AN EXTENSION OF PRATELLI'S INEQUALITY

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Abstract. Let X, Y be càdlàg martingales and let $Y^{\#}$ denote the sharp function of Y. The paper contains the proof of the estimate

$$\left\|\int_0^\infty |d\langle X,Y\rangle_t|\right\|_1 \leq \sqrt{2}\left\|\langle X\rangle^{1/2}Y^{\#}\right\|_1$$

for the total variation between X and Y. The constant $\sqrt{2}$ is shown to be the best possible. The proof rests on the construction of an appropriate special function, enjoying certain size and concavity requirements.

1. Introduction

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, equipped with the discrete-time filtration $(\mathcal{F}_n)_{n\geq 0}$. Suppose further that $f=(f_n)_{n\geq 0}$, $g=(g_n)_{n\geq 0}$ are two real-valued and uniformly integrable martingales, with the corresponding pointwise limits denoted by f_∞ and g_∞. Then $df=(df_n)_{n\geq 0}$, the difference sequence of f, is defined by $df_0=f_0$ and $df_n=f_n-f_{n-1}$. The conditional square function $s(f)$ of f is given by the formula $s(f)=\left(\sum_{n\geq 0}\mathbb{E}(|df_n|^2|\mathcal{F}_{n-1})\right)^{1/2}$, with the convention $\mathcal{F}_{-1}=\mathcal{F}_0$. We will also use the truncated (or localized) version $s_m(f)$, defined by $s_m(f)=\left(\sum_{n=0}^m\mathbb{E}(|df_n|^2|\mathcal{F}_{n-1})\right)^{1/2}$. The martingale f belongs to the space h^1 if and only if its conditional square function is integrable; we define the corresponding h^1-norm by $\|f\|_{h^1}:=\|s(f)\|_{L^1}$. It is well-known that h^1 is contained strictly in the usual Hardy space H^1, defined as the class of martingales f whose maximal function $\sup_{n\geq 0}|f_n|$ lies in L^1. More specifically, there is a finite universal constant C such that $\|f\|_{H^1}:=\mathbb{E}\sup_{n\geq 0}|f_n|\leq C\|f\|_{h^1}$, but the reverse bound fails to hold, unless some additional conditions on the filtration are imposed. See Section 2.2 in [22] for details, consult also [8, 13, 21] for more on the subject.

Following Garsia [8], the martingale g belongs to the class BMO_2, the space of martingales of bounded mean oscillation, if the quantity

$$\|g\|_{BMO_2}=\sup_{n\geq 0}\left\|\mathbb{E}\big(|g_\infty-g_{n-1}|^2|\mathcal{F}_n\big)\right\|_\infty^{1/2}$$

is finite (we use the convention $g_{-1}=g_0$). This space is dual to H^1 (cf. [8]). (For the analytic versions of the above definition and duality, consult John and Nirenberg [12] and Fefferman [7]). In particular, there is a finite constant C such that we have

$$|\mathbb{E}f_ng_n|\leq C\,\|f\|_{H^1}\,\|g\|_{BMO_2},\qquad n=0,\,1,\,2,\,\ldots.$$

Pratelli [18] established a version of this estimate, in which H^1 is replaced by the smaller space h^1. Consider the so-called little space bmo_2, which consists of all martingales g for

2010 *Mathematics Subject Classification.* Primary: 60G44; Secondary: 60G42.
Key words and phrases. Martingale, skew bracket, BMO, sharp function, best constant.

which

$$\|g\|_{bmo_2} = \sup_{n\geq 0} \big\|\mathbb{E}\big(|g_\infty - g_n|^2|\mathcal{F}_n\big)\big\|_\infty^{1/2} < \infty$$

(note the subtle difference in comparison to BMO_2: under the expectation, the variable g_n, instead of g_{n-1}, is subtracted). Then bmo_2 contains BMO_2 properly (cf. Herz [11]). The aforementioned result of Pratelli asserts that

$$\Big\|\sum_{n>0} |df_n||dg_n|\Big\|_{L^1} \leq \sqrt{2}\|f\|_{h^1}\|g\|_{bmo_2}. \tag{1.1}$$

The purpose of this paper is to strengthen this statement. First, we will prove that the constant $\sqrt{2}$ is the best possible. Second, we will show that the product $\|f\|_{h^1}\|g\|_{bmo_2}$ can be replaced by the L^1-norm of a single random variable depending in a bilinear manner on f and g. To explain this precisely, let us recall the definition of the sharp maximal function $g^\#$ of an L^2-bounded martingale g. It is given by the formula

$$g^\# = \sup_{n\geq 0} \Big(\mathbb{E}\big(|g_\infty - g_n|^2|\mathcal{F}_n\big)\Big)^{1/2},$$

and its analytic version appears in various contexts of interpolation theory and, in particular, is of significant importance for the boundedness properties of various classes of operators. For a unified treatment of this topic, we refer the interested reader to Chapter 7 in [10] or Chapter IV in [20].

Our purpose is to establish the following extension of (1.1).

Theorem 1.1. *For any uniformly integrable martingale f and any L^2-bounded martingale g, we have*

$$\Big\|\sum_{n>0} |df_n||dg_n|\Big\|_{L^1} \leq \sqrt{2}\big\|s(f)g^\#\big\|_{L^1}. \tag{1.2}$$

The constant $\sqrt{2}$ is the best possible, it is already optimal in (1.1).

Note that (1.2) does generalize (1.1): indeed, if $g \in bmo_2$, then $g^\# \leq \|g\|_{bmo_2}$ almost surely and hence $\big\|s(f)g^\#\big\|_{L^1} \leq \|f\|_{h^1}\|g\|_{bmo_2}$.

The above statement should be compared to analogous estimates obtained in [16, 17] which, however, do not involve the sharp function under the L^1-norm. Furthermore, it should be emphasized that the results in [16, 17] were established under additional assumptions on the regularity of the processes and/or under a different norming of the Hardy space. Here we do not impose any conditions on the filtration. Thus, by a standard discretization argument (see e.g. Remark 11.1.8 and Lemma 11.1.9 in [3]), we can extend the result to a general, continuous-time setting. Let us briefly discuss this possibility. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, equipped with a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$. Suppose that $X = (X_t)_{t\geq 0}$, $Y = (Y_t)_{t\geq 0}$ are adapted, uniformly integrable martingales, whose trajectories are right-continuous and have limits from the left. For a locally square integrable X, we denote by $\langle X\rangle = (\langle X\rangle_t)_{t\geq 0}$ the associated skew bracket, see Chapters VI and VII in Dellacherie and Meyer [5] or Chapter 4 in Métivier [14] for details. Then $\langle\cdot,\cdot\rangle$ is the quadratic form obtained by polarization and $\int_0^\infty |d\langle\cdot,\cdot\rangle|$ denotes the associated total variation. For an L^2-bounded martingale Y, its sharp function is given by

$$Y^{\#} = \sup_{t\geq 0} \Big(\mathbb{E}\big((Y_\infty - Y_t)^2|\mathcal{F}_t\big)\Big)^{1/2}.$$

Then we have the following version of Theorem 1.1. For related statements with applications to financial mathematics, see e.g. [2, 4].

Theorem 1.2. *For all X, Y as above, we have the sharp estimate*

$$\left\| \int_0^\infty |d\langle X, Y\rangle| \right\|_{L^1} \leq \sqrt{2} \left\| \langle X\rangle^{1/2} Y^{\#} \right\|_{L^1}. \tag{1.3}$$

The constant $\sqrt{2}$ is the best possible. It is already optimal in the estimate

$$\left\| \int_0^\infty |d\langle X, Y\rangle| \right\|_{L^1} \leq \sqrt{2} \left\| \langle X\rangle^{1/2} \right\|_{L^1} \|Y^{\#}\|_{L^\infty}, \tag{1.4}$$

even if X and Y are assumed to have continuous trajectories.

We take the advantage to mention here another related result, obtained in [1, Theorem 1.1 (iii)], [6, Lemma 1.6] and [9, Corollary 5.19], under some additional assumptions on the filtration and for continuous martingales:

$$\left\| \int_0^\infty |\mathrm{d}\langle X, Y\rangle| \right\|_{L^p} \leq \sqrt{2p} \left\| \langle X\rangle^{1/2} \right\|_{L^p} \|Y^{\#}\|_{L^\infty}.$$

Furthermore, the order $O(\sqrt{p})$ as $p \to \infty$ in the above estimate is optimal.

Consider the following application of (1.3): if $1 < p \leq 2$ and $Y \in L^p$ has mean zero, then we have

$$\begin{aligned}
\|Y\|_{L^p} = \sup_{\|X\|_{L^{p'}}\leq 1} \mathbb{E} X_\infty Y_\infty &\leq \sup_{\|X\|_{L^{p'}}\leq 1} \left\| \int_0^\infty |\mathrm{d}\langle X, Y\rangle| \right\|_{L^1} \\
&\leq \sup_{\|X\|_{L^{p'}}\leq 1} \sqrt{2} \left\| \langle X\rangle^{1/2} Y^{\#} \right\|_{L^1} \\
&\leq \sup_{\|X\|_{L^{p'}}\leq 1} \sqrt{2} \left\| \langle X\rangle^{1/2} \right\|_{L^{p'}} \left\|Y^{\#}\right\|_{L^p} \leq \sqrt{\frac{p}{p-1}} \left\|Y^{\#}\right\|_{L^p}.
\end{aligned}$$

Here the last inequality follows from the bound $\left\|\langle X\rangle^{1/2}\right\|_{L^{p'}} \leq \sqrt{p'/2}\|X\|_{L^p}$ (see [21]). Note that the estimate $\|Y\|_{L^p} \leq c_p \left\|Y^{\#}\right\|_{L^p}$ fails for $p > 2$, even for discrete-time martingales. To see this, fix such a p, take an arbitrary $\varepsilon > 0$ and consider the martingale $g = (g_0, g_1, g_1, g_1, \ldots)$ with $g_0 \equiv 0$ and $\mathbb{P}(g_1 = -\varepsilon) = 1 - \mathbb{P}(g_1 = \varepsilon^{-1}) = (1+\varepsilon^2)^{-1}$. Then $g^{\#} = \mathbb{E}|g_1|^2 = 1$ almost surely and $\|g\|_{L^p} \gtrsim \varepsilon^{-1+2/p} \to \infty$ as $\varepsilon \to 0$.

Our proof of (1.2) will exploit Burkholder's method (sometimes referred to as the Bellman function method): we will deduce the validity of the estimate from the existence of a certain special function enjoying an appropriate concavity. The sharpness of (1.4) will be obtained by the explicit construction of extremal examples.

2. Proof of Theorems 1.1 and 1.2

2.1. **Proof of** (1.2) **and** (1.3)**.** A central role in our considerations is played by the special function $B : (0, \infty) \times \mathbb{R}^3 \to \mathbb{R}$, given by the formula

$$B(w, x, y, z) = \frac{x(y^2 - z + 2w^2)}{w}.$$

We will show that this object enjoys the following property.

Lemma 2.1. *For any $w > 0$ and $x, y, z \in \mathbb{R}$ such that $0 \leq z - y^2 \leq w^2$ and any $d, e, f \in \mathbb{R}$ we have the inequality*

$$\begin{aligned} &B\big(w \vee (z+f-(y+e)^2)_+^{1/2}, (x^2+d^2)^{1/2}, y+e, z+f\big) \\ &\geq B(w,x,y,z) + (x^2+d^2)^{1/2}\frac{2ye-f}{w} + \frac{d^2 w}{2(x^2+d^2)^{1/2}} + (x^2+d^2)^{1/2}\frac{e^2}{w}. \end{aligned} \tag{2.1}$$

Proof. We consider two cases. If $z+f-(y+e)^2 < w^2$, then the estimate is equivalent to

$$\Big((x^2+d^2)^{1/2} - x\Big)\frac{y^2-z+2w^2}{w} \geq \frac{d^2 w}{2(x^2+d^2)^{1/2}},$$

which is obvious: we have $y^2 - z + 2w^2 \geq w^2$ and $\big(x^2+d^2\big)^{1/2} - x \geq d^2/(2(x^2+d^2)^{1/2})$. If the reverse inequality $z+f-(y+e)^2 \geq w^2$ holds, then we rewrite (2.1) in the form

$$\begin{aligned} (x^2+d^2)^{1/2}&\left[(z+f-(y+e)^2)^{1/2} + \frac{z+f-(y+e)^2-2w^2}{w}\right] \\ &\geq (x-(x^2+d^2)^{1/2})\frac{y^2-z+2w^2}{w} + \frac{d^2 w}{2(x^2+d^2)^{1/2}}. \end{aligned}$$

Observe that the left-hand side decreases as e increases: therefore, it is enough to show the claim for $z+f-(y+e)^2 = w^2$, which follows from the previous case and continuity. Let us observe that by Schwarz' inequality, (2.1) gives the slightly weaker bound

$$\begin{aligned} &B\big(w \vee (z+f-(y+e)^2)_+^{1/2}, (x^2+d^2)^{1/2}, y+e, z+f\big) \\ &\qquad \geq B(w,x,y,z) + (x^2+d^2)^{1/2}\frac{2ye-f}{w} + \sqrt{2}|d||e|, \end{aligned} \tag{2.2}$$

which will be more suitable for our purposes below. □

Proof of (1.2) *and* (1.3). It is enough to establish (1.2), the second estimate follows by standard discretization. Fix f, g as in the statement and introduce the auxiliary sequences $h_n = \mathbb{E}(g^2|\mathcal{F}_n)$ and $w_n = \max_{0\leq k\leq n}(h_k - g_k^2)^{1/2}$ for $n = 0, 1, 2, \ldots$ (note that $h_k \geq g_k^2$ by Schwarz' inequality, so w_n is well-defined). We may assume that $w_0 > 0$ almost surely (and hence also $\mathbb{P}(w_n > 0) = 1$ for all n), by simple perturbation argument. The key observation is that for any $n \geq 1$ we have

$$\mathbb{E}B(w_n, s_n(f), g_n, h_n) \geq \mathbb{E}B(w_{n-1}, s_{n-1}(f), g_{n-1}, h_{n-1}) + \sqrt{2}\mathbb{E}|df_n||dg_n|. \tag{2.3}$$

To see this, we apply (2.2) with $w = w_{n-1}$, $x = s_{n-1}(f)$, $y = g_{n-1}$, $z = h_{n-1}$ and $d = \big(\mathbb{E}(|df_n|^2|\mathcal{F}_{n-1})\big)^{1/2}$, $e = dg_n$, $f = dh_n$. The required condition $0 \leq z - y^2 \leq w$ follows from Schwarz' inequality and the definition of the sharp function. Integrating both sides gives (2.3), since g, h are martingales. Therefore, by induction, we obtain

$$\begin{aligned} \sqrt{2}\left\|\sum_{n=1}^N |df_n||dg_n|\right\|_{L^1} &\leq \sqrt{2}\mathbb{E}\sum_{n=1}^N |df_n||dg_n| + \mathbb{E}B(w_0, s_0(f), g_0, h_0) \\ &\leq \mathbb{E}B(w_N, s_N(f), g_N, h_N). \end{aligned}$$

Since $h_N \geq g_N^2$, we have $\mathbb{E}B(w_N, s_N(f), g_N, h_N) \leq 2\mathbb{E}s_N(f)w_N \leq 2\mathbb{E}s(f)g^{\#}$ and the estimate follows. □

2.2. **Sharpness.** We turn our attention to the optimality of the constant $\sqrt{2}$ in (1.1), (1.2), (1.3) and (1.4). Of course, it is enough to focus on the last estimate. Let δ, $\lambda \in (0,1)$ be fixed parameters. We start with a well-known property of Brownian motion (see e.g. [19], p. 73), which can be proved by a simple martingale argument.

Lemma 2.2. *Let Y be a standard Brownian motion started at zero. Then the stopping time $\sigma = \inf\{t > 0 : |Y_t| = \delta\}$ satisfies $\mathbb{E}\exp(\lambda\sigma) = (\cos(\sqrt{2\lambda}\delta))^{-1}$.*

We are ready for the construction of extremal processes in (1.4). Let $W = (W_t)_{t\geq 0}$ be a standard Brownian motion and introduce the stopping time $\tau_0 \equiv 0$. To describe the appropriate BMO martingale, we will construct first an appropriate two-dimensional martingale, taking values in a parabolic domain $\{(y,z) : y^2 \leq z \leq y^2+1\}$, and then restrict ourselves to the first coordinate of this process (this representation goes back to the works [15, 16]). So, consider the two-dimensional martingale (Y,Z), starting from $(0,1)$, whose evolution is governed by the inductive use of the following two rules.

1° Suppose that we have successfully defined τ_{2n}. If $(Y_{\tau_{2n}}, Z_{\tau_{2n}})$ belongs to the parabola $z = y^2$, then the process (Y,Z) does not move and we set $\tau_{2n+1} := \tau_{2n}$. On the other hand, suppose that $(Y_{\tau_{2n}}, Z_{\tau_{2n}})$ lies on the parabola $z = y^2+1$ and denote $y_0 = Y_{\tau_{2n}}$, $z_0 = Z_{\tau_{2n}}$. Then for $t > \tau_{2n}$, the pair (Y_t, Z_t) evolves along the line segment passing through (y_0, z_0) and tangent to the parabola $z = y^2+1$, until Y gets to $y_0-\delta$ or $y_0+\delta$. More precisely, we let $Y_t = y_0 + (W_t - W_{\tau_{2n}})$ and $Z_t = z_0 + 2y_0(Y_t - y_0)$ for $t \in (\tau_{2n}, \tau_{2n+1}]$, where $\tau_{2n+1} = \inf\{t > \tau_{2n} : W_t = W_{\tau_{2n}} \pm \delta\}$. For the further use, let us record that

$$Z_{\tau_{2n+1}} - Y^2_{\tau_{2n+1}} = z_0 - y_0^2 - (Y_{\tau_{2n+1}} - y_0)^2 = 1-\delta^2 \tag{2.4}$$

and go to 2°.

2° Suppose that we have successfully defined τ_{2n+1}. If $(Y_{\tau_{2n+1}}, Z_{\tau_{2n+1}})$ belongs to the parabola $z = y^2$, then the process (Y,Z) does not move and we set $\tau_{2n+2} := \tau_{2n+1}$. Otherwise, the process (Y,Z) evolves vertically until it gets to one of the parabolas $z = y^2$ or $z = y^2+1$. Formally, for $t \in (\tau_{2n+1}, \tau_{2n+2}]$ we set $Y_t = Y_{\tau_{2n+1}}$ and $Z_t = Z_{\tau_{2n+1}} + (W_t - W_{\tau_{2n+1}})$, where $\tau_{2n+2} = \inf\big\{t > \tau_{2n+1} : W_t - W_{\tau_{2n+1}} \in \{\delta^2-1, \delta^2\}\big\}$. The identity (2.4) implies that the probability of reaching the upper parabola is equal to $1-\delta^2$. Go to 1°.

Let us gather some information about the pair (Y,Z) which follows directly from the above construction. First, as we have announced before, (Y,Z) is a martingale taking values in the parabolic domain $\{(y,z) : y^2 \leq z \leq y^2+1\}$, in particular, Y is L^2 bounded: for any $t \geq 0$ we have $\mathbb{E}Y_t^2 \leq \mathbb{E}Z_t = \mathbb{E}Z_0 = 1$. Furthermore, by the final sentence in the description of 2°,

$$\mathbb{P}(Z_{\tau_{2n+2}} > Y^2_{\tau_{2n+2}} | Z_{\tau_{2n}} > Y^2_{\tau_{2n}}) = 1-\delta^2.$$

This implies that the pair terminates at the lower parabola $z = y^2$, i.e., we have $Z_\infty = Y^2_\infty$ almost surely: indeed, by the above identity, we see that $\mathbb{P}(Z_{\tau_{2n}} > Y^2_{\tau_{2n}}) = (1-\delta^2)^n \to 0$. These observations imply that Y is a BMO martingale: we have $Y^\# \leq 1$ almost surely. To show this, note that for an arbitrary time $t \geq 0$ we have

$$\mathbb{E}((Y_\infty - Y_t)^2|\mathcal{F}_t) = \mathbb{E}(Y^2_\infty|\mathcal{F}_t) - Y_t^2 = \mathbb{E}(Z_\infty|\mathcal{F}_t) - Y_t^2 = Z_t - Y_t^2 \leq 1$$

almost surely. Next, observe that Y behaves like a Brownian motion on the intervals $[\tau_{2n}, \tau_{2n+1}]$ and is constant on the intervals $[\tau_{2n+1}, \tau_{2n+2}]$, $n = 0, 1, 2, \ldots$. Consequently, for $t \in [\tau_{2n}, \tau_{2n+1}]$ we have

$$\langle Y\rangle_t = t - \tau_{2n} + \sum_{k=0}^{n-1}(\tau_{2k+1} - \tau_{2k}).$$

Let $\tau = \lim_{n\to\infty} \tau_n$ be the lifetime of (Y, Z) and define $N = \inf\{n : \tau_{2n+2} = \tau\}$. Moreover, put $\sigma_n = \tau_{2n+1} - \tau_{2n}$ for $n = 0, 1, 2, \ldots$. Then N follows the geometric law: $\mathbb{P}(N \geq n) = (1 - \delta^2)^n$ for $n = 0, 1, 2, \ldots$. Furthermore, by standard properties of Brownian motion and Lemma 2.2, we have

$$\mathbb{E}\exp\left(\lambda\sum_{n=0}^{N}\sigma_n\right) = \sum_{n=0}^{\infty}(\cos(\sqrt{2\lambda}\delta))^{-n-1}(1-\delta^2)^n\delta^2.$$

This can be checked by conditioning with respect to N: indeed, conditionally on the set $\{N = n\}$, the random variables $\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_n$ are independent copies of the variable σ appearing in Lemma 2.2. Note that the series on the right can be made arbitrarily large, by taking first λ sufficiently close to 1 and then δ close to zero: indeed, $\cos x = 1 - x^2/2 + o(x^2)$ as $x \to 0$.

We proceed to the definition of the process X, which is given by the stochastic integral

$$X_t = \int_{0+}^{t} e^{\lambda\langle Y\rangle_s}\mathrm{d}Y_s, \qquad t \geq 0.$$

By the above discussion, we have

$$\langle X\rangle_\infty = \sum_{n=0}^{\infty}\int_{\tau_{2n}}^{\tau_{2n+1}} e^{2\lambda\langle Y\rangle_s}\mathrm{d}s \leq \frac{1}{2\lambda}\exp\left(2\lambda\sum_{n=0}^{N}\sigma_n\right)$$

and hence

$$\mathbb{E}\langle X\rangle_\infty^{1/2} \leq \frac{1}{\sqrt{2\lambda}}\mathbb{E}\exp\left(\lambda\sum_{n=0}^{N}\sigma_n\right).$$

A similar calculation shows that

$$\mathbb{E}\int_0^\infty |d\langle X, Y\rangle_t| = \mathbb{E}X_\infty Y_\infty = \mathbb{E}\sum_{n=0}^{\infty}\int_{\tau_{2n}}^{\tau_{2n+1}} e^{\lambda\langle Y\rangle_s}\mathrm{d}s = \frac{\mathbb{E}\left(\exp\left(\lambda\sum_{n=0}^{N}\sigma_n\right) - 1\right)}{\lambda}.$$

However, as we have discussed above, for appropriately chosen λ and δ, the expectation $\mathbb{E}\exp\left(\lambda\sum_{n=0}^{N}\sigma_n\right)$ can be made arbitrarily large. This immediately implies that the ratio $\mathbb{E}\int_0^\infty |d\langle X, Y\rangle_t|/\mathbb{E}\langle X\rangle_\infty^{1/2}$ can be made as close to $\sqrt{2}$ as we wish. This is precisely the desired sharpness of (1.4).

Acknowledgments

The author would like to thank the anonymous Referees for the careful reading of the first version of the paper and many helpful suggestions. The research was supported by Narodowe Centrum Nauki (Poland), grant number 2018/30/Q/ST1/00072.

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