\alpha + 1))$, the inequalities (4.15) and (4.16) give

$$\begin{aligned} V(x_0, y_0) &\geq \frac{1 - x_0}{2} V(-1, x_0 + y_0 + 1) \geq \frac{1 - x_0}{2} V(-1 + \delta, x_0 + y_0 + 1 + \alpha\delta) \\ &\geq \frac{1 - x_0}{2} \Big(1 - \frac{\delta}{2}\Big) V(-1, x_0 + y_0 + 1 + (\alpha + 1)\delta) \\ &\geq \frac{1 - x_0}{2} \Big(1 - \frac{\delta}{2}\Big)^N V(-1, x_0 + y_0 + 1 + N(\alpha + 1)\delta) \\ &= \frac{1 - x_0}{2} \Big(1 - \frac{\lambda - 3 - x_0 - y_0}{2N(\alpha + 1)}\Big)^N V(-1, \lambda - 2) \\ &= \frac{(1 - x_0)(1 + \alpha)}{4} \Big(1 - \frac{\lambda - 3 - x_0 - y_0}{2N(\alpha + 1)}\Big)^N. \end{aligned}$$

Now take $N \to \infty$ to obtain $V_\lambda(x_0, y_0) \geq U_\lambda(x_0, y_0)$.

Finally, if $(x_0, y_0) \in E_\lambda$ we use the pair $(f, g)$ used in the proof of the case $(x_0, y_0) \in E_\lambda$, $\lambda \in (2, 4)$, with $\omega = 1$. Then the process $(f, |g|)$ ends at the points $(1, 0)$ and $(-1, 2)$ with probabilities, which can be made arbitrarily close to $H(x_0, y_0, 1)$ and $1 - H(x_0, y_0, 1)$, respectively. It suffices to apply Lemma 4.1 and check that it gives $V_\lambda(x_0, y_0) \geq U_\lambda(x_0, y_0)$. □

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