Weighted weak-type inequalities for some fractional integral operators

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Abstract: For $0 < \alpha < 1$, let W_α and R_α denote Weyl fractional integral operator and Riemann-Liouville fractional integral operator, respectively. We establish sharp versions of Muckenhoupt-Wheeden conjecture for these operators. Specifically, we prove that for any weight w on $[0,\infty)$, we have

$$||W_\alpha f||_{L^{1/(1-\alpha),\infty}(w)} \leq \alpha^{-1}||f||_{L^1((M_- w)^{1-\alpha})}$$

and

$$||R_\alpha f||_{L^{1/(1-\alpha),\infty}(w)} \leq \alpha^{-1}||f||_{L^1((M_+ w)^{1-\alpha})}.$$

Here M_-, M_+ denote the one-sided Hardy-Littlewood maximal operators on $[0,\infty)$. In each of the estimates, the constant α^{-1} is the best possible.

Key words: fractional integral; weight; Muckenhoupt-Wheeden conjecture; best constant.

1. Introduction. The purpose of this paper is to investigate certain weighted inequalities which arise naturally in the context of fractional integral operators. Our starting point is the classical result obtained by Fefferman and Stein in 1971. Let w be a weight (i.e., a nonnegative, locally integrable function) on $\mathbb{R}^d$ and let M stand for the usual Hardy-Littlewood maximal operator. Then, as shown in [4], there exists a universal constant c such that

$$\lambda w\left(\left\{x \in \mathbb{R}^d : Mf(x) \geq \lambda\right\}\right) \leq c||f||_{L^1(Mw)},$$

for any locally integrable function f on $\mathbb{R}^d$ and any $\lambda > 0$. Here we have used the standard notation $w(E) = \int_E w(x)\mathrm{d}x$ and $||f||_{L^1(Mw)} = \int_{\mathbb{R}^d} |f(x)|Mw(x)\mathrm{d}x$. The above statement gave rise to the following natural question, formulated by Muckenhoupt and Wheeden in the seventies. Suppose that T is a Calderón-Zygmund singular integral operator. Is there a constant c, depending only on T, such that for each $\lambda > 0$,

$$\lambda w\left(\left\{x \in \mathbb{R}^d : Tf(x) \geq \lambda\right\}\right) \leq c||f||_{L^1(Mw)}\,? \tag{1.1}$$

This problem, called the Muckenhoupt-Wheeden conjecture, remained open for a long time, and many mathematicians contributed to interesting partial results in this direction. In particular, Chanillo and Wheeden proved in [3] that the estimate holds true for the square function; Buckley [2] showed that the conjecture is true for the weights $w_\delta(x) = |x|^{-d(1-\delta)}$, $0 < \delta < 1$; Pérez showed that if M^2 denotes the second iteration of M, then

$$\lambda w\left(\left\{x \in \mathbb{R}^d : Tf(x) \geq \lambda\right\}\right) \leq c||f||_{L^1(M^2 w)}, \quad \lambda > 0.$$

Actually, he proved a stronger statement, in which the operator M^2 was replaced by a smaller object $M_{L(\log L)^\varepsilon}$. We refer the interested reader to [10] for details. Consult also the recent works of Lerner, Ombrosi and Pérez [6, 7, 8] for further results concerning the weaker form of (1.1). In 2010, the Muckenhoupt-Wheeden conjecture was finally shown to be false. See the counterexamples by Reguera, Thiele, Nazarov, Reznikov, Vasyunin and Volberg, presented in [9, 11, 12].

The purpose of this note is to study an appropriate version of Muckenhoupt-Wheeden conjecture in the setting of classical fractional integral operators on $[0,\infty)$. Let us recall the necessary definitions. For $0 < \alpha < 1$, Weyl fractional integral operator W_α and Riemann-Liouville fractional integral operator R_α are defined by

$$W_\alpha f(x) = \int_x^\infty f(t)(t-x)^{\alpha-1}\mathrm{d}t$$

and

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$$R_\alpha f(x) = \int_0^x f(t)(x-t)^{\alpha-1}\mathrm{d}t,$$

where f is a locally integrable function on the positive halfline. These operators are fundamental objects of fractional calculus and play important role in applications. Note that W_α is the adjoint to R_α and vice versa, in the sense that

$$\int_0^\infty f(x)W_\alpha g(x)\mathrm{d}x = \int_0^\infty R_\alpha f(x)g(x)\mathrm{d}x$$

provided f and g are sufficiently regular (e.g., both are nonnegative).

We go back to (1.1). The first question we need to answer concerns the appropriate form of this estimate for W_α and R_α. To gain some intuition, note that Muckenhoupt-Wheeden conjecture is a weighted weak-type (1,1) estimate, and can be understood as a boundary of the classical L^p-boundedness of Calderón-Zygmund operators, $1 < p < \infty$. Let us inspect the corresponding results for fractional integrals. If $1 < p < 1/\alpha$, $1/q = 1/p - \alpha$ and T is W_α or R_α, then it is well-known (see e.g. Theorem 383 in Hardy, Littlewood and Polya [5]) that

$$\left(\int_0^\infty |Tf(x)|^q\mathrm{d}x\right)^{1/q} \leq C_{p,q}\left(\int_0^\infty |f(x)|^p\mathrm{d}x\right)^{1/p}$$

for some finite $C_{p,q}$ which depends only on the parameters indicated. See also [1] for related results in the weighted setting. In the limit case $p = 1$, $q = 1/(1-\alpha)$ the above $L^p \to L^q$ estimate does not hold with any finite constant, but we have the corresponding weak-type substitute. This suggests that the version of (1.1) should describe the action between the weighted spaces L^1 and $L^{1/(1-\alpha),\infty}$. The next problem concerns the maximal operator to be used in the weight on the right-hand side. It turns out that W_α and R_α will require different objects, the so called Hardy-Littlewood one-sided maximal operators M_- and M_+. These act on locally integrable functions $f : [0,\infty) \to \mathbb{R}$ by the formulae

$$M_-f(x) = \sup_{0\leq u<x} \frac{1}{x-u}\int_u^x f(t)\mathrm{d}t,$$

and

$$M_+f(x) = \sup_{u>x} \frac{1}{u-x}\int_x^u f(t)\mathrm{d}t.$$

The final comment is that instead of the standard norming of weak spaces, we will work under slightly different, but equivalent norms. For any $1 < p < \infty$, any weight w and any locally integrable function f on $[0,\infty)$, we put

$$||f||_{L^{p,\infty}(w)} = \sup\left\{\frac{1}{w(I)^{1-1/p}}\int_I |f(x)|w(x)\mathrm{d}x\right\},$$

the supremum taken over all subsets $I \subset [0,\infty)$ such that $0 < w(I) < \infty$.

We are ready to formulate our main results.

Theorem 1.1. *For any weight w on $[0,\infty)$ and any $0 < \alpha < 1$, we have*

$$||W_\alpha f||_{L^{1/(1-\alpha),\infty}(w)} \leq \alpha^{-1}||f||_{L^1((M_-w)^{1-\alpha})}.$$

The constant α^{-1} is the best possible.

Theorem 1.2. *For any weight w on $[0,\infty)$ and any $0 < \alpha < 1$, we have*

$$||R_\alpha f||_{L^{1/(1-\alpha),\infty}(w)} \leq \alpha^{-1}||f||_{L^1((M_+w)^{1-\alpha})}.$$

The constant α^{-1} is the best possible.

Note that both weights M_-w and M_+w written in the L^1 norm on the right are raised to the power $1-\alpha$. This is a necessary modification in the setting of fractional integral operators, due to the appearance of $L^{1/(1-\alpha),\infty}$ norm on the left.

2. Proofs. We start with the following auxiliary fact, a Hardy-type inequality on $[0,1]$.

Lemma 2.1. *For any nonnegative function φ on $[0,1]$ and any $1 < q < \infty$, we have*

$$\begin{aligned}&\int_0^1 t^{-1/q}\varphi(t)\,dt\\ &\leq q' \sup_{0<x\leq 1}\left(\frac{1}{x}\int_0^x \varphi(t)\,dt\right)^{1/q}\left(\int_0^1 \varphi(t)\,dt\right)^{1/q'},\end{aligned} \tag{2.1}$$

where $q' = q/(q-1)$ is the harmonic conjugate to q. The constant q' on the right is the best possible.

Proof. By homogeneity, we may assume that $\sup_{0<x\leq 1}\frac{1}{x}\int_0^x \varphi(t)\mathrm{d}t = 1$. Clearly, this implies $\int_0^x \varphi(t)\mathrm{d}t \leq \int_0^x \mathrm{d}t$ for all $x \in (0,1]$. Hence, by a classical lemma of Hardy, if f is a nonnegative, nonincreasing function on $(0,1]$, then

$$\int_0^1 f(t)\varphi(t)\mathrm{d}t \leq \int_0^1 f(t)\mathrm{d}t.$$

Let us apply this bound to the function $f(t) = (t^{-1/q} - c^{-1/q})_+$, where $c > 0$ is a fixed constant. As the result, we get

$$\begin{aligned}&\int_0^1 (t^{-1/q} - c^{-1/q})_+\varphi(t)\mathrm{d}t\\ &\quad\leq \int_0^1 (t^{-1/q} - c^{-1/q})_+\mathrm{d}t\end{aligned}$$

$$= \begin{cases} (q-1)^{-1}c^{1-1/q} & \text{if } 0<c<1,\\ \dfrac{q}{q-1} - c^{-1/q} & \text{if } c \geq 1. \end{cases}$$

Denote the latter expression by $F(c)$. We have proved that

$$\int_0^1 t^{-1/q}\varphi(t)\mathrm{d}t \leq c^{-1/q}\int_0^1 \varphi(t)\mathrm{d}t + F(c).$$

One easily checks that the right-hand side, considered as a function of c, attains its minimum for $c = \int_0^1 \varphi(t)\mathrm{d}t$. Plugging this particular value of c, we get

$$\int_0^1 t^{-1/q}\varphi(t)\mathrm{d}t \leq \frac{q}{q-1}\left(\int_0^1 \varphi(t)\mathrm{d}t\right)^{1-1/q},$$

which is (2.1). The optimality of q' will follow from the sharpness of the constant α^{-1} in Theorems 1.1 and 1.2: it will be clear that if q' could be decreased, we would obtain an improvement of α^{-1} in both theorems. □

Proof of Theorem 1.1. With no loss of generality, we may and do assume that f is nonnegative: indeed, the passage from f to $|f|$ does not affect the L^1 norm of f, and does not decrease the weak norm of $W_\alpha f$. Pick $I \subset [0,\infty)$ with $0 < w(I) < \infty$. Using the fact that W_α and R_α are adjoint to each other, we get

$$\begin{aligned}&\int_I W_\alpha f(x)w(x)\mathrm{d}x\\ &\quad = \int_0^\infty f(x)R_\alpha(\chi_I w)(x)\mathrm{d}x\\ &\quad \leq ||f||_{L^1((M_-w)^{1-\alpha})}||(M_-w)^{\alpha-1}R_\alpha(\chi_I w)||_{L^\infty}.\end{aligned}$$

To analyze the second factor, observe that for any $x>0$ we have

$$\begin{aligned}&(M_-w)^{\alpha-1}R_\alpha(\chi_I w)(x)\\ &= \frac{\int_0^x (x-t)^{\alpha-1}w(t)\chi_I(t)\mathrm{d}t}{\left(\sup_{u\in(0,x]}\frac{1}{x-u}\int_u^x w(t)\mathrm{d}t\right)^{1-\alpha}}\\ &\leq \frac{\int_0^x (x-t)^{\alpha-1}w(t)\chi_I(t)\mathrm{d}t}{\left(\sup_{u\in(0,x]}\frac{1}{x-u}\int_u^x \chi_I(t)w(t)\mathrm{d}t\right)^{1-\alpha}}\\ &= \frac{x^\alpha\int_0^1 (1-t)^{\alpha-1}\chi_I(tx)w(tx)\mathrm{d}t}{\left(\sup_{u\in(0,1]}\frac{1}{1-u}\int_u^1 \chi_I(tx)w(tx)\mathrm{d}t\right)^{1-\alpha}}\\ &= \frac{x^\alpha\int_0^1 t^{\alpha-1}\chi_I((1-t)x)w((1-t)x)\mathrm{d}t}{\left(\sup_{u\in(0,1]}\frac{1}{u}\int_0^u \chi_I((1-t)x)w((1-t)x)\mathrm{d}t\right)^{1-\alpha}}.\end{aligned}$$

By the preceding lemma, applied to $q = 1/(1-\alpha)$ and $\varphi(t) = \chi_I((1-t)w)w((1-t)x)$, $t\in[0,1]$, the latter expression does not exceed

$$\frac{1}{\alpha}x^\alpha\left(\int_0^1 w((1-t)x)\chi_I((1-t)x)\mathrm{d}t\right)^\alpha \leq \frac{1}{\alpha}w(I)^\alpha.$$

Since x was arbitrary, the combination of the above arguments gives

$$\frac{1}{w(I)^\alpha}\int_I W_\alpha f(x)w(x)\mathrm{d}x \leq \alpha^{-1}||f||_{L^1((M_-w)^{1-\alpha})},$$

and it remains to take the supremum over all I to get the weak-type bound.

To show that the constant α^{-1} is optimal, take $I=[0,1]$, $w=\chi_{[0,1]}$ and $f=\chi_{[b,1]}$, where b is an arbitrary number belonging to $(0,1)$. One easily derives that $M_-w = 1$ on $(0,1]$, which implies $||f||_{L^1((M_-w)^{1-\alpha})} = 1-b$. Furthermore,

$$W_\alpha f(x) = \begin{cases} \alpha^{-1}((1-x)^\alpha - (b-x)^\alpha) & \text{if } x\in(0,b),\\ \alpha^{-1}(1-x)^\alpha & \text{if } x\in[b,1),\\ 0 & \text{if } x\geq 1\end{cases}$$

and hence $\int_I W_\alpha f(x)w(x)\mathrm{d}x = (1-b^{\alpha+1})/(\alpha(\alpha+1))$. Consequently, we see that

$$\frac{||W_\alpha f||_{L^{1/(1-\alpha),\infty}(w)}}{||f||_{L^1((M_-w)^{1-\alpha})}} \geq \frac{1-b^{\alpha+1}}{\alpha(\alpha+1)(1-b)},$$

and the latter expression converges to α^{-1} as $b\to 1$. This shows the desired sharpness. □

Proof of Theorem 1.2. The argument is similar to that used in the proof of Theorem 1.1. As previously, we may restrict ourselves to nonnegative f. Given $I\subset[0,\infty)$ with $0<w(I)<\infty$, we write

$$\begin{aligned}&\int_I R_\alpha f(x)w(x)\mathrm{d}x\\ &\quad = \int_0^\infty f(x)W_\alpha(\chi_I w)(x)\mathrm{d}x\\ &\quad \leq ||f||_{L^1((M_+w)^{1-\alpha})}||(M_+w)^{\alpha-1}W_\alpha(\chi_I w)||_{L^\infty}.\end{aligned}$$

Now, for any $x\geq 0$,

$$\begin{aligned}&(M_+w(x))^{\alpha-1}W_\alpha(\chi_I w)(x)\\ &\quad = \frac{\int_x^\infty (t-x)^{\alpha-1}\chi_I(t)w(t)\mathrm{d}t}{\left(\sup_{u>x}\frac{1}{u-x}\int_x^u w(t)\mathrm{d}t\right)^{1-\alpha}}\\ &\quad \leq \frac{\int_x^\infty (t-x)^{\alpha-1}\chi_I(t)w(t)\mathrm{d}t}{\left(\sup_{u>x}\frac{1}{u-x}\int_x^u \chi_I(t)w(t)\mathrm{d}t\right)^{1-\alpha}}\\ &\quad = \frac{x^\alpha\int_1^\infty (t-1)^{\alpha-1}\chi_I(tx)w(tx)\mathrm{d}t}{\left(\sup_{u>1}\frac{1}{u-1}\int_1^u \chi_I(tx)w(tx)\mathrm{d}t\right)^{1-\alpha}}\end{aligned}$$

$$= \frac{x^\alpha \int_0^\infty t^{\alpha-1}\chi_I((t+1)x)w((t+1)x)\mathrm{d}t}{\left(\sup_{u>0}\frac{1}{u}\int_0^u \chi_I((t+1)x)w((t+1)x)\mathrm{d}t\right)^{1-\alpha}}.$$

Fix a positive number K. Note that

$$\frac{\int_0^K t^{\alpha-1}\chi_I((t+1)x)w((t+1)x)\mathrm{d}t}{\left(\sup_{u>0}\frac{1}{u}\int_0^u \chi_I((t+1)x)w((t+1)x)\mathrm{d}t\right)^{1-\alpha}}$$
$$\leq \frac{\int_0^K t^{\alpha-1}\chi_I((t+1)x)w((t+1)x)\mathrm{d}t}{\left(\sup_{u\in(0,K]}\frac{1}{u}\int_0^u \chi_I((t+1)x)w((t+1)x)\mathrm{d}t\right)^{1-\alpha}}$$
$$= K^\alpha \frac{\int_0^1 t^{\alpha-1}\varphi(t)\mathrm{d}t}{\left(\sup_{u\in(0,1]}\frac{1}{u}\int_0^u \varphi(t)\mathrm{d}t\right)^{1-\alpha}},$$

where $\varphi(t) = \chi_I((Kt+1)x)w((Kt+1)x)$. By the preceding lemma, the latter expression above does not exceed

$$\alpha^{-1}K^\alpha\left(\int_0^1 \varphi(t)\mathrm{d}t\right)^\alpha$$
$$= \alpha^{-1}\left(\int_0^K \chi_I((t+1)x)w((t+1)x)\mathrm{d}t\right)^\alpha$$
$$\leq \alpha^{-1}\left(\int_0^\infty \chi_I((t+1)x)w((t+1)x)\mathrm{d}t\right)^\alpha$$
$$\leq \alpha^{-1}x^{-\alpha}w(I)^\alpha.$$

Since the numbers K and x were arbitrary, we get

$$\int_I R_\alpha f(x)w(x)\mathrm{d}x \leq \alpha^{-1}||f||_{L^1((M_+w)^{1-\alpha})}w(I)^\alpha,$$

which gives the weak-type bound for R_α.

It remains to prove that the constant α^{-1} cannot be improved. Take $I = [0,1]$, $w = \chi_I$ and $f = \chi_{[0,b)}$, where b is an arbitrary number from $(0,1)$. We easily check that $M_+w = 1$ on $[0,1)$ and hence $||f||_{L^1((M_+w)^{1-\alpha})} = b$. Furthermore, we have

$$R_\alpha f(x) = \begin{cases} \alpha^{-1}x^\alpha & \text{if } x \leq b, \\ \alpha^{-1}(x^\alpha - (x-b)^\alpha) & \text{if } x > b, \end{cases}$$

so $\int_I R_\alpha f(x)\mathrm{d}x = (1-(1-b)^{\alpha+1})/(\alpha(\alpha+1))$. These calculations give

$$\frac{||R_\alpha f||_{L^{1/(1-\alpha),\infty}(w)}}{||f||_{L^1((M_+w)^{1-\alpha})}} \geq \frac{1-(1-b)^{\alpha+1}}{\alpha(\alpha+1)b},$$

and the expression on the right can be made arbitrarily close to α^{-1} by choosing appropriately small b. This proves the claimed optimality of the constant α^{-1} and completes the proof of the theorem. □

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