SHARP L^p-BOUNDS FOR THE MARTINGALE MAXIMAL FUNCTION

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Abstract. The paper studies sharp weighted L^p inequalities for the martingale maximal function. Proofs exploit properties of certain special functions of four variables and self-improving properties of A_p weights.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, filtered by $(\mathcal{F}_t)_{t\geq 0}$, a nondecreasing family of sub-σ-fields of $\mathcal{F}$, such that $\mathcal{F}_0$ consists of all events of probability 0 and all events of probability 1. Let X be an adapted, real-valued, uniformly integrable martingale with right-continuous trajectories that have limits from the left. Then $X^* = \sup_{s\geq 0}|X_s|$ denotes the maximal function of X; we will also use the notation $X_t^* = \sup_{0\leq s\leq t}|X_s|$ for the corresponding truncated maximal function. Assume further that W is a positive, uniformly integrable martingale; this process will be called a weight (sometimes, with no risk of confusion, the word "weight" will refer to the terminal variable W_∞ of the martingale W). For example, one can take an exponential martingale $\mathcal{E}(M) = \big(\exp\big[M_t - \frac{1}{2}\langle M\rangle_t\big]\big)_{t\geq 0}$ corresponding to an adapted, continuous-path martingale M satisfying $\mathbb{E}\exp(\langle M\rangle_\infty/2) < \infty$ (cf. [15]).

In the paper, we will be interested in sharp inequalities between weighted norms of X^* and X. This type of estimates, motivated by corresponding results of Muckenhoupt [11] from the analytic setting, has gathered a lot of interest in the literature. Let us assume for a moment that both X and W have continuous paths. Following Izumisawa and Kazamaki [9], we say that W satisfies Muckenhoupt's condition A_p (where $1 < p < \infty$ is a fixed parameter), if

$$||W||_{A_p} := \sup_\tau \left|\left|\mathbb{E}\Big[\big\{W_\tau/W_\infty\big\}^{1/(p-1)}\big|\mathcal{F}_\tau\Big]^{p-1}\right|\right|_\infty < \infty,$$

where the supremum is taken over the class of all adapted stopping times τ. There is also a version of this condition for $p=1$: W is an A_1 weight if there is a constant c such that $W^* \leq cW$ almost surely; the least c with this property is denoted by $||W||_{A_1}$.

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Tsuchikura [21] proved that if the weight $W = \mathcal{E}(M)$ satisfies $||W||_{A_p} < \infty$ for some $1 \leq p < \infty$, then there is a finite constant C_p such that the inequality

$$\mathbb{E}W_\infty 1_{\{X^*>1\}} \leq C_p \mathbb{E}|X_\infty|^p W_\infty 1_{\{X^*>1\}} \tag{1}$$

holds for all uniformly integrable martingales X. Two years later, Uchiyama [22] proved the converse: if for some p there is a constant $C_p < \infty$ such that the above bound holds for every X, then W belongs to the class A_p. Izumisawa and Kazamaki [9] extended this statement to the case of L^p estimates as follows. Let $W = \mathcal{E}(M)$ be a weight and let $p \in (1, \infty)$. Then there is a finite constant C_p such that

$$||X^*||_{L^p(W)} \leq C_p ||X||_{L^p(W)} \tag{2}$$

holds for any uniformly integrable martingale X if and only if W belongs to the class A_p. Here $||X||_{L^p(W)} = (\mathbb{E}|X_\infty|^p W_\infty)^{1/p}$ is the usual weighted L^p norm of X. For a survey of related results, consult e.g. the book [10] by Kazamaki and see the papers [16], [18] and [20] for more on the recent progress in this direction.

We come back to the general setting in which X and W are assumed to be càdlàg and uniformly integrable martingales. One of the questions we want to study is the following: what are the possible substitutes for (1) and (2) if we consider general weights W (i.e., not satisfying the A_p condition or any assumption of this kind)? This problem was studied by Fefferman and Stein in the analytic context of Hardy-Littlewood maximal operator on $\mathbb{R}^d$, and the solution presented in [5] suggests to replace the random variable W appearing on the right of (1) and (2) by the maximal function W^*. It is easy to see that then the weak-type estimate holds with constant 1. Indeed, if we take $\tau = \inf\{t : X_t > 1\}$, then for every $p \geq 1$,

$$\begin{aligned}\mathbb{E}W_\infty 1_{\{X^*>1\}} &= \mathbb{E}W_\infty 1_{\{|X_\tau|\geq 1\}} \leq \mathbb{E}|X_\tau|^p W_\infty 1_{\{|X_\tau|\geq 1\}} = \mathbb{E}|X_\tau|^p W_\tau 1_{\{|X_\tau|\geq 1\}} \\ &\leq \mathbb{E}|X_\infty|^p W_\tau 1_{\{|X_\tau|\geq 1\}} \leq \mathbb{E}|X_\infty|^p W^* 1_{\{|X_\tau|\geq 1\}} = \mathbb{E}|X_\infty|^p W^* 1_{\{X^*>1\}}.\end{aligned}$$

Now use a straightforward interpolation-type argument: apply the above bound for weak L^1 and the martingale $(X_t/\lambda)_{t\geq 0}$ (where $\lambda > 0$ is fixed) to obtain

$$\mathbb{E}W_\infty 1_{\{X^*>\lambda\}} \leq \lambda^{-1}\mathbb{E}|X_\infty| W^* 1_{\{X^*>\lambda\}}.$$

Fix $p > 1$, multiply both sides by λ^{p-1}, integrate over λ from 0 to ∞ and use Hölder's inequality. As the result of these operations, we obtain the estimate

$$||X^*||_{L^p(W)} \leq \frac{p}{p-1} ||X||_{L^p(W^{1-p}(W^*)^p)}.$$

This is slightly weaker than what we have expected, since $W_\infty^{1-p}(W^*)^p \geq W^*$. One of our main results asserts that the stronger Fefferman-Stein inequality is also true.

THEOREM 1.1. *Fix $1 < p < \infty$. Then for any uniformly integrable martingale X and any weight W we have the estimate*

$$||X^*||_{L^p(W)} \leq \frac{p}{p-1}||X||_{L^p(W^*)}. \tag{3}$$

The constant $p/(p-1)$ is the best possible.

Note that the further improvement of the weight on the right is not possible, i.e., we cannot replace W^* by $W^{1-r}(W^*)^r$ for any $r < 1$. Indeed, consider the following easy example. Let X be a Brownian motion started at 1 and stopped upon leaving $[0, 2]$, and let $W = (\mathbb{E}(W_\infty|\mathcal{F}_t))_{t\geq 0}$ be the weight generated by $W_\infty = 1_{\{X_\infty=0\}}$. Then $||X||_{L^p(W^{1-r}(W^*)^r)} = 0$ and $||X^*||_{L^p(W)} > 0$.

We will also be interested in the following bound, which can be regarded as the dual to (3) (see below for the explanation).

THEOREM 1.2. *Fix $1 < p < \infty$. Then for any uniformly integrable martingale X and any positive weight W we have the estimate*

$$||X^*||_{L^p(W(W^*)^{-p})} \leq \frac{p}{p-1}||X||_{L^p(W^{1-p})}. \tag{4}$$

The constant $p/(p-1)$ is the best possible. The weight on the left cannot be improved in the sense that for any $1 < p < \infty$, any $r > 0$ and any constant c there is a weight W and a martingale X such that

$$||X^*||_{L^p(W^{1-r}(W^*)^{r-p})} > c||X||_{L^p(W^{1-p})}. \tag{5}$$

There is a very interesting application of the above results. Martínez [13] studied the following: given $1 < p < \infty$ and a weight V, find necessary and sufficient conditions for V in order to guarantee the existence of a weight U such that the operator $X \mapsto X^*$ is bounded from $L^p(V\mathrm{d}\mathbb{P})$ to $L^p(U\mathrm{d}\mathbb{P})$. This is a classical problem from the point of view of harmonic analysis: see the monograph [6] for an overview of the results in this direction and consult the references therein.

The above problem (in the probabilistic setting) was completely solved in [13]: the required condition on V is that $\mathbb{E}V^{1/(1-p)} < \infty$. The corresponding weight U was found there with the use of the nonconstructive method developed by Rubio de Francia. The inequality (4) shows an alternative, very simple choice for U: one can take $U = W(W^*)^{-p}$, where $W = V^{1/(1-p)}$. In this spirit, the inequality (3) can be regarded as a dual statement, in which the starting weight V is considered in the target space: the choice $U = V^*$ guarantees that the operator $X \mapsto X^*$ is bounded from $L^p(U\mathrm{d}\mathbb{P})$ to $L^p(V\mathrm{d}\mathbb{P})$.

The next problem that we will study concerns a more precise version of the inequality (2) and is motivated by related results in the analytic setting. Assume that $(\mathcal{F}_t)_{t\geq 0}$ has the property that any adapted local martingale has continuous trajectories: this is the usual setup in which one studies martingale inequalities involving A_p weights (cf. [10]).

What is the least exponent $\kappa(p)$ such that for any A_p weight W and any martingale X we have

$$||X^*||_{L^p(W)} \leq \beta_p ||W||_{A_p}^{\kappa(p)} ||X||_{L^p(W)},$$

with β_p depending only on p? We will give a full answer to this question, along with a number of related weighted inequalities.

Theorem 1.3. *Let $1 < p < \infty$, $1 < q < \infty$ and pick $\alpha \in [q^{-1}, pq^{-1})$. Assume further that W is a weight satisfying the condition A_q and X is a uniformly integrable martingale. Then we have the estimate*

$$||X^*||_{L^p(W^{(q\alpha-1)/(q-1)})} \leq \left(\frac{p}{p-q\alpha}\right)^{1/q\alpha} ||W||_{A_q}^{1/q} ||X||_{L^p(W^{(q\alpha-1)/(q-1)})}. \tag{6}$$

In particular, this implies

$$||X^*||_{L^p(W)} \leq \beta_p ||W||_{A_p}^{1/(p-1)} ||X||_{L^p(W)} \tag{7}$$

for some β_p depending only on p. The exponent $1/(p-1)$ cannot be improved.

This type of results, which focuses on extracting the sharp dependence of the constant on the A_p characteristics of a weight, has gathered a considerable interest in the last fifteen years, especially in the analytic context. See e.g. Buckley [1], Cruz-Uribe et al. [3], Hukovič et al. [7], Hytönen et al. [8], Lacey et al. [12] and Petermichl and Volberg [19], and consult the references therein.

Let us say a few words about the proofs of Theorems 1.1 and 1.2. Clearly, the constant $p/(p-1)$ is already optimal in the unweighted setting (i.e., for $W \equiv 1$), so all we need is to establish the estimates (3) and (4). Our approach will rest on the so-called Bellman function method (or Burkholder's method). This technique allows to deduce the validity of a given martingale bound from the existence of a certain special function, enjoying appropriate majorization and concavity; for a description of the method from different perspectives, we refer the interested reader to the works [2], [14], [17] and the references therein. The proof of Theorem 1.3 will also exploit this method, combined with some structural ("self-improving") properties of A_p weights.

We have organized the rest of the paper as follows. In the next section we study the inequalities (4) and (5). Section 3 is devoted to the proof of Theorem 1.1. In the final part of the paper we address our last result, Theorem 1.3.

2. Proof of Theorem 1.2

2.1. **A special function.** As we have mentioned in the previous section, a crucial role in the proof of (4) is played by a certain special function. Fix $p \in (1,\infty)$ and consider the function $B : [0,\infty)^2 \times (0,\infty)^2 \to \mathbb{R}$ given by

$$B(x,y,w,v) = w\left[\left(\frac{y}{v}\right)^p - \frac{p}{p-1}\frac{x}{w}\left(\frac{y}{v}\right)^{p-1}\right].$$

Clearly, B is continuous on its domain (actually, it is even of class C^∞ on $(0,\infty)^4$). Further important properties of B are studied in two lemmas below.

LEMMA 2.1. *(i) For any $x \geq 0$ and any $w > 0$ we have*

$$B(x,x,w,w) \leq 0. \tag{8}$$

(ii) For any x, $y \geq 0$ and w, $v > 0$ we have the majorization

$$B(x,y,w,v) \geq p^{-1}\left[\left(\frac{y}{v}\right)^p - \left(\frac{p}{p-1}\right)^p\left(\frac{x}{w}\right)^p\right]w. \tag{9}$$

PROOF. The inequality (8) is trivial: $B(x,x,w,w) = -\frac{x^p}{(p-1)w^{p-1}} \leq 0$. To show (9), substitute $\alpha = y/v$ and $\beta = px/(w(p-1))$. Then it is easy to check that the majorization is equivalent to

$$p\alpha^{p-1}(\alpha-\beta) \geq \alpha^p - \beta^p,$$

which follows immediately from the mean-value theorem. □

The key property of B is the following condition, which can be regarded as a certain kind of concavity.

LEMMA 2.2. *Fix x, $y \geq 0$ and w, $v > 0$. Then for any $h \geq -x$ and any $k > -w$ we have*

$$\begin{aligned} &B(x+h, y\vee(x+h), w+k, v\vee(w+k)) \\ &\qquad \leq B(x,y,w,v) + B_x(x,y,w,v)h + B_w(x,y,w,v)k. \end{aligned} \tag{10}$$

PROOF. We must show that

$$\begin{aligned} &\left(\frac{y\vee(x+h)}{v\vee(w+k)}\right)^p(w+k) - \frac{p}{p-1}(x+h)\left(\frac{y\vee(x+h)}{v\vee(w+k)}\right)^{p-1} \\ &\qquad \leq \left(\frac{y}{v}\right)^p(w+k) - \frac{p}{p-1}(x+h)\left(\frac{y}{v}\right)^{p-1}. \end{aligned} \tag{11}$$

We consider separately four cases.

Case I: $w+k \leq v$, $y < x+h$. Under these assumptions, (11) becomes

$$(x+h)^p(w+k) - \frac{p}{p-1}(x+h)^p v \leq y^p(w+k) - \frac{p}{p-1}(x+h)y^{p-1}v,$$

or

$$((x+h)^p - y^p)(w+k) \leq \frac{p}{p-1}(x+h)\big((x+h)^{p-1} - y^{p-1}\big)v.$$

Since $y < x+h$, it is enough to show the above bound for $w+k = v$. Plugging $\alpha = (x+h)^{p-1}$ and $\beta = y^{p-1}$, we see that the estimate is equivalent to $\alpha^{p/(p-1)} - \beta^{p/(p-1)} \leq \frac{p}{p-1}\alpha^{1/(p-1)}(\alpha-\beta)$. But this is true, due to the mean-value property.

Case II: $w+k > v$, $y > x+h$. Then the inequality (11) reads

$$\left(\frac{y}{w+k}\right)^p(w+k) - \frac{p}{p-1}(x+h)\left(\frac{y}{w+k}\right)^{p-1} \leq \left(\frac{y}{v}\right)^p(w+k) - \frac{p}{p-1}(x+h)\left(\frac{y}{v}\right)^{p-1}.$$

By the assumptions of the case, we have

$$\frac{p}{p-1}(x+h)\left[\left(\frac{y}{w+k}\right)^{p-1}-\left(\frac{y}{v}\right)^{p-1}\right]\geq\frac{p}{p-1}y\left[\left(\frac{y}{w+k}\right)^{p-1}-\left(\frac{y}{v}\right)^{p-1}\right]$$

and hence it is enough to show that

$$\frac{p}{p-1}y\left[\left(\frac{y}{w+k}\right)^{p-1}-\left(\frac{y}{v}\right)^{p-1}\right]\geq\left(\frac{y}{w+k}\right)^{p}(w+k)-\left(\frac{y}{v}\right)^{p}(w+k).$$

This is equivalent to the estimate

$$\left(\frac{w+k}{v}\right)^{p}-1\geq\frac{p}{p-1}\left[\left(\frac{w+k}{v}\right)^{p-1}-1\right],$$

which again follows from the mean-value property.

Case III. $w+k>v$, $y<x+h$. Then the estimate (11) takes the form

$$\frac{(x+h)^p}{(w+k)^{p-1}}-\frac{p}{p-1}\frac{(x+h)^p}{(w+k)^{p-1}}\leq\left(\frac{y}{v}\right)^{p}(w+k)-\frac{p}{p-1}(x+h)\left(\frac{y}{v}\right)^{p-1},$$

which, after the substitution $\alpha=(x+h)/(w+k)$, $\beta=y/v$, is equivalent to

$$\alpha^{p/(p-1)}-\beta^{p/(p-1)}\leq\frac{p}{p-1}\alpha^{1/(p-1)}(\alpha-\beta).$$

The latter bound holds true due to the mean-value property.

Case IV. $w+k<v$, $x+h<y$. Then the desired inequality becomes

$$\left(\frac{y}{v}\right)^{p}(w+k)-\frac{p}{p-1}\left(\frac{y}{v}\right)^{p-1}(x+h)\leq\left(\frac{y}{v}\right)^{p}(w+k)-\frac{p}{p-1}\left(\frac{y}{v}\right)^{p-1}(x+h),$$

which is clearly true: actually, both sides are equal. □

2.2. **Proof of** (4)**.** We will first prove the inequality for discrete-time martingales: suppose that the time set is the set of nonnegative integers. Clearly, we may assume that the process X is nonnegative: indeed, the passage from X to $|X|$ does not alter the right-hand side, while the maximal function of X can only increase. Actually, one can even assume that X is strictly positive, by a simple continuity argument. The key ingredient of the proof is to show that the process $(B(X_n,X_n^*,W_n,W_n^*))_{n\geq 0}$ is a supermartingale. This is an immediate consequence of (10). Indeed, denoting $H_{n+1}=X_{n+1}-X_n$ and $K_{n+1}=W_{n+1}-W_n$, we see that $\mathbb{E}(H_{n+1}|\mathcal{F}_n)=\mathbb{E}(K_{n+1}|\mathcal{F}_n)=0$ and hence

$$\begin{aligned}&\mathbb{E}\Big[B(X_{n+1},X_{n+1}^*,W_{n+1},W_{n+1}^*)|\mathcal{F}_n\Big]\\&=\mathbb{E}\Big[B\big(X_n+H_{n+1},X_n^*\vee(X_n+H_{n+1}),W_n+K_{n+1},W_n^*\vee(W_n+K_{n+1})\big)|\mathcal{F}_n\Big]\\&\leq B(X_n,X_n^*,W_n,W_n^*).\end{aligned}$$

Consequently, by (9),

$$p^{-1}\mathbb{E}\left[\left(\frac{X_n^*}{W_n^*}\right)^p-\left(\frac{p}{p-1}\right)^p\left(\frac{X_n}{W_n}\right)^p\right]W_n\leq \mathbb{E}B(X_n,X_n^*,W_n,W_n^*)$$
$$\leq \mathbb{E}B(X_0,X_0^*,W_0,W_0^*)\leq 0,$$

where in the last passage we have used (8) and the equalities $X_0^*=X_0$, $W_0^*=W_0$. Now, observe that

$$\mathbb{E}\left(\frac{X_n^*}{W_n^*}\right)^p W_n=\mathbb{E}\left(\frac{X_n^*}{W_n^*}\right)^p W\geq \mathbb{E}\left(\frac{X_n^*}{W^*}\right)^p W.$$

Furthermore, note that the function $G(x,y)=x^py^{1-p}$ is convex on $[0,\infty)\times(0,\infty)$ (indeed, the Hessian matrix

$$D^2G(x,y)=\begin{bmatrix} p(p-1)x^{p-2}y^{1-p} & -p(p-1)x^{p-1}y^{-p}\\ -p(p-1)x^{p-1}y^{-p} & p(p-1)x^py^{-1-p}\end{bmatrix}$$

is nonnegative-definite). This implies

$$\mathbb{E}X_n^pW_n^{1-p}=\mathbb{E}G\big(\mathbb{E}((X_\infty,W_\infty)|\mathcal{F}_n)\big)\leq \mathbb{E}\big[\mathbb{E}(G(X_\infty,W_\infty)|\mathcal{F}_n)\big]=\mathbb{E}X_\infty^pW_\infty^{1-p}.$$

Putting all the above facts together, we obtain

$$\mathbb{E}\left(\frac{X_n^*}{W^*}\right)^p W_\infty\leq\left(\frac{p}{p-1}\right)^p\mathbb{E}X_\infty^pW_\infty^{1-p}.$$

Letting $n\to\infty$ and using Lebesgue's monotone convergence theorem, we get the desired inequality (4).

It remains to establish the claim in the general, continuous-time setting. This is straightforward: given a continuous-time càdlàg martingale $(X_t)_{t\geq0}$, a weight $(W_t)_{t\geq0}$ and a positive integer N, we apply (4) to the discrete-time processes $(X_{n2^{-N}})_{n=0,1,2,\ldots}$ and $(W_{n2^{-N}})_{n=0,1,2,\ldots}$, obtaining

$$\mathbb{E}\left(\frac{\sup_n X_{n2^{-N}}}{W^*}\right)^p W_\infty\leq \mathbb{E}\left(\frac{\sup_n X_{n2^{-N}}}{\sup_n W_{n2^{-N}}}\right)^p W_\infty\leq\left(\frac{p}{p-1}\right)^p\mathbb{E}X_\infty^pW_\infty^{1-p}.$$

It remains to let $N\to\infty$ and use Lebesgue's monotone convergence theorem. The proof is finished.

2.3. **On the estimate** (5)**.** Fix $p\in(1,\infty)$, $r>0$ and a number $\alpha>1$. Consider a Brownian motion $\beta=(\beta_t)_{t\geq0}$ started at 1 and set $\tau=\inf\{t\geq0:\beta_t^*=\alpha\beta_t\}$. Clearly, the stopped process $\beta^\tau=(\beta_{\tau\wedge t})_{t\geq0}$ is a positive martingale; furthermore, as Wang showed in [23], this process is bounded in L^q for any $q<\alpha/(\alpha-1)$. In particular, this implies that if we put $X=W=\beta^\tau$, then W is a weight and X is uniformly integrable. In addition,

$$||X^*||_{L^p(W^{1-r}(W^*)^{r-p})}=(\mathbb{E}(W^*/W_\infty)^rW_\infty)^{1/p}=\alpha^{r/p}(\mathbb{E}W_\infty)^{1/p}$$

and $||X||_{L^p(W^{1-p})}=(\mathbb{E}W_\infty)^{1/p}$. This shows the second part of Theorem 1.2, since α was arbitrary.

3. Proof of Theorem 1.1

3.1. A special function. Let $B:[0,\infty)^4\to\mathbb{R}$ be given by

$$B(x,y,w,v)=\begin{cases} y^pw+(p-1)y^pv-\frac{p^2}{p-1}xy^{p-1}v & \text{if } x\geq \frac{p-1}{p}y,\\ y^pw-\left(\frac{p}{p-1}\right)^px^pv & \text{if } x<\frac{p-1}{p}y.\end{cases}$$

Let us establish several crucial properties of B, analogous to those studied in the previous section.

Lemma 3.1. *(i) For any nonnegative x and w we have*

$$B(x,x,w,w)\leq 0. \tag{12}$$

(ii) For any nonnegative numbers x, y, w and v we have the estimate

$$B(x,y,w,v)\geq y^pw-\left(\frac{p}{p-1}\right)^px^pv. \tag{13}$$

Proof. The inequality (12) is evident: $B(x,x,w,w)=-\frac{p}{p-1}x^pw\leq 0$. In the proof of (13), we may restrict ourselves to $x\geq\frac{p-1}{p}y$. Then the inequality reads

$$(p-1)y^pv-\frac{p^2}{p-1}xy^{p-1}v\geq -\left(\frac{p}{p-1}\right)^px^pv,$$

which, after the substitution $s=px/(p-1)y$ becomes $s^p-1\geq p(s-1)$. By mean-value property, this is true and the proof is finished. □

Lemma 3.2. *Fix x, y, w, $v\geq 0$ with $y\geq x$ and $v\geq w$. Then for any $h\geq -x$ and any $k\geq -w$, we have*

$$\begin{aligned} &B(x+h,y\vee(x+h),w+k,v\vee(w+k))\\ &\qquad\leq B(x,y,w,v)+B_x(x,y,w,v)h+B_w(x,y,w,v)k.\end{aligned} \tag{14}$$

Proof. It is convenient to split the argumentation into a few separate parts.

Case I. $x+h\leq y$, $w+k\leq v$. Then (14) becomes

$$B(x+h,y,w+k,v)\leq B(x,y,w,v)+B_x(x,y,w,v)h+B_w(x,y,w,v)k.$$

This follows at once from the fact that for fixed y and v, the function $(x,w)\mapsto B(x,y,w,v)$ is concave (as a sum of a concave function in x and a linear function in w).

Case II. $x+h\geq y$, $w+k\geq v$. Suppose first that $x\leq\frac{p-1}{p}y$; then $h\geq y-x\geq x/(p-1)$ and the inequality takes the form

$$\frac{p}{p-1}(x+h)^p(w+k)+y^p(w+k)-\left(\frac{p}{p-1}\right)^px^pv-p\left(\frac{p}{p-1}\right)^px^{p-1}vh\geq 0.$$

Clearly, it suffices to show this bound for $w+k=v$, since the expression on the left is a nondecreasing function of k. Next, fix x, y, w, v, put $k=v-w$ and denote the left-hand side above by $F(h)$. Then for $h\geq x/(p-1)$ we have

$$F'(h)=\frac{p^2}{p-1}(x+h)^{p-1}v-p\left(\frac{p}{p-1}\right)^p x^{p-1}v\geq 0,$$

so $F\geq 0$ on $[x/(p-1),\infty)$, which is the desired estimate. Now, suppose that $x\geq \frac{p-1}{p}y$. Then (14) becomes

$$\frac{p}{p-1}(x+h)^p(w+k)+y^p(w+k)+(p-1)y^pv-\frac{p^2}{p-1}(x+h)y^{p-1}v\geq 0.$$

As previously, we may assume that $w+k=v$; if we denote the left-hand side by $F(h)$, we compute that $F'(h)=p^2v\left((x+h)^{p-1}-y^{p-1}\right)/(p-1)\geq 0$. This proves that it is enough to prove the above bound under the additional assumption $x+h=y$, but this brings us back to Case I.

Case III. $x+h\geq y$, $w+k\leq v$. Put all the terms of (14) on the left-hand side and observe that the inequality becomes

$$\big[(x+h)^p-y^p\big](w+k)+\big(\text{expressions not depending on } k\big)\leq 0.$$

Since the expression in the square brackets is positive, it is enough to show the above bound for the largest value of $w+k$, i.e., for $w+k=v$. However, this has already been verified in Case II.

Case IV. $x+h\leq y$, $w+k\geq v$. As in Case III, we put all the terms on the left-hand side, denote the expression on the left by I and analyze its dependence on the variable k. If $x+h\geq\frac{p-1}{p}y$, then

$$I=\left[(p-1)y^p-\frac{p^2}{p-1}(x+h)y^{p-1}\right](w+k)+\big(\text{terms not depending on } k\big)$$

and the expression in the square brackets does not exceed $-y^p$. Hence, it suffices to show the estimate for the least k, that is, under the additional assumption $w+k=v$. This has been already analyzed in Case I. So, suppose that $x+h\leq\frac{p-1}{p}y$; then the situation is even simpler, since

$$I=-\left(\frac{p}{p-1}\right)^p(x+h)^p(w+k)+\big(\text{terms not depending on } k\big).$$

Hence, as previously, we may assume that $w+k=v$, and this brings us back to Case I.

We have considered all the possible cases. The proof is complete. □

3.2. **Proof of** (3)**.** The argumentation is similar to that used in the proof of (4). We restrict ourselves to positive processes, start with the discrete-time martingales $(X_n)_{n=0,1,2,\ldots}$, $(W_n)_{n=0,1,2,\ldots}$ and deduce from (14) that the sequence $(B(X_n, X_n^*, W_n, W_n^*))_{n\geq 0}$ is a supermartingale. This property, combined with (12) and (14), yields the estimate

$$\mathbb{E}(X_n^*)^p W_n \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}X_n^p W_n^*.$$

However, we have $\mathbb{E}(X_n^*)^p W_n = \mathbb{E}(X_n^*)^p W_\infty$ and, by Jensen's inequality, $\mathbb{E}X_n^p W_n^* \leq \mathbb{E}X^p W_n^* \leq \mathbb{E}X_\infty^p W_\infty^*$. This implies

$$\mathbb{E}(X_n^*)^p W_\infty \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}X_\infty^p W_\infty^*$$

and letting $n\to\infty$ establishes (3) in the discrete-time case. The passage to the general continuous-time processes is carried out in a similar manner as previously. The proof is complete.

4. One-weight inequalities for the martingale maximal function

Throughout this section we assume that the filtration $(\mathcal{F}_t)_{t\geq 0}$ has the property that all adapted local martingales have continuous trajectories (e.g., the Brownian filtration satisfies this condition). This is equivalent to saying that any adapted stopping time is integrable. This assumption guarantees that the notion of A_p weights leads to an interesting class of processes.

If W belongs to A_1 class, then $W^* \leq ||W||_{A_1} W$ almost surely and (3), (4) yield the estimates

$$||X^*||_{L^p(W)} \leq \frac{p}{p-1}||W||_{A_1}||X||_{L^p(W)}, \qquad ||X^*||_{L^p(W^{1-p})} \leq \frac{p}{p-1}||W||_{A_1}^p||X||_{L^p(W^{1-p})}.$$

In other words, the inequalities (3) and (4), or rather the corresponding Bellman functions introduced in the preceding two sections, form an appropriate tool to the study of one-weight bounds involving weights belonging to the class A_1. As we will show now, a similar phenomenon occurs for $1<p<\infty$; the inequality (6) can be established with the use of certain special functions.

4.1. A_p **weights and their properties.** We start from recalling the notion of probabilistic A_p condition, $1<p<\infty$. Following Izumisawa and Kazamaki [9], W belongs to the class A_p if

$$||W||_{A_p} := \sup_\tau \left|\left|\mathbb{E}\Big[\big\{W_\tau/W_\infty\big\}^{1/(p-1)}\big|\mathcal{F}_\tau\Big]^{p-1}\right|\right|_\infty < \infty,$$

where the supremum is taken over all stopping times τ. There is a nice alternative two-dimensional description of this condition. Namely, consider the additional process $V_t = \mathbb{E}(W_\infty^{1/(1-p)}|\mathcal{F}_t)$, $t\geq 0$. Then (W,V) is a uniformly integrable martingale terminating at the lower boundary of this set (i.e., satisfying $W_\infty V_\infty^{p-1}=1$ almost surely) and taking values in the hyperbolic strip $\{(w,v) \in (0,\infty)^2 : 1 \leq wv^{p-1} \leq ||W||_{A_p}\}$. Indeed, the inequality $WV^{p-1} \geq 1$ follows at once from Hölder's inequality, while $WV^{p-1} \leq ||W||_{A_p}$ is a direct consequence of the A_p condition. It is not difficult to see that the converse correspondence is also true: any uniformly integrable martingale (W,V) taking values in a set $D_c = \{(w,v) : 1 \leq wv^{p-1} \leq c\}$ and terminating at its lower boundary gives rise to an A_p weight W with $||W||_{A_p} \leq c$, and the least c allowed (leading to the smallest set D_c) equals the A_p characteristics of W.

In our further considerations, we will need certain self-improving properties of A_p weights. It follows directly from Jensen's inequality that if a weight satisfies the A_p condition for some p, then it automatically satisfies A_q for all $q > p$ (more precisely, we have $||Y||_{A_q} \leq ||Y||_{A_p}$). It turns out that if $p > 1$, then such a weight satisfies A_q also for some $q < p$ (see Kazamaki [10, Corollary 3.3]). We will need a little more precise information on this phenomenon. To this end, recall the result of Uchiyama [22], who proved the following connection between A_q characteristics and the weak-type constants:

$$||W||_{A_q} = \sup_X \left[\mathbb{E}W_\infty 1_{\{X^*>1\}}/||X_\infty||^q_{L^q(W)}\right]. \tag{15}$$

The supremum on the right has been studied by the author in [18]. Given $p > 1$ and $c \geq 1$, let $d = d(p,c) \in [0, p-1)$ be the unique number satisfying

$$F(d) := (1+d)\left(1 - \frac{d}{p-1}\right)^{p-1} = c^{-1}. \tag{16}$$

As showed in [18], if $||W||_{A_p} = c$ and $q \in (d+1, p)$, then we have the sharp bound

$$||W||_{A_q} = \sup_X \left[\mathbb{E}W_\infty 1_{\{X^*>1\}}/||X_\infty||^q_{L^q(W)}\right] \leq \left(1 - \frac{d}{q-1}\right)^{1-q}(1+d)^{-1}. \tag{17}$$

Let us try to get a more explicit estimate.

LEMMA 4.1. *Let W be an A_p weight. If $q = p - (p-1)(p||W||_{A_p})^{-1/(p-1)}/2$, then*

$$||W||_{A_q} \leq 2^{q-1}p^2||W||_{A_p}^{(q-1)/(p-1)}.$$

PROOF. Set $c := ||W||_{A_p}$ and note that we have $d(p,c) \leq p - 1 - (p-1)(pc)^{-1/(p-1)}$. This follows at once from the fact that the function F defined in (16) is decreasing, $F(0) = 1 \geq c^{-1}$ and

$$F\big(p-1-(p-1)(pc)^{-1/(p-1)}\big) = (p-(p-1)(pc)^{-1/(p-1)})(pc)^{-1} < c^{-1}.$$

Therefore, we have $q > d(p,c)+1$ and we may apply (17). Since the expression on the right of this estimate is an increasing function of d, we obtain

$$\begin{aligned}||W||_{A_q} &\leq \left(1 - \frac{p-1-(p-1)(pc)^{-1/(p-1)}}{q-1}\right)^{1-q}(1+p-1-(p-1)(pc)^{-1/(p-1)})\\ &\leq \left(\frac{(pc)^{-1/(p-1)}/2}{1-(pc)^{-1/(p-1)}/2}\right)^{1-q} p \leq \frac{p^{(q-1)/(p-1)+1}}{2^{1-q}}c^{(q-1)/(p-1)} \leq 2^{q-1}p^2c^{(q-1)/(p-1)}.\end{aligned}$$

This proves the claim. $\square$

4.2. **A special function and its properties.** Now we will introduce the special function corresponding to the estimate (6). Throughout this subsection, $c \geq 1$, $1 < p < \infty$, $1 < q < \infty$ and $\alpha \in [q^{-1}, pq^{-1})$ are fixed parameters. Let $B_c : [0,\infty)^2 \times (0,\infty)^2 \to \mathbb{R}$ be given by the formula

$$B_c(x,y,w,v) = y^p w^\alpha v^{1-\alpha} - \frac{p}{p-q\alpha} c^\alpha x^{q\alpha} y^{p-q\alpha} v^{1-q\alpha}.$$

We will show the following properties of this object.

LEMMA 4.2. *(i) For any $x \geq 0$ and any w, $v > 0$ satisfying $wv^{q-1} \leq c$ we have*

$$B_c(x,x,w,v) \leq 0. \tag{18}$$

(ii) For any x, $y \geq 0$ and any $w > 0$ we have

$$B_c(x,y,w,w^{1/(1-q)}) \geq \frac{q\alpha}{p}\left[y^p - \left(\frac{p}{p-q\alpha}\right)^{p/(q\alpha)} c^{p/q} x^p\right] w^{(q\alpha-1)/(q-1)}. \tag{19}$$

PROOF. The inequality (18) is very easy:

$$B_c(x,x,w,v) = x^p v^{1-q\alpha}\left[(wv^{q-1})^\alpha - \frac{p}{p-q\alpha}c^\alpha\right] \leq x^p v^{1-q\alpha}\left[c^\alpha - \frac{p}{p-q\alpha}c^\alpha\right] \leq 0.$$

The majorization (19) follows at once from the mean-value property of the convex function $s \mapsto s^{p/q\alpha}$: indeed, after the substitution $x' = \left(\frac{p}{p-q\alpha}c^\alpha\right)^{1/q\alpha} x$, the inequality is equivalent to $y^{p/q\alpha} - (x')^{p/(q\alpha)} \leq (p/q\alpha) y^{p/q\alpha - 1}(y - x')$. $\square$

The second property can be regarded as a "differential" version of Lemmas 2.2 and 3.2.

LEMMA 4.3. *(i) For any $x \geq 0$ and any $w, v > 0$ satisfying $wv^{q-1} \leq c$ we have $B_{cy}(x,x,w,v) \leq 0$.*

(ii) For any x, $y \geq 0$ and w, $v > 0$, the matrix

$$\mathcal{M}_c(x,y,w,v) = \begin{bmatrix} B_{cxx} & B_{cxw} & B_{cxv} \\ B_{cwx} & B_{cww} & B_{cwv} \\ B_{cvx} & B_{cvw} & B_{cvv} \end{bmatrix}(x,y,w,v)$$

is nonpositive-definite.

PROOF. The first property is easy to prove: we have

$$B_{cy}(x,x,w,v) = px^{p-1}v^{1-q\alpha}\left((wv^{q-1})^\alpha - c^\alpha\right) \leq 0.$$

To show the second property, write $B_c = y^p B_c^1 + \frac{pc^\alpha}{p-q\alpha} y^{p-q\alpha} B_c^2$, where

$$B_c^1(x,y,w,v) = w^\alpha v^{1-\alpha} \quad \text{and} \quad B_c^2(x,y,w,v) = -x^{q\alpha} v^{1-q\alpha}.$$

Then the matrix $\mathcal{M}_c$ equals $y^p\mathcal{M}_c^1+\frac{pc^\alpha}{p-q\alpha}y^{p-q\alpha}\mathcal{M}_c^2$, where $\mathcal{M}_c^1$, $\mathcal{M}_c^2$ are the 3×3 "partially-Hessian" matrices corresponding to B_c^1, B_c^2. It remains to note that

$$\mathcal{M}_c^1(x,y,w,v)=\begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha(\alpha-1)w^{\alpha-2}v^{1-\alpha} & \alpha(\alpha-1)w^{\alpha-1}v^{-\alpha} \\ 0 & \alpha(\alpha-1)w^{\alpha-1}v^{-\alpha} & \alpha(\alpha-1)w^{\alpha}v^{-1-\alpha}\end{bmatrix}$$

and

$$\mathcal{M}_c^2(x,y,w,v)=-\begin{bmatrix} q\alpha(q\alpha-1)x^{q\alpha-2}v^{1-q\alpha} & 0 & q\alpha(1-q\alpha)x^{q\alpha-1}v^{-q\alpha} \\ 0 & 0 & 0 \\ q\alpha(1-q\alpha)x^{q\alpha-1}v^{-q\alpha} & 0 & q\alpha(q\alpha-1)x^{q\alpha}v^{-1-q\alpha}\end{bmatrix}$$

are both nonpositive-definite. □

4.3. **Proof of** (6)**.** We will apply Itô's formula (cf. [4]) to the composition of B_c and the process $A_t=(X_t,X_t^*,W_t,V_t)$, $t\geq 0$, where $c=||W||_{A_q}$. As the result, we obtain that

$$B_c(A_t)=I_0+I_1+I_2+I_3/2,$$

where

$$\begin{aligned} I_0&=B_c(A_0),\\ I_1&=\int_0^t B_{cx}(A_s)\mathrm{d}X_s+\int_0^t B_{cw}(A_s)\mathrm{d}W_s+\int_0^t B_{cv}(A_s)\mathrm{d}V_s,\\ I_2&=\int_0^t B_{cy}(A_s)\mathrm{d}X_s^*,\\ I_3&=\int_0^t \mathcal{M}_c(A_s)\mathrm{d}[X,W,V]_s.\end{aligned}$$

Here in I_3 we have used a shortened notation for the sum of integrals corresponding to second-order terms, i.e.,

$$I_3=\int_0^t B_{cxx}(A_s)\mathrm{d}[X]_s+2\int_0^t B_{cxw}(A_s)\mathrm{d}[X,W]_s+\dots$$

and so on. Let us study the terms I_0 through I_3. First, by (18), we see that $I_0\leq 0$. The term I_1 is a local martingale, by the properties of stochastic integrals. Next, we have $I_2\leq 0$ in view of Lemma 4.3 (i): indeed, the continuous part of the process X^* increases only for s satisfying $X_s=X_s^*$, and then $B_{cy}(A_s)\leq 0$. Finally, the term I_3 is also nonpositive, which follows directly from the second part of Lemma 4.3. Putting all the above facts together, we see that if $(\tau_n)_{n\geq 1}$ is a localizing sequence for I_1, then $\mathbb{E}B_c(A_{\tau_n\wedge t})\leq 0$, $n=1,2,\dots$. This is equivalent to saying that for each n,

$$\mathbb{E}(X^*_{\tau_n\wedge t})^pW^\alpha_{\tau_n\wedge t}V^{1-\alpha}_{\tau_n\wedge t}\leq \frac{p}{p-q\alpha}c^\alpha\mathbb{E}X^{q\alpha}_{\tau_n\wedge t}(X^*_{\tau_n\wedge t})^{p-q\alpha}V^{1-q\alpha}_{\tau_n\wedge t}. \tag{20}$$

Now we carry out a limiting procedure. First, note that (X,V) is a pair of uniformly integrable martingales and the function $G(x,v)=x^{q\alpha}v^{1-q\alpha}$ is convex on $[0,\infty)\times(0,\infty)$ (one easily checks that the Hessian matrix of G is nonnegative-definite). Hence, by Jensen's inequality and Doob's optional sampling theorem, we get

$$\mathbb{E}X_{\tau_n\wedge t}^{q\alpha}(X_{\tau_n\wedge t}^*)^{p-q\alpha}V_{\tau_n\wedge t}^{1-q\alpha} \leq \mathbb{E}X_\infty^{q\alpha}(X_{\tau_n\wedge t}^*)^{p-q\alpha}V_\infty^{1-q\alpha} \leq \mathbb{E}X_\infty^{q\alpha}(X_\infty^*)^{p-q\alpha}V_\infty^{1-q\alpha}.$$

Furthermore, by Fatou's lemma,

$$\mathbb{E}(X_\infty^*)^pW_\infty^\alpha V_\infty^{1-\alpha} \leq \liminf_{n\to\infty,t\to\infty}\mathbb{E}(X_{\tau_n\wedge t}^*)^pW_{\tau_n\wedge t}^\alpha V_{\tau_n\wedge t}^{1-\alpha}.$$

Combining these observations with (20), we obtain

$$\mathbb{E}(X_\infty^*)^pW_\infty^\alpha V_\infty^{1-\alpha} \leq \frac{p}{p-q\alpha}c^\alpha\mathbb{E}X_\infty^{q\alpha}(X_\infty^*)^{p-q\alpha}V_\infty^{1-q\alpha},$$

which is equivalent to $\mathbb{E}B_c(X_\infty, X_\infty^*, W_\infty, W_\infty^{1/(1-q)}) \leq 0$, by virtue of the equation $V_\infty^{q-1} = W_\infty^{-1}$. It remains to apply (19) to obtain the desired estimate (6).

4.4. **Proof of** (7) **and the optimality of the exponent** $1/(p-1)$**.** Pick an A_p weight W and use Lemma 4.1. Then the inequality (6), applied to $\alpha = 1$ and q defined in the lemma, gives

$$||X^*||_{L^p(W)} \leq \left(\frac{p}{p-q}\right)^{1/q}||W||_{A_q}^{1/q}||X||_{L^p(W)}.$$

Plugging the value of q and using the bound from Lemma 4.1, we get

$$\begin{aligned}||X^*||_{L^p(W)} &\leq \left(\frac{2p^{p/(p-1)}}{p-1}\right)^{1/q}\left(2^{q-1}p^2\right)^{1/q}||W||_{A_p}^{1/(p-1)q}||W||_{A_p}^{(q-1)/(p-1)q}||X||_{L^p(W)}\\ &\leq \frac{2p^{(3p-2)/(p-1)}}{p-1}||W||_{A_p}^{1/(p-1)}||X||_{L^p(W)}.\end{aligned}$$

It remains to show that the exponent $1/(p-1)$ cannot be improved. Suppose on contrary that there is $1 < p < \infty$, a constant β_p and a number $m < 1/(p-1)$ such that for all A_p weights W and all martingales X we have

$$||X^*||_{L^p(W)} \leq \beta_p||W||_{A_p}^m||X||_{L^p(W)}. \tag{21}$$

The construction and the analysis of the appropriate counterexample is quite elaborate and consists of several steps, so it is convenient to split the reasoning into separate parts.

1° *Special points in the first quadrant.* For two given numbers b and c satisfying $1 < b < c$, draw three curves: $\gamma_1 = \{(w,v) : wv^{p-1} = 1\}$, $\gamma_b = \{(w,v) : wv^{p-1} = b\}$ and $\gamma_c = \{(w,v) : wv^{p-1} = c\}$. Next, consider the line passing through $(1,c)$, tangent to γ_c. This line intersects γ_b in two points: $P_0 = (w_+, v_+)$ and $P_1 = (w_-, v_-)$, where $w_+ > 1$ and $w_- < 1$. Furthermore, it intersects γ_1 at a point $Z_1 = (z, z^{1/(1-p)})$ satisfying $z > 1$. Next, construct inductively the sequences $(P_n)_{n\geq 2}$ and $(Z_n)_{n\geq 2}$ of points as follows. Having constructed P_{n-1} and P_{n-2}, consider a line passing through P_{n-1}, tangent to γ_c, different from $P_{n-2}P_{n-1}$; this line intersects γ_b in P_{n-1} and yet another point, which we denote by P_n. Furthermore, let Z_n be the point of intersection of the line $P_{n-1}P_n$ with γ_1, having a bigger x-coordinate than P_n. We hope that the Figure 1 below clarifies the construction.

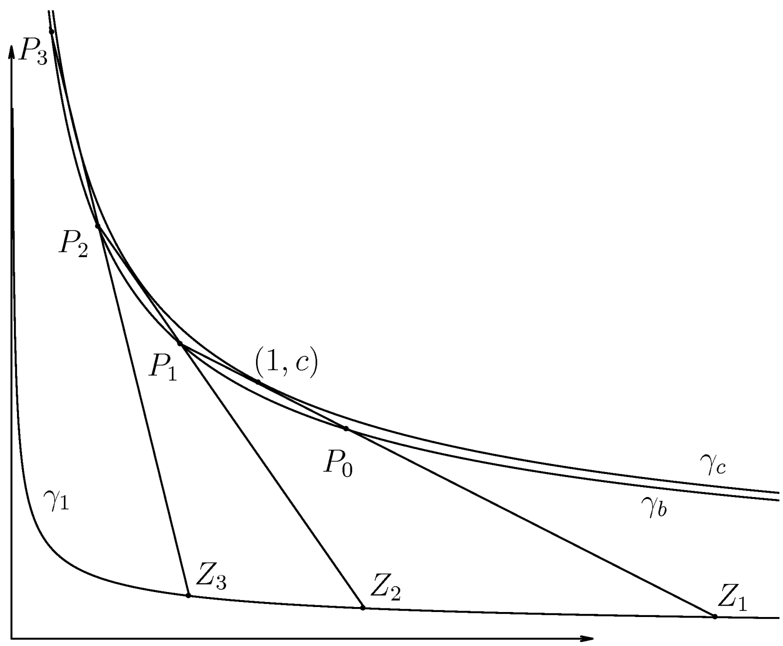

FIGURE 1. Special points $P_0 = (w_+, v_+)$, $P_1 = (w_-, v_-)$, P_2, ... and $Z_1 = (z, z^{1/(1-p)})$, Z_2, Z_3, ...

It is clear that $w_\pm$ are functions of b, c and p; furthermore, if we keep c and p fixed, and let $b \uparrow c$, then $w_+ \downarrow 1$ and $w_- \uparrow 1$, so in particular the difference $w_+ - w_-$ converges to 0. Next, the line P_0P_1 has the equation $v = (1-p)^{-1}c^{1/(p-1)}w + p(p-1)^{-1}c^{1/(p-1)}$ and hence $z > 1$ is the unique solution to the equation

$$z((1-p)^{-1}c^{1/(p-1)}z + p(p-1)^{-1}c^{1/(p-1)})^{p-1} = 1,$$

or

$$z(p-z)^{p-1} = c^{-1}(p-1)^{p-1}. \tag{22}$$

Observe also that the picture has a self-similarity property. Clearly, for any $\lambda > 0$ we have $(w, v) \in \gamma_c$ if and only if $(\lambda w, \lambda^{1/(1-p)}v) \in \gamma_c$, and a similar equivalence holds for γ_b and γ_1. In consequence, for each $n \geq 0$ we have

$$P_n = \left(w_+(w_-/w_+)^n, v_+(w_+/w_-)^{n/(p-1)}\right)$$

and

$$Z_{n+1} = \left(z(w_-/w_+)^n, z^{1/(1-p)}(w_+/w_-)^{n/(p-1)}\right).$$

In particular, this implies that for each $n \geq 0$ the point P_n splits the segment $P_{n+1}Z_{n+1}$ in the same ratio:

$$\frac{|Z_{n+1} - P_n|}{|Z_{n+1} - P_{n+1}|} = \frac{z - w_+}{z - w_-}. \tag{23}$$

2° Construction of the weight W. Consider the two-dimensional continuous-path martingale (W,V) whose distribution is uniquely determined by the following conditions.

- W is a stopped Brownian motion,
- $(W_0,V_0)=P_0$ almost surely.
- The range of (W,V) is equal to the union of the segments P_nZ_n, $n=1,2,\ldots$.

A more explicit description is in order. The process (W,V) starts from P_0 and first, it evolves along the line segment P_1Z_1, hitting eventually one of the endpoints. Denote

$$\tau_1=\inf\{t:(W_t,V_t)\in\{P_1,Z_1\}\}.$$

If the ending point is Z_1, then the process (W,V) stops and we define its lifetime to be $\tau=\tau_1$. Otherwise, it continues its movement, but now it evolves along the line segment P_2Z_2, ending after some time in the set $\{P_2,Z_2\}$. Let

$$\tau_2=\inf\{t:(W_t,V_t)\in\{P_2,Z_2\}\}.$$

If $(W_{\tau_2},V_{\tau_2})=Z_2$, then the evolution is over and the lifetime τ of (W,V) equals τ_2. If $(W_{\tau_2},V_{\tau_2})=P_2$, the process starts moving along P_3Z_3, and so on. Thus, we end up with a sequence $(\tau_n)_{n\geq 0}$ of stopping times (we set $\tau_0\equiv 0$) and the lifetime variable $\tau=\sup_{n\geq 0}\tau_n$. Furthermore, directly from the self-similarity of the picture mentioned above (see (23)), we get

$$\mathbb{P}(\tau>\tau_n)=\left(\frac{z-w_+}{z-w_-}\right)^k, \tag{24}$$

so in particular τ is finite with probability 1 (since all τ_n's are). Finally, one easily checks that the pair (W,V) is uniformly integrable with values in $\{(w,v):1\leq wv^{p-1}\leq c\}$, and hence W is an A_p weight satisfying $||W||_{A_p}\leq c$. Actually, the A_p characteristics is equal to c, since the trajectory of (W,V) touches the curve γ_c.

3° Construction of the martingale X. As we will see, the process X will be a "partially-affine" transformation of W. Set $\delta=(1-c^{-m})(w_+-w_-)/(z-w_+)$ and define the points $\tilde{P}_n=(1+\delta)^n$, $\tilde{Z}_{n+1}=(1+\delta)^nc^{-m}$ for $n=0,\,1,\,2,\,\ldots$. Then $\tilde{Z}_n<\tilde{P}_{n-1}<\tilde{P}_n$ for each $n\geq 1$ and

$$\frac{|P_{n-1}-Z_n|}{|P_n-Z_n|}=\frac{|\tilde{P}_{n-1}-\tilde{Z}_n|}{|\tilde{P}_n-\tilde{Z}_n|}=\frac{z-w_+}{z-w_-}. \tag{25}$$

Let us construct X separately on each $[\tau_n,\tau_{n+1}]$ (where τ_n's are the stopping times introduced in the construction of the weight W). First, we let X start from $\tilde{P}_0$ and on the interval $[\tau_0,\tau_1]$, let it move along $[\tilde{Z}_1,\tilde{P}_1]$ so that $\tau_1=\inf\{t:X_t\in\{\tilde{P}_1,\tilde{Z}_1\}\}$. Clearly, this is possible because of (25); actually, we may even require that

$$\{X_{\tau_1}=\tilde{P}_1\}=\{(W_{\tau_1},V_{\tau_1})=P_1\}\quad\text{ and }\quad\{X_{\tau_1}=\tilde{Z}_1\}=\{(W_{\tau_1},V_{\tau_1})=Z_1\}.$$

Indeed, it suffices to put $X_t=\varphi(W_t)$, $t\in[\tau_0,\tau_1]$, where $\varphi:\mathbb{R}\to\mathbb{R}$ is an affine mapping sending z to c^{-m} and w_- to $1+\delta$.

If $X_{\tau_1} = \tilde{Z}_1$, the process stops (and so does (W, V)); otherwise, on the set $\{\tau > \tau_1\}$, the movement is continued, along the segment $[\tilde{Z}_2, \tilde{P}_2]$ on the time interval $[\tau_1, \tau_2]$ so that $\tau_2 = \inf\{t > \tau_1 : X_t \in \{\tilde{P}_2, \tilde{Z}_2\}$ and

$$\{X_{\tau_2} = \tilde{P}_2\} = \{(W_{\tau_2}, V_{\tau_2}) = P_2\} \quad \text{and} \quad \{X_{\tau_2} = \tilde{Z}_2\} = \{(W_{\tau_2}, V_{\tau_2}) = Z_2\}.$$

This can be guaranteed due to (25): if φ is an affine mapping which sends zw_-/w_+ to $(1+\delta)c^{-m}$ and w_-^2/w_+ to $(1+\delta)^2$, then the formula $X_t = \varphi(W_t)$ on $\{\tau > \tau_1\}$, $t \in [\tau_1, \tau_2]$, does the job. We continue the construction using this pattern on each $[\tau_n, \tau_{n+1}]$, requiring

$$\{X_{\tau_n} = \tilde{P}_n\} = \{(W_{\tau_n}, V_{\tau_n}) = P_n\} \quad \text{and} \quad \{X_{\tau_n} = \tilde{Z}_n\} = \{(W_{\tau_n}, V_{\tau_n}) = Z_n\}$$

for all $n \geq 1$.

4° Calculation. We start from some facts which follow directly from the above construction. First, note that on the set $\{\tau_n < \tau_{n+1} = \tau\}$ we have $X_{\tau_n} = \tilde{P}_n$ and $X_{\tau_{n+1}} = \tilde{Z}_{n+1}$, which means that

$$X^* \geq \tilde{P}_n = (1+\delta)^n = c^m \tilde{Z}_{n+1} = c^m X_\infty. \tag{26}$$

Furthermore, by (24), we have

$$\mathbb{P}(X_\infty = \tilde{Z}_{n+1}) = \mathbb{P}((W_\infty, V_\infty) = Z_{n+1}) = \mathbb{P}(\tau_n < \tau_{n+1} = \tau) = \left(\frac{z - w_+}{z - w_-}\right)^n \frac{w_+ - w_-}{z - w_-},$$

which implies

$$||X||^p_{L^p(W)} = \sum_{n=0}^{\infty} \left[(1+\delta)^n c^{-m}\right]^p z \left(\frac{w_-}{w_+}\right)^n \cdot \left(\frac{z - w_+}{z - w_-}\right)^n \frac{w_+ - w_-}{z - w_-}. \tag{27}$$

We will show that if b is sufficiently close to c, then the above series converges. Assume for a moment that this holds, and let us quickly see how this contradicts (21). Combining (21) with (26), we get

$$||W||^m_{A_p} ||X||_{L^p(W)} \leq ||X^*||_{L^p(W)} \leq \beta_p ||W||^{\kappa(p)}_{A_p} ||X||_{L^p(W)}$$

and since $||X||_{L^p(W)} < \infty$, we must have $||W||^{m-\kappa(p)}_{A_p} < \beta_p$. But the above construction allows arbitrarily large values of $||W||_{A_p}$; this enforces $\kappa(p) \geq m$. Since m was an arbitrary constant smaller that $1/(p-1)$, the exponent in (7) is indeed the best possible.

5° Convergence of the geometric series (27). To prove the convergence, it suffices to show that the corresponding geometric ratio is smaller than 1:

$$(1+\delta)^p \cdot \frac{w_-}{w_+} \cdot \frac{z - w_+}{z - w_-} = \left(1 + \frac{(1 - c^{-m})(w_+ - w_-)}{z - w_+}\right)^p \cdot \frac{w_-}{w_+} \cdot \frac{z - w_+}{z - w_-} < 1.$$

However, when $b \to c$, then $w_+ - w_- \to 0$ and the ratio is of the order

$$\begin{aligned}&\left(1+\frac{(1-c^{-m})(w_+-w_-)}{z-w_+}\right)^p \cdot \frac{w_-}{w_+} \cdot \frac{z-w_+}{z-w_-}\\ &= 1+p(1-c^{-m})(w_+-w_-)/(z-w_+) - \frac{w_+-w_-}{w_+} - \frac{w_+-w_-}{z-w_-} + o(w_+-w_-).\end{aligned}$$

So, we will be done if we show that

$$p(1-c^{-m}) - \frac{z-w_+}{w_+} - \frac{z-w_+}{z-w_-} < 0$$

for b sufficiently close to c. But $\lim_{b\to c} w_+ = \lim_{b\to c} w_- = 1$, so the left-hand side above can be made arbitrarily close to $p(1-c^{-m})-z$. If $p(1-c^{-m}) \leq 1$, then $p(1-c^{-m})-z<0$ (and we are done). If $p(1-c^{-m})>1$, then this inequality is also true. Indeed, by (22), it suffices to show that

$$c^{-1}(p-1)^{p-1} < p(1-c^{-m})(p-p(1-c^{-m}))^{p-1} = p(1-c^{-m})p^{p-1}c^{-m(p-1)}$$

(since the function $s \mapsto s(p-s)^{p-1}$ is decreasing on $[1,p]$). But this clearly holds, since we have assumed that $m<1/(p-1)$ and $p(1-c^{-m})>1$.

This implies the desired convergence of (27) and completes the proof of Theorem 1.3.

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