# SHARP INEQUALITIES FOR MONOTONE BASES IN $L^1$

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ABSTRACT. We introduce a novel method which can be used to establish general sharp maximal inequalities for monotone bases and contractive projections in $L^1$. The technique enables to deduce such estimates from the existence of the upper solutions to the corresponding nonlinear problems. As an application, we identify the best unconditional-type constants in certain maximal and weak-type inequalities for monotone bases in $L^1$.

## 1. Introduction

The motivation for the results obtained in this paper comes from a very natural question about monotone bases and contractive projections in $L^1$. We start with introducing the necessary background and notation. Recall that a sequence $e = (e_n)_{n\geq 0}$ with values in a given real Banach space $X$ is a basis, if for every $f \in X$ there is a unique sequence $a = (a_n)_{n\geq 0} \subset \mathbb{R}$ satisfying $||f - \sum_{k=0}^n a_k e_k||_X \to 0$. The basis $(e_k)_{k\geq 0}$ is unconditional, if for any $f \in X$ the corresponding series converges unconditionally. This is equivalent to the condition $\sup\{||P_E|| : E \subset \mathbb{N} \text{ finite}\} < \infty$, where, for a given $E$, the symbol $P_E$ stands for the associated projection defined by $P_E f = \sum_{i\in E} a_i e_i$. A basis is called monotone if for each $n$ the projection $P_n := P_{\{0,1,\ldots,n\}}$ is contractive. This is equivalent to saying that for any nonnegative integer $n$ and any real numbers $a_0$, $a_1$, $\ldots$, $a_n$, $a_{n+1}$,

$$\left|\left|\sum_{k=0}^{n} a_k e_k\right|\right|_X \leq \left|\left|\sum_{k=0}^{n+1} a_k e_k\right|\right|_X .$$

We will be particularly interested in monotone bases of $L^p(\Omega, \mathcal{F}, \mu)$, where the underlying measure $\mu$ is assumed to be positive and nonatomic. Suppose first that $1 < p < \infty$. Then, as observed by Ando [1], every non-vanishing contractive projection of $L^p$ is isometrically equivalent to a conditional expectation. This argument can be pushed further to yield that every nondecreasing sequence $(P_n)_{n\geq 0}$ of contractive projections (i.e., satisfying $P_m P_n = P_{m\wedge n}$ for all $m$, $n$) gives rise to a sequence of conditional expectations with respect to a nondecreasing family of sub-$\sigma$-algebras, which in turn links this subject with the theory of martingales. Then, as shown by Dor and Odell [7], the use of the inequalities for martingale transforms (see Burkholder [2]) yields the following statement.

**Theorem 1.1.** *Assume that $(\Omega, \mathcal{F}, \mu)$ is a positive measure space. Let $P_{-1} = 0$, $P_0$, $P_1$, $P_2$, $\ldots$ be a nondecreasing sequence of contractive projections in $L^p(\Omega, \mathcal{F}, \mu)$,*

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*$1 < p < \infty$. If $f \in L^p(\Omega, \mathcal{F}, \mu)$, then for any sequence $\varepsilon_0$, $\varepsilon_1$, $\varepsilon_2, \ldots$ of signs,*

$$\left|\left|\sum_{k=0}^{\infty} \varepsilon_k (P_k - P_{k-1}) f\right|\right|_p \leq C_p ||f||_p, \tag{1.1}$$

*for some universal constant $C_p$ which depends only on $p$.*

It turns out that the optimal choice for the constant $C_p$ in (1.1) equals $p^* - 1$, where $p^* = \max\{p, p/(p-1)\}$. This follows from a related sharp inequality for martingales shown by Burkholder in [3] (see also [4]). In particular, the above theorem implies that every monotone basis in $L^p$ is unconditional provided $1 < p < \infty$. Further combination with the results of Olevskiǐ [10], [11] gives that the unconditional constant of any monotone basis $e$ of $L^p$ ($1 < p < \infty$) equals $p^* - 1$. That is, for any $n$, any sequence $a_0$, $a_1$, $a_2$, $\ldots$, $a_n$ of real numbers and any sequence $\varepsilon_0$, $\varepsilon_1$, $\varepsilon_2$, $\ldots$, $\varepsilon_n$ of signs we have

$$\left|\left|\sum_{k=0}^{n} \varepsilon_k a_k e_k\right|\right|_p \leq (p^* - 1) \left|\left|\sum_{k=0}^{n} a_k e_k\right|\right|_p, \qquad 1 < p < \infty, \tag{1.2}$$

and the constant $p^* - 1$ cannot be improved. Consult also the paper of Choi [6], in which the unconditional constant is defined in a slightly different manner.

There is a very interesting question about the validity of the inequality (1.2) in the limit case $p = 1$. A well-known result, due to Paley [12] (consult also Marcinkiewicz [9]), states that the Haar basis, a fundamental monotone basis of $L^1([0,1], \mathcal{B}([0,1]), |\cdot|)$, is not unconditional. Thus there is a further question about an appropriate version for the inequality (1.2) for $p = 1$, which will serve as a substitute for the unconditionality. A typical approach to such a problem is to study the corresponding weak-type inequality

$$\left|\left|\sum_{k=0}^{n} \varepsilon_k a_k e_k\right|\right|_{1,\infty} \leq c \left|\left|\sum_{k=0}^{n} a_k e_k\right|\right|_1, \tag{1.3}$$

where $||f||_{1,\infty} = \sup_{\lambda > 0} \lambda \mu(\{\omega \in \Omega : |f(\omega)| \geq \lambda\})$ denotes the weak norm of $f$. However, one encounters a difficulty here. Namely, Ando's theorem fails to hold for $p = 1$ and the structure of a contractive projection in $L^1$ is more complicated. Indeed, as shown by Douglas [8], such a projection is isometrically equivalent to the sum of a conditional expectation and an appropriate nilpotent operator. This in turn implies that the monotone sequence $(P_n)_{n \geq 0}$ of projections in $L^1$ cannot be represented as a martingale any more. In addition, while the version of (1.3) for martingale transforms holds true (see [2] and [3]), the inequality (1.3) *is not* valid in general with any finite constant $c$. This can be easily seen by considering the following example: fix a large positive integer $N$ and let $e$ be a basis of $L^1([0,1], \mathcal{B}([0,1]), |\cdot|)$, given as follows. Put $e_0 = \chi_{[0,1/N)}$ and $e_k = -\chi_{[(k-1)/N, k/N)} + \chi_{[k/N,(k+1)/N)}$ for $k = 1, 2, \ldots, N-1$. Finally, complete this sequence to a basis, by using a copy of the Haar system on each of the intervals $[(k-1)/N, k/N)$, $k = 1, 2, \ldots, N$. Then it is not difficult to verify that $e$ is monotone,

$$\sum_{k=0}^{N-1} e_k = \chi_{[(N-1)/N, 1)}$$

and

$$\sum_{k=0}^{N-1}(-1)^k e_k = 2\sum_{k=1}^{N-1}(-1)^k \chi_{[(k-1)/N,k/N)} + (-1)^N \chi_{[(N-1)/N,1)}.$$

This implies $c \geq N$ and shows that no finite constant suffices in (1.3).

Thus, we see that the theory of martingale transforms and that of contractive projections in $L^1$ are no longer parallel. One of the objectives of this paper is, in a sense, to fill this gap. We introduce an approach which enables the successful treatment of the inequalities for monotone bases in $L^1$. Namely, we will see how a certain class of estimates can be reduced to finding the upper solutions to some novel nonlinear boundary value problems. This will allow us to establish the following sharp maximal version of (1.2).

**Theorem 1.2.** *Suppose that $e$ in a monotone basis of $L^1(\Omega, \mathcal{F}, \mu)$. Then for any sequences $a_0$, $a_1$, $a_2$, ... of real numbers and $\varepsilon_0$, $\varepsilon_1$, $\varepsilon_2$, ... of signs we have the sharp inequalities*

$$\left\|\sum_{k=0}^{\infty}\varepsilon_k a_k e_k\right\|_1 \leq \beta \left\|\sup_{n\geq 0}\left|\sum_{k=0}^{n} a_k e_k\right|\right\|_1 \tag{1.4}$$

*and*

$$\left\|\sum_{k=0}^{\infty}\varepsilon_k a_k e_k\right\|_{1,\infty} \leq \beta \left\|\sup_{n\geq 0}\left|\sum_{k=0}^{n} a_k e_k\right|\right\|_1. \tag{1.5}$$

*Here $\beta = 2.536\ldots$ is the unique positive solution to the equation*

$$\beta = 3 - \exp\frac{1-\beta}{2}. \tag{1.6}$$

We would like to mention here that the method we plan to develop has its analogue in martingale theory (see Burkholder [5]), but the interplay between the two is non-trivial. Namely, if we compare the above statement to its version for martingale transforms, we have that the constant $\beta$ is also optimal in the probabilistic counterpart of (1.4) (cf. [5]); on contrary, quite surprisingly, the best constant in the martingale version of (1.5) is strictly smaller than $\beta$ (in fact it does not exceed 2, see e.g. [3]).

A few words about the organization of the paper are in order. The next section contains the description of the structure of a monotone basis in $L^1$. Section 3 is devoted to the detailed presentation of the method which allows to study general maximal inequalities for a certain class of monotone bases. In Section 4 we construct the special function which yields the validity of the inequalities (1.4) and (1.5). In the final part of the paper we address the question about the optimality of the constant $\beta$.

## 2. On the structure of monotone bases in $L^1$

The material presented in this section is, essentially, taken from the paper [7] of Dor and Odell. We will show how to construct an isometry of $L^1(\Omega, \mathcal{F}, \mu)$ onto a certain $L^1(\Omega, \mathcal{F}, \nu)$, which sends a given monotone basis $e$ onto a simple basis (for the necessary definition, see below). The new thing which will be proved here is that this isometry preserves the $L^1$ norm of the maximal function $\sup_{n\geq 0}|\sum_{k=0}^n a_k e_k|$.

Let us start with definitions.

**Definition 2.1.** A system of sets $\{A_{n,i} : i = 1, 2, \ldots, 2^n, n = 0, 1, 2, \ldots\}$ is called a *dyadic tree* if for all $n$ and $1 \leq i \leq 2^n$ we have

$$A_{n+1,2i-1} \cap A_{n+1,2i} = \emptyset$$

and

$$A_{n+1,2i-1} \cup A_{n+1,2i} = A_{n,i}.$$

**Definition 2.2.** Given a dyadic tree of sets satisfying $\mu(A_{n,i}) > 0$ for all $n$ and $i$, we define the associated generalized Haar sequence $h = (h_k)_{k\geq 0}$ by $h_0 = h_{0,1} = \chi_{A_{0,1}}/||\chi_{A_{0,1}}||_1$ and

$$h_{2^{n-1}+i-1} = h_{n,i} = H_{n,i}/||H_{n,i}||_1,$$

where

$$H_{n,i} = \chi_{A_{n,2i-1}}/\mu(A_{n,2i-1}) - \chi_{A_{n,2i}}/\mu(A_{n,2i}), \qquad i \leq 2^n,\ n = 1, 2, \ldots.$$

If $h$ forms a basis, it will be referred to as a *generalized Haar basis.*

The generalized Haar sequence $(h_n)_{n\geq 0}$ is uniquely determined by a dyadic tree $\{A_{n,i}\}$ and the following condition: for each $n \geq 1$ and $1 \leq i \leq 2^n$, the function $h_{n,i}$ is a linear combination of $\chi_{A_{n,2i-1}}$ and $\chi_{A_{n,2i}}$, such that

$$||h_{n,i}||_1 = 1 \quad \text{and} \quad \int_\Omega h_{n,i} = 0 \quad \text{for } n \geq 1. \tag{2.1}$$

Observe that if $\{A_{n,i}\}$ is the family of the dyadic subintervals of $[0, 1]$ and $\mu$ is the Lebesgue's measure, then the above definition yields the usual Haar system in $L^1$.

The final notion we need is the following.

**Definition 2.3.** A basis $d = (d_k)_{k\geq 0}$ in $L^1(\Omega, \mathcal{F}, \nu)$ is called *simple*, if there is a sequence (possibly finite) of disjoint sets $E_n \in \mathcal{F}$ covering $\Omega$, so that $(d_k)_{k\geq 0}$ is the union of disjoint subsequences $(d_i^n)_{i\geq 1}$, $n = 1, 2, \ldots$, satisfying the following two conditions.

(i) For each $n$ the sequence $\chi_{E_n}/||\chi_{E_n}||_1$, $d_2^n$, $d_3^n$, $\ldots$ is a generalized Haar basis for $L^1(E_n)$.

(ii) For each $n$ we have $d_1^n = c_n\chi_{E_n} + \psi_n$, where $||d_1^n||_1 = 1$, $||\psi_n||_1 \leq ||c_n\chi_{E_n}||_1$ and $\psi_n$ is a combination of the elements of $(d_k)_{k\geq 0}$ which precede $d_1^n$.

Next, we recall Theorem 3.1 from [7], which shows that monotone bases of $L^1$ are equivalent to simple bases.

**Theorem 2.4.** *Let $(e_k)_{k\geq 0}$ be a normalized monotone basis for $L^1(\Omega, \mathcal{F}, \mu)$. Then there is an isometry $T$ of $L^1(\Omega, \mathcal{F}, \mu)$ onto some $L^1(\Omega, \mathcal{F}, \nu)$, which sends $(e_k)_{k\geq 0}$ to some simple basis $(d_k)_{k\geq 0}$.*

The proof of this statement, presented in [7], shows that one can take $\mathrm{d}\nu = |\varphi|\mathrm{d}\mu$ and $Tf = f/\varphi$ for an appropriately chosen measurable function $\varphi : \Omega \to \mathbb{R} \setminus \{0\}$. Thus, we see that for each nonnegative integer $n$ and any numbers $a_0$, $a_2$, $\ldots$, $a_n$,

$$T\left(\max_{0\leq m\leq n}\left|\sum_{k=0}^m a_k e_k\right|\right) = \max_{0\leq m\leq n}\left|\sum_{k=0}^m a_k d_k\right|$$

and hence

$$\left|\left|\max_{0\leq m\leq n}\left|\sum_{k=0}^m a_k e_k\right|\right|\right|_{L^1(\mu)} = \left|\left|\max_{0\leq m\leq n}\left|\sum_{k=0}^m a_k d_k\right|\right|\right|_{L^1(\nu)}.$$

In consequence, to show (1.4), it suffices to establish it for simple bases only. In the next section we introduce a tool to handle this problem.

## 3. An upper class of functions

In this section, $(e_k)_{k\geq 0}$ will always be a simple basis of $L^1(\Omega,\mathcal{F},\mu)$. For any $f=\sum_{k=0}^\infty a_ke_k$, we will write $f_n=P_nf=\sum_{k=0}^n a_ke_k$ for the projection on the subspace generated by $e_0$, $e_1$, $\ldots$, $e_n$ and we say that $g\in L^1(\Omega,\mathcal{F},\mu)$ is a $\pm 1$-transform of $f$, if there is a sequence $\varepsilon_0$, $\varepsilon_1$, $\varepsilon_2$, $\ldots$ of signs such that $g=\sum_{k=0}^\infty \varepsilon_k a_k e_k$. Moreover, we will use the notation $f_n^*(\omega)=\max_{0\leq k\leq n}|f_k(\omega)|$, $\omega\in\Omega$, $n=0,\,1,\,2,\,\ldots$, for the "maximal function" of $f$.

We are ready to describe the method. Let

$$D=\{(x,y,z,w)\in\mathbb{R}\times\mathbb{R}\times[0,\infty)\times[0,\infty):|x|\vee z>0,\ |y|\vee w>0\}\cup\{(0,0,0,0)\}$$

and suppose that $V:D\to\mathbb{R}$ is a given function, satisfying $V(0,0,0,0)=0$ and

$$V(x,y,z,w)=V(x,y,|x|\vee z,|y|\vee w),\qquad (x,y,z,w)\in D. \tag{3.1}$$

This function need not be Borel or even measurable. Assume we are interested in proving that

$$\int_\Omega V\left(f_n(\omega),g_n(\omega),f_n^*(\omega),g_n^*(\omega)\right)\,\mathrm{d}\mu(\omega)\leq 0,\qquad n=0,\,1,\,2,\,\ldots, \tag{3.2}$$

for all $f$, $g\in L^1(\Omega)$ such that $g$ is a $\pm1$ transform of $f$. To handle this problem, we consider the class $\mathcal{U}(V)$ which consists of all functions $U$ which satisfy the following four conditions.

1° For all $(x,y,z,w)\in D$ we have

$$U(x,y,z,w)=U(x,y,|x|\vee z,|y|\vee w). \tag{3.3}$$

2° For all $(x,y,z,w)\in D$ we have

$$U(x,y,z,w)\geq V(x,y,z,w). \tag{3.4}$$

3° If $|x|\leq z$, $|y|\leq w$, $\varepsilon\in\{-1,1\}$, and $\alpha_1$, $\alpha_2\in(0,1)$, $t_1$, $t_2\in\mathbb{R}$ satisfy $\alpha_1+\alpha_2=1$, $\alpha_1t_1+\alpha_2t_2=0$, then

$$U(x,y,z,w)\geq\alpha_1U(x+t_1,y+\varepsilon t_1,z,w)+\alpha_2U(x+t_2,y+\varepsilon t_2,z,w). \tag{3.5}$$

4° If $|x|\leq z$, $|y|\leq w$, $\varepsilon\in\{-1,1\}$ and $t_1$, $t_2\in\mathbb{R}$, then

$$|t_2|U(x,y,z,w)\geq|t_2|U(x+t_1,y+\varepsilon t_1,z,w)+|t_1|U(t_2,\varepsilon t_2,|t_2|,|t_2|). \tag{3.6}$$

A few comments about these conditions are in order. The property 1° is a technical assumption which will make an appropriate induction argument work: see below. Concerning 2°, we will see that the properties 1°, 3° and 4° imply the validity of (3.2), but with $V$ replaced by $U$; then the application of the majorization will yield the claim. The condition 3° is a concavity-type property. In particular, it implies that for each $z$, $w>0$, the function $U(\cdot,\cdot,z,w)$ is diagonally concave on $[-z,z]\times[-w,w]$, i.e., concave along any line segment of slope $\pm1$ contained in this rectangle. More generally, 3° means that for each $\varepsilon\in\{-1,1\}$ and any $(x,y,z,w)\in D$ with $|x|\leq z$, $|y|\leq w$, the function $\Phi(t)=U(x+t,y+\varepsilon t,z,w)$, $t\in\mathbb{R}$, is majorized by a linear function $\Psi$ satisfying $\Psi(0)=\Phi(0)$. Finally, 4° can be regarded as a bound for the slopes of all such functions $\Psi$'s.

In particular, these conditions imply that

$$U(0,0,0,0)=0 \tag{3.7}$$

and

$$U(t, \pm t, |t|, |t|) \leq 0, \qquad t \in \mathbb{R}. \tag{3.8}$$

Indeed, plugging $t_2 = 0$ into (3.6) gives $U(0,0,0,0) \leq 0$, while 1° together with $V(0,0,0,0) = 0$ implies the reverse bound. Thus (3.7) follows. To see (3.8), fix $\varepsilon \in \{-1,1\}$, $t \neq 0$ and apply 4° to $x = t$, $y = \varepsilon t$, $z = w = |t|$, $t_1 = -t$ and $t_2 = t$. As the result, we get an estimate which is equivalent to $U(0,0,|t|,|t|) \leq 0$. Now apply 4° again, this time with $x = y = 0$, $z = w = |t|$ and $t_1 = t_2 = t$ to obtain (3.8).

To explain the interplay between the inequality (3.2) and the class $\mathcal{U}(V)$, we will prove the following fact.

**Theorem 3.1.** *If the class $\mathcal{U}(V)$ is nonempty, then (3.2) is valid.*

*Proof.* By simplicity, each $e_k$ is either a generalized Haar function, or it can be written in the form $c\chi_{E_k} + \psi_k$, where $\psi_k$ is a combination of $e_0$, $e_1$, …, $e_{k-1}$, the set $E_k$ is disjoint from the union of the supports of these functions and $||\psi_k||_1 \leq ||c\chi_{E_k}||_1$. Pick $f$, $g \in L^1(\Omega, \mathcal{F}, \mu)$ such that $g$ is a $\pm 1$-transform of $f$, and let $a_0$, $a_1$, $a_2$, …, $\varepsilon_0$, $\varepsilon_1$, $\varepsilon_2$, … denote the corresponding coefficients and the signs appearing in their expansions. The key part of the proof is to show that for any $n \geq 0$,

$$\int_\Omega U\left(f_n, g_n, f_n^*, g_n^*\right) \mathrm{d}\mu \geq \int_\Omega U\left(f_{n+1}, g_{n+1}, f_{n+1}^*, g_{n+1}^*\right) \mathrm{d}\mu. \tag{3.9}$$

To do this, fix $n \geq 0$ and assume first that $e_{n+1}$ is a generalized Haar function, with the support $E$ contained in the union of the supports of $e_0$, $e_1$, …, $e_n$. Then the quadruples $(f_n, g_n, f_n^*, g_n^*)$, $(f_{n+1}, g_{n+1}, f_{n+1}^*, g_{n+1}^*)$ coincide outside $E$ and hence it suffices to show that

$$\int_E U\left(f_n, g_n, f_n^*, g_n^*\right) \mathrm{d}\mu \geq \int_E U\left(f_{n+1}, g_{n+1}, f_{n+1}^*, g_{n+1}^*\right) \mathrm{d}\mu. \tag{3.10}$$

But $f_n$, $g_n$, $f_n^*$ and $g_n^*$ are constant on $E$, because of the structure of the simple basis $e$. Denoting the corresponding values by $x$, $y$, $z$ and $w$, we see that $|x| \leq z$ and $|y| \leq w$. By 1°, we have

$$U\left(f_{n+1}, g_{n+1}, f_{n+1}^*, g_{n+1}^*\right) = U\left(f_{n+1}, g_{n+1}, f_n^*, g_n^*\right) \qquad \text{on } E,$$

which allows us to transform the previous estimate into

$$\frac{1}{\mu(E)} \int_E U(x + a_{n+1}e_{n+1}, y + \varepsilon_{n+1}a_{n+1}e_{n+1}, z, w)\mathrm{d}\mu \leq U(x,y,z,w).$$

This inequality follows immediately from 3°, since $e_{n+1}$ is a generalized Haar function (see the second equation in (2.1)). Next, assume that $e_{n+1}$ is of the second type, i.e. $e_{n+1} = c\chi_{E_{n+1}} + \psi_{n+1}$, for appropriate $c \neq 0$, $E_{n+1}$ and $\psi_{n+1}$. Let $E$ denote the support of $e_{n+1}$. Again, the quadruples $(f_n, g_n, f_n^*, g_n^*)$, $(f_{n+1}, g_{n+1}, f_{n+1}^*, g_{n+1}^*)$ coincide outside $E$; furthermore, $U(f_n, g_n, f_n^*, g_n^*) = 0$ on $E_{n+1}$, see (3.7). Therefore, (3.9) reduces to

$$\int_{E \setminus E_{n+1}} U\left(f_n, g_n, f_n^*, g_n^*\right) \mathrm{d}\mu \geq \int_E U\left(f_{n+1}, g_{n+1}, f_{n+1}^*, g_{n+1}^*\right) \mathrm{d}\mu. \tag{3.11}$$

The right-hand side of this inequality equals

$$\int_{E_{n+1}} U(c,\varepsilon_{n+1}c,c,c)\mathrm{d}\mu + \int_{E\setminus E_{n+1}} U\left(f_n+\psi_{n+1}, g_n+\varepsilon_{n+1}\psi_{n+1}, f_{n+1}^*, g_{n+1}^*\right)\mathrm{d}\mu$$
$$= \mu(E_{n+1})\,U(c,\varepsilon_{n+1}c,c,c) + \int_{E\setminus E_{n+1}} U\left(f_n+\psi_{n+1}, g_n+\varepsilon_{n+1}\psi_{n+1}, f_n^*, g_n^*\right)\mathrm{d}\mu,$$

by virtue of 1°. Applying 4°, we get the pointwise estimate

$$U\left(f_n+\psi_{n+1}, g_n+\varepsilon_{n+1}\psi_{n+1}, f_n^*, g_n^*\right) \leq U\left(f_n, g_n, f_n^*, g_n^*\right) - \frac{|\psi_{n+1}|}{c} U\left(c,\varepsilon_{n+1}c,c,c\right).$$

By (3.8), we have the inequality $U(c,\varepsilon_{n+1}c,c,c)\leq 0$. Moreover, $||\psi_{n+1}||_1 \leq c\mu(E_{n+1})$, which follows from the form of $e_{n+1}$. This gives

$$\int_{E\setminus E_{n+1}} \frac{|\psi_{n+1}|}{c} U(c,\varepsilon_{n+1}c,c,c)\mathrm{d}\mu \geq \mu(E_{n+1})U(c,\varepsilon_{n+1}c,c,c).$$

Combining the above facts yields (3.11) and thus the sequence

$$\left(\int_\Omega U\left(f_n,g_n,f_n^*,g_n^*\right)\mathrm{d}\mu\right)_{n\geq 0}$$

is nonincreasing. In consequence, by 2°, we obtain that for any $n\geq 0$,

$$\int_\Omega V\left(f_n,g_n,f_n^*,g_n^*\right)\mathrm{d}\mu \leq \int_\Omega U\left(f_n,g_n,f_n^*,g_n^*\right)\mathrm{d}\mu \leq \int_\Omega U\left(f_0,g_0,f_0^*,g_0^*\right)\mathrm{d}\mu.$$

It remains to note that $f_0^*=g_0^*=|f_0|=|g_0|$ and use (3.8) to get the desired estimate (3.2). □

A very interesting fact is that the implication of the above theorem can be reversed. For a given measurable space $(\Omega,\mathcal{F},\mu)$ with $\mu(\Omega)\geq 1$, we let $\mathcal{E}(\Omega,\mathcal{F},\mu)$ denote the family of all simple bases $(e_k)_{k\geq 0}$ of $L^1(\Omega,\mathcal{F},\mu)$ such that $e_0$ is the characteristic function of a set of measure 1. Of course, this family is nonempty. Next, for a given $e\in\mathcal{E}(\Omega,\mathcal{F},\mu)$ and $x$, $y\in\mathbb{R}$, let $\mathcal{M}(x,y,e)$ be the class of all pairs $(f,g)$ of functions which admit the expansions

$$f = xe_0+\sum_{k=1}^n a_n e_n, \qquad g = ye_0+\sum_{k=1}^n \varepsilon_n a_n e_n$$

for some $n$ and some sequences $a_1, a_2, \ldots, a_n\in\mathbb{R}$, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\in\{-1,1\}$.

Define $U^0: D\to\mathbb{R}\cup\{\infty\}$ by the formula

$$U^0(x,y,z,w) = \sup\left\{\int_\Omega V(f,g,f^*\vee ze_0, g^*\vee we_0)\mathrm{d}\mu\right\}, \tag{3.12}$$

where the supremum is taken over all measurable spaces $(\Omega,\mathcal{F},\mu)$ with $\mu(\Omega)\geq 1$, all $e\in\mathcal{E}(\Omega,\mathcal{F},\mu)$ and all $(f,g)\in\mathcal{M}(x,y,e)$.

**Theorem 3.2.** *If the inequality* (3.2) *is valid, then the function* $U^0$ *belongs to the class* $\mathcal{U}(V)$.

*Proof.* The condition 1° follows from the pointwise bounds $f^*\geq|f_0|=|x|e_0$, $g^*\geq|g_0|=|y|e_0$, which imply that

$$f^*\vee ze_0 = f^*\vee((|x|\vee z)e_0), \qquad g^*\vee we_0 = g^*\vee((|y|\vee w)e_0).$$

To prove 2°, let us compute the integral in (3.12) for $n = 0$ and some $e \in \mathcal{E}(\Omega, \mathcal{F}, \mu)$. Since $e_0$ is the indicator function of a set of measure one, the integral equals $V(x, y, |x| \vee z, |y| \vee w) = V(x, y, z, w)$, by (3.1). This implies the desired majorization. To show 3°, pick $x$, $y$, $z$, $w$, $\varepsilon$, $\alpha_1$, $\alpha_2$, $t_1$ and $t_2$ as in the statement. Take two bases $e^1 \in \mathcal{E}(\Omega^1, \mathcal{F}^1, \mu^1)$, $e^2 \in \mathcal{E}(\Omega^2, \mathcal{F}^2, \mu^2)$ and two pairs $(f^1, g^1)$, $(f^2, g^2)$ of functions which have the following finite expansions in $e^1$ and $e^2$:

$$f^i = (x + t_i)e_0^i + \sum_{k=1}^{n} a_n^i e_n^i, \qquad g^i = (y + \varepsilon t_i)e_0^i + \sum_{k=1}^{n} \varepsilon_n^i a_n^i e_n^i \tag{3.13}$$

(we may assume that the length of the expansion is the same for both pairs, enlarging one of them if necessary). Suppose that $\Omega^1$ and $\Omega^2$ are disjoint and let us splice the measure spaces $(\Omega^i, \mathcal{F}^i, \mu^i)$ into one space $(\Omega, \mathcal{F}, \mu)$, with $\Omega = \Omega^1 \cup \Omega^2$, $\mathcal{F} = \sigma(\mathcal{F}^1, \mathcal{F}^2)$ and $\mu(A^1 \cup A^2) = \alpha_1 \mu^1(A^1) + \alpha_2 \mu^2(A^2)$ for all $A^i \in \mathcal{F}^i$, $i = 1, 2$. Next, we splice $e^1$ and $e^2$ into one base $e \in \mathcal{E}(\Omega, \mathcal{F}, \mu)$, by putting $e_0 = e_0^1 \chi_{\Omega^1} + e_0^2 \chi_{\Omega^2}$, $e_1 = \frac{1}{2}\alpha_1^{-1} e_0^1 \chi_{\Omega^1} - \frac{1}{2}\alpha_2^{-1} e_0^2 \chi_{\Omega^2}$ and, for $k \geq 1$,

$$e_{2k} = \alpha_1^{-1} e_k^1 \chi_{\Omega^1}, \quad e_{2k+1} = \alpha_2^{-1} e_k^2 \chi_{\Omega^2}.$$

Clearly, this new sequence forms a simple basis of $L^1(\Omega, \mathcal{F}, \mu)$. Furthermore, $e_0$ is the indicator function of a certain set of measure 1, so $e \in \mathcal{E}(\Omega, \mathcal{F}, \mu)$. Using (3.13), it is easy to check that the functions

$$f = f^1 \chi_{\Omega^1} + f^2 \chi_{\Omega^2}, \quad g = g^1 \chi_{\Omega^1} + g^2 \chi_{\Omega^2} \tag{3.14}$$

admit the following expansions in the basis $e$:

$$f = xe_0 + 2\alpha_1 t_1 e_1 + \sum_{k=1}^{n} \left( \alpha_1 a_k^1 e_{2k} + \alpha_2 a_k^2 e_{2k+1} \right),$$

$$g = ye_0 + 2\alpha_1 \varepsilon t_1 e_1 + \sum_{k=1}^{n} \left( \varepsilon_k^1 \alpha_1 a_k^1 e_{2k} + \varepsilon_k^2 \alpha_2 a_k^2 e_{2k+1} \right).$$

Consequently, by the definition of $U$ and the formula (3.14) for $f$ and $g$, we have

$$\begin{aligned} U^0(x, y, z, w) &\geq \int_\Omega V(f, g, f^* \vee ze_0, g^* \vee we_0) \mathrm{d}\mu \\ &= \alpha_1 \int_{\Omega^1} V(f^1, g^1, (f^1)^* \vee ze_0^1, (g^1)^* \vee we_0^1) \mathrm{d}\mu^1 \\ &\quad + \alpha_2 \int_{\Omega^2} V(f^2, g^2, (f^2)^* \vee ze_0^2, (g^2)^* \vee we_0^2) \mathrm{d}\mu^2. \end{aligned}$$

It remains to take the supremum over all triples $(\Omega^i, \mathcal{F}^i, \mu^i)$, all $n$ and all pairs $(f^i, g^i)$ to get (3.5). Finally, to establish (3.6), we may assume that $t_1$, $t_2 \neq 0$. Pick two bases $e^i \in \mathcal{E}(\Omega^i, \mathcal{F}^i, \mu^i)$ with $(\Omega^i, \mathcal{F}^i, \mu^i)$ as previously and two pairs $f^i, g^i$ of functions of the form

$$f^1 = (x + t_1)e_0^1 + \sum_{k=1}^{n} a_n^1 e_n^1, \qquad g^1 = (y + \varepsilon t_1)e_0^1 + \sum_{k=1}^{n} \varepsilon_n^1 a_n^1 e_n^1,$$

$$f^2 = t_2 e_0^2 + \sum_{k=1}^{n} a_n^2 e_n^2, \qquad g^2 = \varepsilon t_2 e_0^2 + \sum_{k=1}^{n} \varepsilon_n^2 a_n^2 e_n^2.$$

This time we use the splicing

$$\Omega = \Omega^1 \cup \Omega^2, \quad \mathcal{F} = \sigma(\mathcal{F}^1, \mathcal{F}^2), \quad \mu(A^1 \cup A^2) = \mu^1(A^1) + \frac{|t_1|}{|t_2|}\mu^2(A^2),$$

for all $A^1 \in \mathcal{F}^1$, $A^2 \in \mathcal{F}^2$. Furthermore, we put $e_0 = e_0^1\chi_{\Omega^1}$, $e_1 = \frac{1}{2}e_0^1\chi_{\Omega^1} + \frac{t_2}{2t_1}e_0^2\chi_{\Omega^2}$ and, for $k \geq 1$, we let

$$e_{2k} = e_k^1\chi_{\Omega^1} \quad \text{and} \quad e_{2k+1} = \frac{t_2}{t_1}e_k^2\chi_{\Omega^2}.$$

Then it is straightforward to check that $e$ is a simple basis, which follows immediately from the simplicity of $e^1$ and $e^2$. The only thing which needs to be verified is whether $e_1$ satisfies the condition (ii) of Definition 2.3. But this amounts to checking that

$$\left|\left|\frac{1}{2}e_0^1\chi_{\Omega^1}\right|\right|_1 \leq \left|\left|\frac{t_2}{2t_1}e_0^2\chi_{\Omega^2}\right|\right|_1,$$

which is evident: in fact, both sides are equal. Now, we easily see that the functions $f$, $g$ given by $f = f^1\chi_{\Omega^1} + f^2\chi_{\Omega^2}$ and $g = g^1\chi_{\Omega^1} + g^2\chi_{\Omega^2}$ have the expansions

$$f = xe_0 + 2t_1e_1 + \sum_{k=1}^n \left(a_k^1 e_{2k} + \frac{t_1a_k^2}{t_2}e_{2k+1}\right),$$

$$g = ye_0 + \varepsilon \cdot 2t_1e_1 + \sum_{k=1}^n \left(\varepsilon_k^1 a_k^1 e_{2k} + \frac{\varepsilon_k^2 t_1 a_k^2}{t_2}e_{2k+1}\right).$$

Therefore, by the definition of $U^0$, we obtain

$$\begin{aligned}
U^0(x,y,z,w) &\geq \int_\Omega V(f, g, f^* \vee ze_0, g^* \vee we_0)\mathrm{d}\mu \\
&= \int_{\Omega^1} V(f^1, g^1, (f^1)^* \vee ze_0^1, (g^1)^* \vee we_0^1)\mathrm{d}\mu^1 \\
&\quad + \frac{|t_1|}{|t_2|}\int_{\Omega^2} V(f^2, g^2, (f^2)^* \vee 0, (g^2)^* \vee 0)\mathrm{d}\mu^2.
\end{aligned}$$

However,

$$\begin{aligned}
V(f^2, g^2, (f^2)^* \vee 0, (g^2)^* \vee 0) &= V(f^2, g^2, (f^2)^* \vee |f_0^2|, (g^2)^* \vee |g_0^2|) \\
&= V(f^2, g^2, (f^2)^* \vee |t_2|e_0^2, (g^2)^* \vee |t_2|e_0^2),
\end{aligned}$$

so it suffices to take supremum over all $(f^i, g^i) \in L^1(\Omega^i, \mathcal{F}^i, \mu^i)$ to obtain (3.6). $\square$

We conclude this section by two important observations.

**Remark 3.3.** (i) If one of the maximal functions does not appear in the estimate under investigation, then we may consider $U$, $V$ defined on the appropriate three-dimensional domain. Simply remove the variable corresponding to the non-existing maximal function.

(ii) In certain cases, the function $U^0$ inherits some crucial properties from the function $V$, which in turn simplifies the search for its explicit formula. For example, if $V$ is homogeneous of order $p$, then so is $U^0$. To see this, pick arbitrary $(\Omega, \mathcal{F}, \mu)$

with $\mu(\Omega)\geq 1$, $e\in\mathcal{E}(\Omega,\mathcal{F},\mu)$, $(f,g)\in\mathcal{M}(x,y,e)$ and $\lambda>0$. Then $(\lambda f,\lambda g)\in\mathcal{M}(\lambda x,\lambda y,e)$ and hence

$$\begin{aligned}U^0(\lambda x,\lambda y,\lambda z,\lambda w)&\geq\int_\Omega V\big(\lambda f,\lambda g,\lambda(f^*\vee ze_0),\lambda(g^*\vee we_0)\big)\mathrm{d}\mu\\&=\lambda^p\int_\Omega V\big(f,g,f^*\vee ze_0,g^*\vee we_0\big)\mathrm{d}\mu.\end{aligned}$$

Taking the supremum over all the parameters gives the inequality

$$U^0(\lambda x,\lambda y,\lambda z,\lambda w)\geq\lambda^pU^0(x,y,z,w)\qquad\text{for }(x,y,z,w)\in D,$$

and switching from $\lambda$ to $\lambda^{-1}$ yields the reverse. Using a similar reasoning one can show that if $V$ satisfies the symmetry condition

$$V(x,y,z,w)=V(-x,y,z,w)=V(x,-y,z,w)\qquad\text{for all }(x,y,z,w)\in D,$$

then the same is true for $U^0$.

## 4. Proof of (1.4) and (1.5)

As an application of the method described in the previous section, let us present the proofs of the maximal estimates formulated in the Introduction. Obviously, it suffices to focus on the $L^1$-inequality (1.4); then the weak-type bound follows immediately by the use of Chebyshev's inequality. In view of Lebesgue's monotone convergence theorem and Fatou's lemma, it suffices to prove that for any monotone basis $e$ of $L^1(\Omega,\mathcal{F},\mu)$, any $n$ and all $a_0$, $a_1$, $a_2$, $\ldots a_n\in\mathbb{R}$, $\varepsilon_0$, $\varepsilon_1$, $\varepsilon_2$, $\ldots\varepsilon_n\in\{-1,1\}$, we have

$$\int_\Omega\left|\sum_{k=0}^n\varepsilon_ka_ke_k(\omega)\right|\mathrm{d}\mu(\omega)\leq\beta\int_\Omega\sup_{0\leq m\leq n}\left|\sum_{k=0}^m a_ke_k(\omega)\right|\mathrm{d}\mu(\omega).$$

This can be rewritten in the more compact form

$$\int_\Omega V(f_n,g_n,f_n^*)\mathrm{d}\mu\leq 0,\tag{4.1}$$

where $V(x,y,z)=|y|-\beta(|x|\vee z)$ and $f_n$, $g_n$ and $f_n^*$ are as previously. Thus the problem is of the form (3.2) and hence it can be treated by means of Theorems 3.1 and 3.2.

To introduce the corresponding special function $U$, first we define an auxiliary object. Let $u:[-1,1]\times\mathbb{R}\to\mathbb{R}$ be given by

$$u(x,y)=\begin{cases}\frac{2}{\beta-1}\left[-|x|-\frac13-\frac13(2-2|x|-|y|)(1-|x|+|y|)^{1/2}\right] & \text{if }|y|<|x|,\\ \frac{2}{\beta-1}\left[|y|-3+(2-|x|)\exp\left(\frac{|x|-|y|}{2}\right)\right] & \text{if }|y|\geq|x|.\end{cases}$$

It is not difficult to check that the function $u$ is of class $C^1$: the partial derivatives match appropriately at the diagonals $\{(x,y)\in[-1,1]\times\mathbb{R}:|x|=|y|\}$. Furthermore, the function $u$ enjoys the following properties.

**Lemma 4.1.** *(i) The function $u(1,\cdot):y\mapsto u(1,y)$ is convex on $\mathbb{R}$.*

*(ii) The function $u$ is concave along any line segment of slope $\pm1$ contained in $[-1,1]\times\mathbb{R}$.*

*(iii) For any $(x,y)\in[-1,1]\times\mathbb{R}$ we have*

$$u(x,y)\geq|y|-\beta.\tag{4.2}$$

*Proof.* (i) The convexity is evident on $(-\infty,-1]$, $[-1,1]$ and $[1,\infty)$. Furthermore, the function $u(1,\cdot)$ is of class $C^1$ on $\mathbb{R}$.

(ii) Since $u$ is of class $C^1$ and satisfies

$$u(x,y)=u(x,-y)=u(-x,y) \qquad \text{for all } (x,y)\in[-1,1]\times\mathbb{R}, \tag{4.3}$$

it suffices to show that

$$u_{xx}(x,y)\pm 2u_{xy}(x,y)+u_{yy}(y,y)\leq 0$$

for $(x,y)\in(0,1)\times(0,\infty)$ such that $x\neq y$. It is clear from the definition that $u_{xx}+2u_{xy}+u_{yy}=0$ on $(0,1)\times(0,\infty)$. Furthermore, a little calculation shows that

$$u_{xx}(x,y)-2u_{xy}(x,y)+u_{yy}(x,y)=\begin{cases}-\frac{2y}{\beta-1}(1-x+y)^{-3/2} & \text{if } 0<y<x,\\ -\frac{2x}{\beta-1}\exp\left(\frac{x-y}{2}\right) & \text{if } 0<x<y\end{cases}$$

and both expressions are obviously nonpositive.

(iii) The function $(x,y)\mapsto|y|-\beta$ is convex, so by (ii) and the symmetry condition (4.3), it suffices to prove the majorization (4.2) only for $x=1$ and $y>0$. However, by (i), the function $F(y)=u(1,y)-y+\beta$ is convex on $(0,\infty)$; it suffices to observe that $F(\beta)=F'(\beta)=0$ to complete the proof. $\square$

Now, for any $(x,y)\in\mathbb{R}^2$ and any $z\geq 0$ such that $|x|\vee z>0$, we define

$$U(x,y,z)=(|x|\vee z)\,u\left(\frac{x}{|x|\vee z},\frac{y}{|x|\vee z}\right) \tag{4.4}$$

and set $U(0,0,0)=0$. This is the special function corresponding to the inequality (1.4), which, in view of (4.1) and Theorem 3.1, can be deduced from the following statement.

**Theorem 4.2.** *The function $U$ belongs to $\mathcal{U}(V)$.*

*Proof.* We need to verify the conditions 1°-4°. The first of them is clear in view of the definition of $U$. The majorization 2° follows immediately from (4.2) and (4.4). The main technical difficulty lies in proving the conditions 3° and 4°. To handle these, fix $\varepsilon\in\{-1,1\}$ and a point $(x,y,z)\in\mathbb{R}\times\mathbb{R}\times(0,\infty)$ such that $z\geq|x|$. Introduce the function $\Phi=\Phi_{x,y,z,\varepsilon}:\mathbb{R}\to\mathbb{R}$, given by $\Phi(t)=U(x+t,y+\varepsilon t,z)$. Let us prove that there is $A=A(x,y,z,\varepsilon)\in[-2/(\beta-1),2/(\beta-1)]$ such that

$$\Phi(t)\leq\Phi(0)+At \qquad \text{for all } t\in\mathbb{R}. \tag{4.5}$$

This will clearly yield (3.5); furthermore, it will imply that for all $x$, $y$, $z$, $t_1$, $t_2$, $\varepsilon$ as in the formulation of 4°,

$$\begin{aligned}U(x+t_1,y+\varepsilon t_1,z)&\leq\Phi(0)+At_1\\&\leq U(x,y,z)+\frac{2}{\beta-1}|t_1|\\&=U(x,y,z)-\frac{|t_1|}{|t_2|}U(t_2,\varepsilon t_2,|t_2|,|t_2|),\end{aligned}$$

which is (3.6). Thus, all we need is to prove (4.5). By homogeneity, we may and do assume that $z=1$. Furthermore, we have $\Phi_{x,y,z,\varepsilon}=\Phi_{x,-y,z,-\varepsilon}$, which allows us to consider the case $\varepsilon=1$ only. Finally, by the identity $\Phi_{x,y,z,1}(t)=\Phi_{-x,-y,z,1}(t)$, it suffices to establish (4.5) for $t\leq 0$. After all these reductions, we see that (4.5) will follow if we show that

$$\Phi \text{ is concave on } [-1-x,1-x], \tag{4.6}$$

$$\Phi \text{ is convex on } (-\infty, -1-x), \tag{4.7}$$

$$\lim_{t\to-\infty} \Phi'(t+) \geq \frac{2}{\beta-1} \tag{4.8}$$

and

$$\Phi'(-1-x+) \leq \frac{2}{\beta-1}, \qquad \Phi'(1-x-) \geq -\frac{2}{\beta-1}. \tag{4.9}$$

However, (4.6) is a consequence of the part (ii) of Lemma 4.1. The prove the second condition, fix $\alpha_1$, $\alpha_2 > 0$ satisfying $\alpha_1 + \alpha_2 = 1$, choose $t_1, t_2 \in (-\infty, -1-x]$ and let $t = \alpha_1 t_1 + \alpha_2 t_2$. We have

$$\begin{aligned}
\alpha_1\Phi(t_1) + \alpha_2\Phi(t_2) &= \alpha_1 U(x+t_1, y+t_1, 1) + \alpha_2 U(x+t_2, y+t_2, 1)\\
&= \alpha_1 U(x+t_1, y+t_1, |x+t_1|) + \alpha_2 U(x+t_2, y+t_2, |x+t_2|)\\
&= \alpha_1 |x+t_1|\, u\left(-1, \frac{y+t_1}{|x+t_1|}\right) + \alpha_2 |x+t_2|\, u\left(-1, \frac{y+t_2}{|x+t_2|}\right).
\end{aligned}$$

Using the convexity of $u(1,\cdot)$, this can be bounded from below by

$$|x+t|\, u\left(1, \frac{y+t}{|x+t|}\right) = \Phi(t),$$

which gives (4.7). Using this convexity, we compute that

$$\lim_{t\to-\infty} \Phi'(t+) = \lim_{t\to-\infty} \frac{\Phi(t)}{t} = \lim_{t\to-\infty} \frac{|x+t|u\left(1, \frac{y+t}{|x+t|}\right)}{t} = -u(1,1) = \frac{2}{\beta-1}$$

and hence (4.8) holds true. Finally, we see that

$$\Phi'(-1-x) = \begin{cases} \frac{2}{\beta-1} & \text{if } y-x \geq 1,\\ \frac{2}{\beta-1}(1-(x-y+1)^{1/2}) & \text{if } y-x \in [0,1),\\ \frac{2}{\beta-1}\left(-1+\exp\left(\frac{y-x}{2}\right)\right) & \text{if } y-x < 0. \end{cases}$$

does not exceed $2/(\beta-1)$, which is the first estimate in (4.9). This also yields the second bound, since

$$\Phi'(1-x) = \Phi'_{x,y,1,1}(1-x) = -\Phi'_{-x,-y,1,1}(-1+x) \geq -\frac{2}{\beta-1}.$$

The proof is complete. $\square$

Finally, let us state here the following interesting observation.

**Remark 4.3.** The function $U$ studied above does not coincide with the function $U^0$ corresponding to the problem, but we have $U = U^0$ on the large part of the set $D$. Namely, it can be shown that $U^0$ is given by $U^0(0,0,0) = 0$ and, for $|x| \vee z > 0$,

$$U^0(x,y,z) = (|x| \vee z) u^0\left(\frac{x}{|x|\vee z}, \frac{y}{|x|\vee z}\right),$$

where $u^0(x,y)$ (for $(x,y) \in [-1,1]\times\mathbb{R}$) equals

$$\begin{cases} \frac{2}{\beta-1}\left[-|x| - \frac{1}{3} - \frac{1}{3}(2-2|x|-|y|)(1-|x|+|y|)^{1/2}\right] & \text{if } |y| < |x|,\\ \frac{2}{\beta-1}\left[|y| - 3 + (2-|x|)\exp\left(\frac{|x|-|y|}{2}\right)\right] & \text{if } |x| \leq |y| \leq |x|+\beta-1,\\ |y| - \beta + \frac{3-\beta}{\beta-1}(1-|x|)\exp(|x|-|y|+\beta-1) & \text{if } |y| \geq |x|+\beta-1. \end{cases}$$

## 5. Sharpness

Of course, it suffices to prove that the constant $\beta$ is the best in the weak-type inequality (1.5). This can be obtained by the construction of appropriate examples (see Burkholder [5]), but this approach involves quite elaborate analysis and computations. We take the opportunity to present a completely different method, based on Theorem 3.2, which has the advantage of being much simpler.

So, suppose that $\beta_0$ is the best constant in the estimate (1.5). Then for any $n$, any coefficients $a_0, a_1, \ldots \in \mathbb{R}$ and any signs $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ we have

$$\beta\,\mu\left(\left|\sum_{k=0}^{n}\varepsilon_k a_k e_k\right| \geq \beta\right) \leq \beta_0 \int_\Omega \sup_{0\leq m\leq n}\left|\sum_{k=0}^{m} a_k e_k\right| \mathrm{d}\mu.$$

This is equivalent to the estimate

$$\int_\Omega V(f_n, g_n, f_n^*)\mathrm{d}\mu \leq 0,$$

where $V : D \to \mathbb{R}$ is given by $V(x,y,z) = \beta\chi_{\{|y|\geq\beta\}} - \beta_0(|x|\vee z)$. By Theorem 3.2, the function

$$U^0(x,y,z) = \sup\left\{\int_\Omega V(f_n, g_n, f_n^*\vee ze_0)\mathrm{d}\mu\right\},$$

with the supremum taken over appropriate parameters, belongs to the class $\mathcal{U}(V)$. Since $V(x,y,z) = V(-x,y,z) = V(x,-y,z)$ for all $x$, $y$, $z$, the function $U^0$ inherits this property. Introduce the notation $B(y) = U^0(1,y,1)$ and $A(y) = U^0(0,y,1)$. For the sake of convenience, let us divide the reasoning into few parts.

*Step 1.* First we show that

$$A(0) \geq B(1). \tag{5.1}$$

This follows immediately from 3°, applied to $x = y = 0$, $z = 1$, $\varepsilon = 1$, $t_1 = 1$ and $t_2 = -1$ (note that then we must take $\alpha_1 = \alpha_2 = 1/2$, so that the conditions $\alpha_1 + \alpha_2 = 1$, $\alpha_1 t_1 + \alpha_2 t_2 = 0$ are satisfied).

*Step 2.* The next step is to prove that

$$A(y-1) \geq B(y) + B(1)\left(1 - \exp\left(\frac{1-y}{2}\right)\right) \tag{5.2}$$

for all $y \geq 1$. To do this, fix $\delta > 0$ and apply 4° with $x = z = 1$, $y$, $\varepsilon = -1$, $t_1 = -\delta$ and $t_2 = 1$. As the result, we obtain

$$U^0(1,y,1) \geq U^0(1-\delta, y+\delta, 1) + \delta U^0(1,-1,1) = U^0(1-\delta, y+\delta, 1) + \delta U^0(1,1,1).$$

Next, use the property 3° with $x = 1-\delta$, $y+\delta$, $z = 1$, $\varepsilon = 1$, $t_1 = \delta - 1$, $t_2 = \delta$ (then we are forced to take $\alpha_1 = \delta$ and $\alpha_2 = 1-\delta$). We arrive at

$$U^0(1-\delta, y+\delta, 1) \geq \delta U^0(0, y+2\delta-1, 1) + (1-\delta)U^0(1, y+2\delta, 1)$$

and combining this with the preceding estimate yields

$$B(y) \geq (1-\delta)B(y+2\delta) + \delta A(y+2\delta-1) + \delta B(1). \tag{5.3}$$

Using a similar reasoning, we show that

$$A(y+2\delta-1) \geq \frac{\delta}{1+\delta}B(y) + \frac{\delta}{1+\delta}B(y+2\delta) + \frac{1-\delta}{1+\delta}A(y-1). \tag{5.4}$$

Indeed, it suffices to combine the following inequalities: first,

$$A(y+2\delta-1) \geq \frac{1}{1+\delta}U^0(\delta, y+\delta-1, 1) + \frac{\delta}{1+\delta}B(y+2\delta),$$

coming from 3° with $x=0$, $y+2\delta$, $z=1$, $\varepsilon=-1$, $t_1=\delta$, $t_2=-1$; second,

$$U^0(\delta, y+\delta-1, 1) \geq (1-\delta)A(y-1) + \delta B(y),$$

a consequence of 3° with $x=\delta$, $y+\delta-1$, $z=1$, $\varepsilon=1$, $t_1=-\delta$ and $t_2=1-\delta$.

Now multiply both sides of (5.3) by $1/(1+\delta)$ and add it to (5.4). After some manipulations, we get

$$A(y+2\delta-1) - B(y+2\delta) - B(1) \geq (1-\delta)(A(y-1) - B(y) - B(1)).$$

Therefore, by induction, we see that for any nonnegative integer $N$,

$$A(y+2N\delta-1) - B(y+2N\delta) - B(1) \geq (1-\delta)^N(A(y-1) - B(y) - B(1)).$$

Now fix $s>1$ and set $y=1$, $\delta=(s-1)/(2N)$. Letting $N\to\infty$ yields

$$A(s-1) - B(s) - B(1) \geq (A(0) - 2B(1))\exp\frac{1-s}{2},$$

and using (5.1) we arrive at (5.2).

*Step 3.* Now we come back to (5.3) and insert the bound (5.2) there, obtaining

$$B(y) \geq B(y+2\delta) + 2\delta B(1) - \delta B(1)\exp\frac{1-y-2\delta}{2}.$$

By induction, we get, for any nonnegative integer $N$,

$$\begin{aligned} B(y) &\geq B(y+2N\delta) + 2N\delta B(1) - \delta B(1)\sum_{k=1}^{N}\exp\left(\frac{1-y-2k\delta}{2}\right) \\ &= B(y+2N\delta) + 2N\delta B(1) - \delta B(1)\frac{1-e^{-N\delta}}{1-e^{-\delta}}\exp\left(\frac{1-y-2\delta}{2}\right). \end{aligned}$$

As previously, fix $s>1$ and set $y=1$, $\delta=(s-1)/(2N)$. Letting $N\to\infty$ gives

$$B(1)\left(3-s-\exp\left(\frac{1-s}{2}\right)\right) \geq B(s)$$

and hence $B(\beta)\leq 0$ (see (1.6)). Since $U^0$ majorizes $V$, we obtain that

$$\beta-\beta_0 = V(1,\beta,1) \leq U^0(1,\beta,1) = B(\beta) \leq 0,$$

that is, $\beta_0\geq\beta$. This shows that the constant $\beta$ is indeed the best possible.

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